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THREE SOLUTIONS FOR A CLASS OF NEUMANN DOUBLY EIGENVALUE BOUNDARY VALUE SYSTEMS DRIVEN BY A $(p_1,\dots,p_n)$-LAPLACIAN OPERATOR

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In this paper, we prove the existence of at least three weak solutions for Neumann doubly eigenvalue elliptic systems involving the $(p_1,\dots,p_n)$-Laplacian. The approach is based on a recent three critical points theorem obtained by B. Ricceri [9]. We also give some examples to illustrate the obtained results.

1. Introduction

In this paper, we are interested in ensuring the existence of at least three weak solutions for the following quasilinear elliptic system

$$\begin{cases} -\Delta_{p_i} u_i + a_i(x)|u_i|^{p_i-2}u_i \\ = \lambda F_{u_i}(x,u_1,\dots,u_n) + \mu G_{u_i}(x,u_1,\dots,u_n) & \text{in } \Omega, \\ \dfrac{\partial u_i}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

for $1 \le i \le n$, where $\Omega \subset \mathbb{R}^N (N \ge 1)$ is a non-empty bounded open set with a smooth boundary $\partial\Omega$, $\Delta_{p_i} u_i := \operatorname{div}(|\nabla u_i|^{p_i-2}\nabla u_i)$ is the p_i-Laplacian operator, $p_i > N$, $a_i \in L^\infty(\Omega)$ with $\operatorname{ess\,inf}_\Omega a_i > 0$ for $1 \le i \le n$, λ and μ are positive

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parameters, $F, G : \Omega \times \mathbb{R}^n \to \mathbb{R}$ are measurable functions with respect to $x \in \Omega$ for every $(t_1, \dots, t_n) \in \mathbb{R}^n$ and are C^1 with respect to $(t_1, \dots, t_n) \in \mathbb{R}^n$ for a.e. $x \in \Omega$, F_{u_i} and G_{u_i} denotes the partial derivative of F and G with respect to u_i, respectively, and ν is the outer unit normal to $\partial\Omega$.
Moreover, F and G satisfy the following additional assumptions:

(F_1) for every $M > 0$ and every $1 \le i \le n$,
$$\sup_{|(t_1,\dots,t_n)| \le M} |F_{u_i}(x, t_1, \dots, t_n)| \in L^1(\Omega).$$

(F_2) $F(x, 0, \dots, 0) = 0$ for a.e. $x \in \Omega$.

(G) for every $M > 0$ and every $1 \le i \le n$,
$$\sup_{|(t_1,\dots,t_n)| \le M} |G_{u_i}(x, t_1, \dots, t_n)| \in L^1(\Omega).$$

Throughout this paper, we let X be the Cartesian product of the n Sobolev spaces $W^{1,p_i}(\Omega)$ for $1 \le i \le n$, i.e., $X = W^{1,p_1}(\Omega) \times W^{1,p_2}(\Omega) \times \cdots \times W^{1,p_n}(\Omega)$ equipped with the norm
$$\|u\| := \sum_{i=1}^{n} \|u_i\|_{p_i}, \quad u = (u_1, u_2, \dots, u_n),$$
where
$$\|u_i\|_{p_i} := \Big[\int_\Omega \big(|\nabla u_i|^{p_i} + a_i(x)|u_i|^{p_i} \big) dx \Big]^{\frac{1}{p_i}}$$
for $1 \le i \le n$, which is equivalent to the usual one.

