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ON CERTAIN $q-$BASKAKOV-DURRMEYER OPERATORS

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In this paper we introduce a $q-$ analogue of Baskakaov-beta operators. We establish Voronovskaja-type theorem and obtain local error estimates by these $q-$operators in uniform norm by using the Ditzian-Totik weighted modulus of smoothness for $0 < q < 1$.

1. Introduction

Among various modifications of the celebrated Bernstein polynomials, (cf.[5], [15]) the $q-$ variant based on q-integers was introduced by Phillips in 1997 [19] as

$$B_{n,q}(f,x) = \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right) b_{n,k}(q;x),\ f \in C[0,1],$$

where $b_{n,k}(q;x) = \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{r=0}^{n-k-1}(1-q^r x)$. Since then these operators have been studied by several authors (cf.[17], [20]-[25]). Derriennic [3] introduced a q-analogue of the Durrmeyer operators and established some approximation properties of these operators. In the sequel q-analogue of many well known positive linear operators e.g. Baskakov, modified-beta and Szász operators have been introduced and studied by several authors (cf. [1], [8], [12], [9],[10]). For $f \in C_B[0,\infty)$ (the class of continuous and bounded functions on $[0,\infty)$) we introduce the $q-$ Baskakov-Durrmeyer operators $\mathcal{L}_{n,q}$ as follows:

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$$\mathcal{L}_{n,q}(f,x)=\sum_{k=0}^{\infty}p_{n,k}(q;x)\int_{0}^{\infty/A}q^{k}b_{n,k}(q;u)f(u)\,d_{q}u,$$

where

$$b_{n,k}(q;x)=\frac{q^{k(k-1)/2}x^{k}}{B_{q}(k+1,n)(1+x)_{q}^{n+k+1}},\quad p_{n,k}(q;x)=\begin{bmatrix}n+k-1\\ k\end{bmatrix}_{q}\frac{q^{k(k-1)/2}x^{k}}{(1+x)_{q}^{n+k}}.$$

In the sequel we need some definitions of $q-$ calculus which can be found in [14] and [20]. Let q be a real number in $(0,1)$ and $\mathbb{N}$ be the set of positive integers.
For $n\in\mathbb{N}$, we define the q integer and q-factorial by

$$[n]_{q}=\begin{cases}\frac{1-q^{n}}{1-q}=1+q+q^{2}+...+q^{n-1}, & q\neq 1\\ n, & q=1\end{cases}$$

and

$$[n]_{q}!=\begin{cases}[n]_{q}[n-1]_{q}[n-2]_{q}.....[1]_{q}, & n\in\mathbb{N}\\ 1, & n=0\end{cases}$$

respectively. The $q-$ binomial coefficients $\begin{bmatrix}n\\ k\end{bmatrix}_{q}$ and the q product $(a+b)_{q}^{n}$ are defined by

$$\begin{bmatrix}n\\ k\end{bmatrix}_{q}=\frac{[n]_{q}!}{[k]_{q}![n-k]_{q}!},0\leqslant k\leqslant n$$

and

$$(a+b)_{q}^{n}=\prod_{j=0}^{n-1}(a+q^{j}b)$$

respectively. The $q-$ analogue E_{q}^{x} of classical exponential function which we shall use in this paper is given by

$$E_{q}^{x}=\sum_{j=0}^{\infty}q^{j(j-1)/2}\frac{x^{j}}{[j]_{q}!}.$$

For further properties see [14]. The $q-$ Jackson integrals and $q-$ improper integrals are given by (cf. [13], [16])

$$\int_{0}^{a}f(x)\,d_{q}x=(1-q)\,a\sum_{n=0}^{\infty}f(aq^{n})\,q^{n}$$

and

$$\int_{0}^{\infty/A}f(x)\,d_{q}x=(1-q)\sum_{n=-\infty}^{\infty}f\left(\frac{q^{n}}{A}\right)\frac{q^{n}}{A},A>0$$

respectively, where the sums are assumed to be absolutely convergent.
For $q \in (0,1)$ and any arbitrary real function $f : \mathbb{R} \to \mathbb{R}$, the $q-$derivative $D_q f(t)$ is defined as

$$D_q f(t) = \begin{cases} \dfrac{f(t) - f(qt)}{(1-q)t}; t \neq 0 \\ \lim_{t\to 0} D_q f(t); t = 0. \end{cases}$$