Let
$$c := \max \left\{ \sup_{u_i \in W^{1,p_i}(\Omega) \setminus \{0\}} \frac{\max_{x \in \overline{\Omega}} |u_i(x)|^{p_i}}{\|u_i\|_{p_i}^{p_i}} \; : \; \text{for } 1 \le i \le n \right\}. \tag{2}$$
Since $p_i > N$ for $1 \le i \le n$, the embedding $X \hookrightarrow (C^0(\overline{\Omega}))^n$ is compact, and so $c < +\infty$. It follows from Proposition 4.1 of [2] that
$$\sup_{u_i \in W^{1,p_i}(\Omega) \setminus \{0\}} \frac{\max_{x \in \overline{\Omega}} |u_i(x)|^{p_i}}{\|u_i\|_{p_i}^{p_i}} > \frac{1}{\|a_i\|_1} \quad \text{for } 1 \le i \le n,$$
where $\|a_i\|_1 := \int_\Omega |a_i(x)| dx$ for $1 \le i \le n$, and so $\frac{1}{\|a_i\|_1} \le c$ for $1 \le i \le n$. In addition, if Ω is convex, it is known [2] that
$$\sup_{u_i \in W^{1,p_i}(\Omega) \setminus \{0\}} \frac{\max_{x \in \overline{\Omega}} |u_i(x)|}{\|u_i\|}$$
$$\le 2^{\frac{p_i - 1}{p_i}} \max \left\{ \Big(\frac{1}{\|a_i\|_1} \Big)^{\frac{1}{p_i}}, \frac{\operatorname{diam}(\Omega)}{N^{\frac{1}{p_i}}} \Big(\frac{p_i - 1}{p_i - N} m(\Omega) \Big)^{\frac{p_i - 1}{p_i}} \frac{\|a_i\|_\infty}{\|a_i\|_1} \right\} \tag{3}$$

for $1 \le i \le n$, where $m(\Omega)$ is the Lebesgue measure of the set Ω, and equality occurs when Ω is a ball.

We mean by a (weak) solution of system (1), any $u = (u_1, u_2, \dots, u_n) \in X$ such that

$$\int_\Omega \sum_{i=1}^n |\nabla u_i(x)|^{p_i-2} \nabla u_i(x) \nabla v_i(x) dx + \int_\Omega \sum_{i=1}^n a_i(x)|u_i(x)|^{p_i-2} u_i(x) v_i(x) dx - \lambda \int_\Omega \sum_{i=1}^n F_{u_i}(x, u_1(x), \dots, u_n(x)) v_i(x) dx - \mu \int_\Omega \sum_{i=1}^n G_{u_i}(x, u_1(x), \dots, u_n(x)) v_i(x) dx = 0$$

for all $v = (v_1, v_2, \dots, v_n) \in X$.

In recent years, many publications (see, e.g., [1, 3–6] and the references therein) have appeared about elliptic problems with Neumann boundary conditions which have been used in a great variety of application. For example in [1], Afrouzi et al., using an abstract critical point result of B. Ricceri [8], established the existence of at least three weak solutions for system (1), while in [6], the authors, with the same method, proved the existence of at least three solutions for a class of two-point boundary value systems of the type

$$\begin{cases} -u_i'' + u_i = \lambda F_{u_i}(x, u_1, \dots, u_n) + \mu G_{u_i}(x, u_1, \dots, u_n) & \text{in } (a,b), \\ u_i'(a) = u_i'(b) = 0, \end{cases} \tag{4}$$

for $1 \le i \le n$. Also in [5], G. Bonanno and P. Candito using Ricceri's three critical points theorem [7], established the existence of an interval $\Lambda \subseteq]0, +\infty[$ and a positive real number q such that for each $\lambda \in \Lambda$, the problem

$$\begin{cases} -\Delta_p u + a(x)|u|^{p-2}u = \lambda f(x,u) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{5}$$

where $\Omega \subset \mathbb{R}^N (N \ge 1)$ is a non-empty bounded open set with a boundary $\partial\Omega$ of class C^1, $a \in L^\infty(\Omega)$ with $\operatorname{ess\,inf}_\Omega a > 0$, $p > N$, $\lambda > 0$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function, possesses at least three weak solutions whose norms in $W^{1,p}(\Omega)$ are less than q.

The goal of this work is to establish some new criteria for system (1) to have at least three weak solutions in X, by means of a very recent abstract critical points result of B. Ricceri [9]. First, we recall the following result of [9, Theorem 1], with easy manipulations, that we are going to use in the sequel.

Lemma 1.1. *Let X be a reflexive real Banach space; $\Phi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*,* *bounded on bounded subsets of X; $\Psi : X \to \mathbb{R}$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that*

$$\Phi(0) = \Psi(0) = 0.$$

Assume that there exists $r > 0$ and $\bar{x} \in X$, with $r < \Phi(\bar{x})$, such that

(a$_1$) $\dfrac{\sup_{\Phi(x)\le r}\Psi(x)}{r} < \dfrac{\Psi(\bar{x})}{\Phi(\bar{x})}$;

(a$_2$) *for each $\lambda \in \Lambda_r := \left]\dfrac{\Phi(\bar{x})}{\Psi(\bar{x})}, \dfrac{r}{\sup_{\Phi(x)\le r}\Psi(x)}\right[$, the functional $\Phi - \lambda\Psi$ is coercive.*

Then, for each compact interval $[a,b] \subseteq \Lambda_r$, there exists $\rho > 0$ with the following property: for every $\lambda \in [a,b]$ and every C^1 functional $\Gamma : X \to \mathbb{R}$ with compact derivative, there exists $\delta > 0$ such that, for each $\mu \in [0,\delta]$, the equation

$$\Phi'(x) - \lambda\Psi'(x) - \mu\Gamma'(x) = 0$$

has at least three solutions in X whose norms are less than ρ.

2. Main results

In the present section we discuss the existence of multiple solutions for system (1). For any $\gamma > 0$, we denote by $K(\gamma)$ the set

$$\left\{(t_1,\ldots,t_n) \in \mathbb{R}^n : \sum_{i=1}^{n} \frac{|t_i|^{p_i}}{p_i} \le \gamma\right\}.$$

This set will be used in some of our hypotheses with appropriate choices of γ.

We state our main result as follows.

Theorem 2.1. *Assume that the conditions* (F$_1$), (F$_2$), (G) *are satisfied, and there exist a positive constant r and a function $w = (w_1,\ldots,w_n) \in X$ such that*

(i) $\displaystyle\sum_{i=1}^{n} \frac{\|w_i\|_{p_i}^{p_i}}{p_i} > r$;

(ii) $\displaystyle\left(r\prod_{i=1}^{n} p_i\right) \frac{\int_\Omega F(x,w_1,\ldots,w_n)dx}{\sum_{i=1}^{n}\Big(\prod_{\substack{j=1\\ j\neq i}}^{n} p_j\Big)\|w_i\|_{p_i}^{p_i}} - \int_\Omega \sup_{(t_1,\ldots,t_n)\in K(cr)} F(x,t_1,\ldots,t_n)dx > 0$;

(iii) $$\limsup_{(|t_1|,\dots,|t_n|)\to(+\infty,\dots,+\infty)} \frac{F(x,t_1,\dots,t_n)}{\sum_{i=1}^n \frac{|t_i|^{p_i}}{p_i}} \le 0.$$

Then, setting

$$\Lambda := \left] \frac{\sum_{i=1}^n \big(\prod_{\substack{j=1\\ j\neq i}}^{n} p_j\big)\|w_i\|_{p_i}^{p_i}}{\big(\prod_{i=1}^n p_i\big)\int_\Omega F(x,w_1,\dots,w_n)dx}, \frac{r}{\int_\Omega \sup_{(t_1,\dots,t_n)\in K(cr)} F(x,t_1,\dots,t_n)dx} \right[,$$

for each compact interval $[a,b] \subseteq \Lambda$, there exists $\rho > 0$ with the following property: for every $\lambda \in [a,b]$, there exists $\delta > 0$ such that, for each $\mu \in [0,\delta]$, system (1) admits at least three weak solutions in X whose norms are less than ρ.