The product formula for $q-$differentiation is given by

$$D_q(f(x)g(x)) = f(qx)D_q(g(x)) + g(x)D_q(f(x)).$$

Analogous to the ordinary gamma and beta functions the $q-$gamma $\Gamma_q(t)$ and the $q-$ beta functions $B_q(t,s)$ are defined as

$$\Gamma_q(t) = \int_0^{1/(1-q)} x^{t-1} E_q^{-qx}\, d_q x$$

$$B_q(t,s) = K(A,t) \int_0^{\infty/A} \frac{x^{t-1}}{(1+x)_q^{t+s}}\, d_q x,$$

where $K(x,t) = \frac{1}{1+x} x^t (1+\frac{1}{x})_q^t (1+x)_q^{1-t}$. In case $t \in \mathbb{N}$, we have

$$K(x,n) = q^{\frac{n(n-1)}{2}},\ K(x,0) = 1$$

and

$$B_q(t,s) = \frac{\Gamma_q(t)\Gamma_q(s)}{\Gamma_q(t+s)}.$$

In the limit $q \uparrow 1$ the functions $\Gamma_q(t)$ and $B_q(t,s)$ reduce to $\Gamma(t)$ and $B(t,s)$ respectively. Moreover, these functions also satisfy certain properties, similar to those of $\Gamma(t)$ and $B(t,s)$. By $C_B^r[0,\infty), r \in \mathbb{N}$ we denote the set of r times differentiable functions such that $f^{(r)} \in C_B[0,\infty)$. The space $C_B[0,\infty)$ is normed by $\|f\| = \sup\{|f(x)| : x \in [0,\infty)\}$. The K-functional and the modulus of smoothness used in this paper are given as follows:

$$\overline{K}_{2,\varphi^\lambda}(f,t^2) = \inf_{g\in W^2_{\varphi^\lambda}} \{\|f-g\| + t^2\|\varphi^{2\lambda} g''\| + t^4\|g''\|\},$$

where $0 \leqslant \lambda \leqslant 1$ and $W^2_{\varphi^\lambda} = \{g \in C_B[0,\infty) : g' \in AC_{\text{loc}}[0,\infty), \varphi^{2\lambda} g'' \in C_B[0,\infty)\}$.
It is known [4] that there exist absolute constants $C_1, C_2 > 0$ such that

$$C_1\overline{K}_{2,\varphi}(f,t^2) \leqslant \omega^2_{\varphi^\lambda}(f,t) \leqslant C_2\overline{K}_{2,\varphi}(f,t^2), \tag{1}$$

where

$$\omega^2_{\varphi^\lambda}(f,t) = \sup_{0<h\leqslant t}\ \sup_{x\in[0,\infty)} \|\Delta^2_{h\varphi^\lambda} f(x)\|$$

is the second order modulus of smoothness of f and $\varphi^\lambda(x)$ is the admissible weight function of the Ditzian-Totik modulus of smoothness. It is easy to see that $\varphi^\lambda(x)$ satisfies properties (I)-(III) p.8 [4]. In what follows, we shall use the notations $\varphi(x) = \sqrt{x(1+x)}$ and $\delta_n^2(x) = \left(\varphi^2(x) + \frac{1}{[n-2]_q}\right)$. The symbols $\rightrightarrows$ and $\subset$ stand for uniform convergence and proper inclusion respectively. Further, the constant C is different at each occurrence.

2. Moments

Remark 2.1. Applying the product rule for differentiation, we easily obtain the relation

$$q^k\varphi^2(x)D_q[p_{n,k}(q;x)] = \left([k]_q - q^k[n]_q x\right)p_{n,k}(q;qx),$$

where D_q denotes the $q-$derivative operator.

Remark 2.2. Making use of the $q-$Taylor's formula

$$g(z) = \sum_{k=0}^{\infty} \frac{(z-x)_q^k}{[k]_q!}\left(D_q^k g(z)\right)_{z=x}$$

for the function $g(z) = \frac{1}{(1+z)_q^n}$ and then put $z = 0$, and in view of the relation

$$(-x)_q^k = q^{k(k-1)/2}(-1)^k x^k$$

we obtain the identity

$$\begin{aligned} 1 = g(0) &= \sum_{k=0}^{\infty} \frac{[n+k-1]_q \ldots [n]_q}{[k]_q!} q^{k(k-1)/2} \frac{x^k}{(1+x)_q^{n+k}} \\ &= \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q q^{k(k-1)/2} \frac{x^k}{(1+x)_q^{n+k}} = \sum_{k=0}^{\infty} p_{n,k}(q;x). \qquad (2) \end{aligned}$$

Lemma 2.3. *Let us define* $T_{n,m}(x) = \mathcal{L}_{n,q}(e_m,x)$, $e_m = t^m$, $m = 0,1,2\ldots$ *Then, we have*

$$T_{n,0}(x) = 1,\ T_{n,1}(x) = \frac{1}{q^2[n-1]_q}(q + [n]_q x)$$

and

$$T_{n,2}(x) = \frac{1}{q^6[n-1]_q[n-2]_q}\left(q^3(1+q) + q(q+1)^2[n]_q x + [n]_q[n+1]_q x^2\right).$$

Further, there holds the recurrence relation:

$$T_{n,m+1}(qx)=\frac{\big([n]_q x+[m+1]_q\big)T_{n,m}(qx)+\varphi^2(x)D_qT_{n,m}(x)}{q^{m+1}[n-m-1]_q},n\geqslant m+2. \quad (3)$$

Proof. From the definition, we have

$$\begin{aligned} T_{n,0}(x) &= \sum_{k=0}^{\infty}\frac{x^k q^{k(k-1)/2}}{(1+x)_q^{n+k}}\frac{q^{k(k+1)/2}}{B_q(k+1,n)}\begin{bmatrix} n+k-1\\ k\end{bmatrix}_q\frac{B_q(k+1,n)}{K(A,k+1)}\\ &= \sum_{k=0}^{\infty}p_{n,k}(q;x). \qquad (4)\end{aligned}$$

Therefore, using (2), we get $T_{n,0}(x)=1$. Next, we have

$$\begin{aligned} T_{n,1}(x) &= \sum_{k=0}^{\infty}q^{k(k-1)/2}q^{-k-1}\begin{bmatrix} n+k-1\\ k\end{bmatrix}_q\frac{[k+1]_q}{[n-1]_q}\frac{x^k}{(1+x)_q^{n+k}}\\ &= \sum_{k=0}^{\infty}q^{k(k-1)/2}\begin{bmatrix} n+k-1\\ k\end{bmatrix}_q\frac{q^{-1}}{[n-1]_q}\frac{x^k}{(1+x)_q^{n+k}}\\ &+ \sum_{k=0}^{\infty}q^{k(k-1)/2}q^{-k-1}\begin{bmatrix} n+k-1\\ k\end{bmatrix}_q\frac{[k]_q}{[n-1]_q}\frac{x^k}{(1+x)_q^{n+k}}\\ &= E_1+E_2,\text{ say,}\end{aligned}$$

where we have used the relation $[k+1]_q=q^k+[k]_q$. Now, as in the estimate of $T_{n,0}(x)$, using (2) we get