Proof of Theorem 2.1. For each $u = (u_1,\dots,u_n) \in X$, define $\Phi,\Psi : X \to \mathbb{R}$ as

$$\Phi(u) := \sum_{i=1}^n \frac{\|u_i\|_{p_i}^{p_i}}{p_i}$$

and

$$\Psi(u) := \int_\Omega F(x,u_1(x),\dots,u_n(x))dx.$$

Clearly, Φ is bounded on each bounded subset of X and it is well known that Φ and Ψ are well defined and continuously Gâteaux differentiable functionals whose derivatives at the point $u = (u_1,\dots,u_n) \in X$ are the functionals $\Phi'(u)$ and $\Psi'(u)$ given by

$$\begin{aligned}\Phi'(u)(v)&\\ = \int_\Omega \sum_{i=1}^n |\nabla u_i(x)|^{p_i-2}\nabla u_i(x)\nabla v_i(x)dx &+ \int_\Omega \sum_{i=1}^n a_i(x)|u_i(x)|^{p_i-2}u_i(x)v_i(x)dx\end{aligned}$$

and

$$\Psi'(u)(v) = \int_\Omega \sum_{i=1}^n F_{u_i}(x,u_1(x),\dots,u_n(x))v_i(x)dx$$

for every $v = (v_1,\dots,v_n) \in X$, as well as $\Psi' : X \to X^*$ is a compact operator (see Proposition 26.2 of [11]). Furthermore, by Proposition 25.20 of [11], Φ is sequentially weakly lower semicontinuous. Also, $\Phi' : X \to X^*$ is an uniformly monotone operator in X (for more details, see (2.2) of [10]), and since Φ' is coercive and hemicontinuous in X, by applying Theorem 26.A of [11], Φ' admits a continuous inverse on X^*. Moreover, we have

$$\Phi(0) = \Psi(0) = 0.$$

Also, we see that the required hypothesis $\Phi(\bar{x}) > r$ follows from (i) and the definition of Φ by choosing $\bar{x} = w$. Moreover, since for each $u_i \in W^{1,p_i}(\Omega)$

$$\sup_{x\in\Omega} |u_i(x)|^{p_i} \leq c\|u_i\|_{p_i}^{p_i}$$

for $1 \leq i \leq n$ (see (2)), we have that

$$\sup_{x\in\Omega} \sum_{i=1}^{n} \frac{|u_i(x)|^{p_i}}{p_i} \leq c \sum_{i=1}^{n} \frac{\|u_i\|_{p_i}^{p_i}}{p_i} \tag{6}$$

for each $u = (u_1, \ldots, u_n) \in X$. From (6), for each $r > 0$ we obtain

$$\begin{aligned}
\Phi^{-1}((-\infty, r]) &= \Big\{ u = (u_1, \ldots, u_n) \in X : \Phi(u) \leq r \Big\} \\
&= \Big\{ u = (u_1, \ldots, u_n) \in X : \sum_{i=1}^{n} \frac{\|u_i\|_{p_i}^{p_i}}{p_i} \leq r \Big\} \\
&\subseteq \Big\{ u = (u_1, \ldots, u_n) \in X : \sum_{i=1}^{n} \frac{|u_i(x)|^{p_i}}{p_i} \leq cr \quad \text{for all } \ x \in \Omega \Big\}.
\end{aligned}$$

Then,

$$\begin{aligned}
\sup_{u\in\Phi^{-1}((-\infty,r])} \Psi(u) &= \sup_{u\in\Phi^{-1}((-\infty,r])} \int_\Omega F(x, u_1, \ldots, u_n)dx \\
&\leq \int_\Omega \sup_{(t_1,\ldots,t_n)\in K(cr)} F(x, t_1, \ldots, t_n)dx.
\end{aligned}$$

Therefore, from the condition (ii), we have

$$\begin{aligned}
\sup_{u\in\Phi^{-1}((-\infty,r])} \Psi(u) &\leq \int_\Omega \sup_{(t_1,\ldots,t_n)\in K(cr)} F(x, t_1, \ldots, t_n)dx \\
&< \Big(r \prod_{i=1}^{n} p_i \Big) \frac{\int_\Omega F(x, w_1, \ldots, w_n)dx}{\sum_{i=1}^{n} \big(\prod_{\substack{j=1 \\ j\neq i}}^{n} p_j \big) \|w_i\|_{p_i}^{p_i}} \\
&= r \frac{\int_\Omega F(x, w_1, \ldots, w_n)dx}{\sum_{i=1}^{n} \frac{\|w_i\|_{p_i}^{p_i}}{p_i}} \\
&= r \frac{\Psi(w)}{\Phi(w)},
\end{aligned}$$

from which (a_1) of Lemma 1.1 follows.

Thanks to the assumption (iii), the functional $\Phi - \lambda\Psi$ is coercive for every parameter λ, in particular, for every $\lambda \in \Lambda \subseteq \left]\frac{\Phi(w)}{\Psi(w)}, \frac{r}{\sup_{\Phi(u)\le r}\Psi(u)}\right[$. Then, also condition (a_2) holds.

In addition, since $G : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is a measurable function with respect to $x \in \Omega$ for every $(t_1,\dots,t_n) \in \mathbb{R}^n$ and is C^1 with respect to $(t_1,\dots,t_n) \in \mathbb{R}^n$ for a.e. $x \in \Omega$, satisfying condition (G), the functional

$$\Gamma(u) = \int_\Omega G(x,u_1(x),\dots,u_n(x))dx$$

is well defined and continuously Gâteaux differentiable on X with a compact derivative, and

$$\Gamma'(u)(v) = \int_\Omega \sum_{i=1}^n G_{u_i}(x,u_1(x),\dots,u_n(x))v_i(x)dx$$

for all $u = (u_1,\dots,u_n), v = (v_1,\dots,v_n) \in X$. Thus, all the hypotheses of Lemma 1.1 are satisfied. Also note that the solutions of the equation $\Phi'(u) - \lambda\Psi'(u) - \mu\Gamma'(u) = 0$ are exactly the weak solutions of (1). So, the conclusion follows from Lemma 1.1. $\square$

Next, we want to give a verifiable consequence of Theorem 2.1 for a fixed text function w.

Corollary 2.2. *Assume that there exist $n+1$ positive constants θ_i and τ with $\theta_i < \tau$ for $1 \le i \le n$ such that Assumption (iii) in Theorem 2.1 holds, and*

$$\text{(j)} \qquad \frac{1}{c}\frac{\sum_{i=1}^n \Big(\prod_{\substack{j=1\\ j\neq i}}^n p_j\Big)\theta_i^{p_i}}{\sum_{i=1}^n \Big(\prod_{\substack{j=1\\ j\neq i}}^n p_j\Big)\tau^{p_i}\|a_i\|_1}\int_\Omega F(x,\tau,\dots,\tau)dx - \int_\Omega \sup_{(t_1,\dots,t_n)\in K\left(\sum_{i=1}^n \frac{\theta_i^{p_i}}{p_i}\right)} F(x,t_1,\dots,t_n)dx > 0.$$

Then, setting

$$\Lambda := \left]\frac{\sum_{i=1}^n \big(\prod\limits_{\substack{j=1\\ j\neq i}}^n p_j\big)\tau^{p_i}\|a_i\|_1}{\left(\prod_{i=1}^n p_i\right)\int_\Omega F(x,\tau,\dots,\tau)dx}, \right.$$

$$\left.\frac{\sum_{i=1}^n \big(\prod\limits_{\substack{j=1\\ j\neq i}}^n \big)\theta_i^{p_i}}{\left(c\prod_{i=1}^n p_i\right)\int_\Omega \sup\limits_{(t_1,\dots,t_n)\in K\left(\sum_{i=1}^n \frac{\theta_i^{p_i}}{p_i}\right)} F(x,t_1,\dots,t_n)dx}\right[,$$

for each compact interval $[a,b]\subseteq\Lambda$, there exists $\rho>0$ with the following property: for every $\lambda\in[a,b]$, there exists $\delta>0$ such that, for each $\mu\in[0,\delta]$, system (1) admits at least three weak solutions in X whose norms are less than ρ.

Proof. Choose $w(x)=(w_1(x),\dots,w_n(x))=(\tau,\dots,\tau)$ for every $x\in\Omega$. Then we have $\sum_{i=1}^n \frac{\|w_i\|_{p_i}^{p_i}}{p_i}=\sum_{i=1}^n \frac{\tau^{p_i}}{p_i}\|a_i\|_1$. Put $r:=\frac{1}{c}\sum_{i=1}^n \frac{\theta_i^{p_i}}{p_i}$. Now since $\theta_i<\tau$, bearing in mind that $\frac{1}{\|a_i\|_1}\le c$ for $1\le i\le n$, one has $\sum_{i=1}^n \frac{\|w_i\|_{p_i}^{p_i}}{p_i}>r$ which is (i) of Theorem 2.1. Also, since

$$\begin{aligned}
&\left(r\prod_{i=1}^n p_i\right)\frac{\int_\Omega F(x,w_1(x),\dots,w_n(x))dx}{\sum_{i=1}^n \big(\prod\limits_{\substack{j=1\\ j\neq i}}^n p_j\big)\|w_i\|_{p_i}^{p_i}}\\
&\qquad=\frac{1}{c}\frac{\left(\prod_{i=1}^n p_i\right)\sum_{i=1}^n \frac{\theta_i^{p_i}}{p_i}}{\sum_{i=1}^n \big(\prod\limits_{\substack{j=1\\ j\neq i}}^n p_j\big)\tau^{p_i}\|a_i\|_1}\int_\Omega F(x,\tau,\dots,\tau)dx\\
&\qquad=\frac{1}{c}\frac{\sum_{i=1}^n \big(\prod\limits_{\substack{j=1\\ j\neq i}}^n p_j\big)\theta_i^{p_i}}{\sum_{i=1}^n \big(\prod\limits_{\substack{j=1\\ j\neq i}}^n p_j\big)\tau^{p_i}\|a_i\|_1}\int_\Omega F(x,\tau,\dots,\tau)dx,
\end{aligned}$$

(j) guarantees (ii). Thus, all the assumptions of Theorem 2.1 are satisfied and the proof is complete. □

Now, we give an example to illustrate Corollary 2.2.

Example 2.3. Let $\Omega = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 16\}$. Consider the system

$$\begin{cases} -\Delta_3 u_i + |u_i|u_i \\ = \lambda(x^2+y^2)(u_i^+)^{49} e^{-u_i^+}(50-u_i^+) + \mu G_{u_i}(x,y,u_1,u_2) & \text{in } \Omega, \\ \dfrac{\partial u_i}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{7}$$

for $i=1,2$, where $t^+ := \max\{t,0\}$ and $G : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ is an arbitrary function which is measurable with respect to $(x,y) \in \Omega$ for every $(t_1,t_2) \in \mathbb{R}^2$ and is C^1 with respect to $(t_1,t_2) \in \mathbb{R}^2$ for a.e. $(x,y) \in \Omega$, satisfying

$$\sup_{|(t_1,t_2)| \le M} |G_{u_i}(x,y,t_1,t_2)| \in L^1(\Omega)$$

for every $M > 0$ and $i = 1,2$.

By choosing $\tau = 2$, $\theta_i = 1$, $p_i = 3$, $a_i(x,y) \equiv 1$ for $i = 1,2$, and by a simple calculation, we obtain that $c = \frac{2^8}{\pi}$. So, with

$$F(x,y,t_1,t_2) = \begin{cases} 0 & \text{for } t_i < 0,\ i = 1,2, \\ (x^2+y^2) t_2^{50} e^{-t_2} & \text{for } t_1 < 0,\ t_2 \ge 0, \\ (x^2+y^2) t_1^{50} e^{-t_1} & \text{for } t_1 \ge 0,\ t_2 < 0, \\ (x^2+y^2) \sum_{i=1}^{2} t_i^{50} e^{-t_i} & \text{for } t_i \ge 0,\ i = 1,2 \end{cases}$$

for each $(x,y) \in \Omega$ and $(t_1,t_2) \in \mathbb{R}^2$, we see that

$$\begin{aligned} &\frac{1}{c} \frac{p_2\theta_1^{p_1} + p_1\theta_2^{p_2}}{p_2\tau^{p_1}\|a_1\|_1 + p_1\tau^{p_2}\|a_2\|_1} \iint_\Omega F(x,y,\tau,\tau)\,dxdy \\ &- \iint_\Omega \sup_{(t_1,t_2) \in K(\frac{\theta_1^{p_1}}{p_1} + \frac{\theta_2^{p_2}}{p_2})} F(x,y,t_1,t_2)\,dxdy \ge \frac{1}{2^{15}} \times (2^{51}e^{-2}) \iint_\Omega (x^2+y^2)\,dxdy \\ &- \sup_{|t_1|^3+|t_2|^3 \le 2} \sum_{i=1}^{2} t_i^{50} e^{-t_i} \iint_\Omega (x^2+y^2)\,dxdy \ge 2^{43}\pi e^{-2} - 2^8\pi\big((\sqrt[3]{2})^{50} e^{\sqrt[3]{2}}\big) > 0. \end{aligned}$$

Note that

$$\limsup_{|t_1| \to +\infty, |t_2| \to +\infty} \frac{F(x,y,t_1,t_2)}{\sum_{i=1}^{2} \frac{|t_i|^3}{3}} = 0$$

for $(x,y) \in \Omega$. Hence, Corollary 2.2 is applicable to system (7) for

$$\Lambda \subseteq \left] \frac{e^2}{3 \times 2^{50}}, \frac{1}{2^{15} \times (\sqrt[3]{2})^{50} \times 3e^{\sqrt[3]{2}}} \right[.$$

Corollary 2.4. *Let $F : \mathbb{R}^n \to \mathbb{R}$ be a C^1 function and assume that there exist $n+1$ positive constants θ_i and τ with $\theta_i < \tau$ for $1 \le i \le n$ such that*

(k) $\frac{1}{c}\frac{\sum_{i=1}^n \big(\prod\limits_{\substack{j=1\\ j\neq i}}^n p_j\big)\theta_i^{p_i}}{\sum_{i=1}^n \big(\prod\limits_{\substack{j=1\\ j\neq i}}^n p_j\big)\tau^{p_i}\|a_i\|_1} F(\tau,\dots,\tau) - \max\limits_{(t_1,\dots,t_n)\in K(\sum_{i=1}^n \frac{\theta_i^{p_i}}{p_i})} F(t_1,\dots,t_n) > 0$;

(kk) $\limsup_{(|t_1|,\dots,|t_n|)\to(+\infty,\dots,+\infty)} \frac{F(t_1,\dots,t_n)}{\sum_{i=1}^n \frac{|t_i|^{p_i}}{p_i}} \le 0.$

Then, setting

$$\Lambda := \left]\frac{\sum_{i=1}^n \big(\prod\limits_{\substack{j=1\\ j\neq i}}^n p_j\big)\tau^{p_i}\|a_i\|_1}{m(\Omega)\big(\prod_{i=1}^n p_i\big)F(\tau,\dots,\tau)}, \frac{\sum_{i=1}^n \big(\prod\limits_{\substack{j=1\\ j\neq i}}^n p_j\big)\theta_i^{p_i}}{m(\Omega)\big(c\prod_{i=1}^n p_i\big)\max\limits_{(t_1,\dots,t_n)\in K(\sum_{i=1}^n \frac{\theta_i^{p_i}}{p_i})} F(t_1,\dots,t_n)}\right[,$$

for each compact interval $[a,b] \subseteq \Lambda$, there exists $\rho > 0$ with the following property: for every $\lambda \in [a,b]$, there exists $\delta > 0$ such that, for each $\mu \in [0,\delta]$, the system

$$\begin{cases} -\Delta_{p_i} u_i + a_i(x)|u_i|^{p_i-2}u_i & \\ = \lambda F_{u_i}(u_1,\dots,u_n) + \mu G_{u_i}(x,u_1,\dots,u_n) & \text{in } \Omega, \\ \dfrac{\partial u_i}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{8}$$

for $1 \le i \le n$, admits at least three weak solutions in X whose norms are less than ρ.

Finally, we conclude this paper by giving an immediate consequence of Corollary 2.4 and an example of it when $n = 1$.

Corollary 2.5. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be an L^1-Carathéodory function. Put $F(t) = \int_0^t f(\xi)d\xi$ for each $t \in \mathbb{R}$. Assume that there exist two positive constants θ and τ with $\theta < \tau$ such that*

(l) $\dfrac{1}{c}\dfrac{\theta^p}{\tau^p\|a\|_1}F(\tau) - \max\limits_{t\in[-\theta,\theta]} F(t) > 0$;

(ll) $\limsup\limits_{|t|\to+\infty} \dfrac{F(t)}{|t|^p} \le 0.$

Then, setting

$$\Lambda := \left] \frac{\tau^p \|a\|_1}{p m(\Omega) F(\tau)}, \frac{\theta^p}{m(\Omega)(pc) \max_{t\in[-\theta,\theta]} F(t)} \right[,$$

for each compact interval $[a,b] \subseteq \Lambda$*, there exists* $\rho > 0$ *with the following property: for every* $\lambda \in [a,b]$*, there exists* $\delta > 0$ *such that, for each* $\mu \in [0,\delta]$*, the problem*

$$\begin{cases} -\Delta_p u + a(x)|u|^{p-2}u = \lambda f(u) + \mu g(x,u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{9}$$

admits at least three weak solutions in $W^{1,p}(\Omega)$ *whose norms are less than* ρ*.*

Example 2.6. Let Ω be the open unit ball of $\mathbb{R}^2$. Putting $f(t) = t^{11}e^{-t}(12-t)$ for all $t \in \mathbb{R}$, one has $F(t) = \int_0^t f(\xi)d\xi = t^{12}e^{-t}$ for all $t \in \mathbb{R}$. Now, consider the problem

$$\begin{cases} -\Delta_3 u + \frac{2+x}{\pi}|u|u = \lambda e^{-u}u^{11}(12-u) + \mu g(x,y,u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{10}$$

where $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is an arbitrary L^1-Carathéodory function.

Note that, in this case, $c = \frac{216}{\pi}$. In fact, by choosing $\tau = 3$, $\theta = 1$, $p = 3$ and $a(x,y) = \frac{2+x}{\pi}$ we have

$$\begin{aligned} \frac{1}{c}\frac{\theta^p}{\tau^p\|a\|_1}F(\tau) - \max_{t\in[-\theta,\theta]} F(t) &= \frac{1}{2^4 \times 3^6}F(3) - F(-1) \\ &= \frac{3^6\pi}{2^4 e^3} - e \\ &> 0. \end{aligned}$$

Furthermore, we have that $\limsup_{|t|\to+\infty} \frac{F(t)}{|t|^3} = 0$. Thus, all hypotheses of Corollary 2.5 are satisfied. So, setting $\Lambda \subseteq \left] \frac{2e^3}{3^{10}\pi}, \frac{1}{2^3\times 3^4 e} \right[$, for each compact interval $[a,b] \subseteq \Lambda$, there exists $\rho > 0$ with the following property: for every $\lambda \in [a,b]$, there exists $\delta > 0$ such that, for each $\mu \in [0,\delta]$, problem (10) has at least three weak solutions in $W^{1,3}(\Omega)$ whose norms are less than ρ.

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