$$E_1=\frac{1}{q[n-1]_q}$$

and

$$\begin{aligned} E_2 &= \sum_{k=0}^{\infty}q^{k(k-1)/2}q^{-k-1}\begin{bmatrix} n+k-1\\ k\end{bmatrix}_q\frac{[k]_q}{[n-1]_q}\frac{x^k}{(1+x)_q^{n+k}}\\ &= \frac{[n]_q}{q^2[n-1]_q}x\sum_{k=0}^{\infty}\begin{bmatrix} n+k\\ k\end{bmatrix}_q q^{k(k-1)/2}\frac{x^k}{(1+x)_q^{n+k+1}}=\frac{[n]_q}{q^2[n-1]_q}x.\end{aligned}$$

Therefore, $T_{n,1}(x)=\dfrac{1}{q^2[n-1]_q}(q+[n]_qx)$.

Now, using remark 2.1, we obtain

$$
\begin{aligned}
&\varphi^2(x)D_qT_{n,m}(x)\\
= &\sum_{k=0}^{\infty}\Big(q^{-k}[k]_q-[n]_qx\Big)p_{n,k}(q;qx)\int_0^{\infty/A}q^kb_{n,k}(q;u)u^m\,d_qu\\
= &\sum_{k=0}^{\infty}\Big(q^{-k}[k]_q\Big)p_{n,k}(q;qx)\int_0^{\infty/A}q^kb_{n,k}(q;u)u^m\,d_qu-[n]_qxT_{n,m}(qx)\\
= &\sum_{k=0}^{\infty}p_{n,k}(q;qx)\int_0^{\infty/A}\Big[\Big(q^{-k}[k]_q-\frac{[n+1]_qu}{q}\Big)+\frac{[n+1]_qu}{q}\Big]q^kb_{n,k}(q;u)u^m\,d_qu\\
- &[n]_qxT_{n,m}(qx)\\
= &\sum_{k=0}^{\infty}p_{n,k}(q;qx)\int_0^{\infty/A}\Big(q^{-k}[k]_q-\frac{[n+1]_qu}{q}\Big)q^kb_{n,k}(q;u)u^m\,d_qu\\
+ &\frac{[n+1]_q}{q}T_{n,m+1}(qx)-[n]_qxT_{n,m}(qx).
\end{aligned}
$$

To simplify the integral, we make use of the chain rule (which is applicable only for this particular transformation) for the transformation $u=qz$, which gives $d_qu=qd_qz$ (see page 3-4, [14]). Thus, in view of remark 2.1, we get

$$
\begin{aligned}
I &= q^{m+1}\sum_{k=0}^{\infty}p_{n,k}(q;qx)\int_0^{\infty/A}q^kz^m\varphi^2(z)D_qb_{n,k}(q;z)\,d_qz\\
&= q^{m+1}\sum_{k=0}^{\infty}p_{n,k}(q;qx)\int_0^{\infty/A}q^k\big(z^{m+1}+z^{m+2}\big)D_qb_{n,k}(q;z)\,d_qz\\
&= I_1+I_2.
\end{aligned}
$$

In order to obtain I_1 and I_2 we make use of the q integration by parts

$$\int_a^b u(t)D_q(v(t))d_qt=[u(t)v(t)]_a^b-\int_a^b v(qt)D_q[u(t)]d_qt.$$

Therefore, we get $I_1=-[m+1]_qT_{n,m}(qx)$ and $I_2=-[m+2]_qq^{-1}T_{n,m+1}(qx)$. Combining these expressions and using $q^{-1}\big([n+1]_q-[m+2]_q\big)=q^{m+1}[n-m-1]_q$, we obtain (3). From this recurrence relation, $T_{n,2}$ is easily obtained. □

Corollary 2.4. *The operators $\mathcal{L}_{n,q}(f,x)$ are linear and preserve constants. Using $T_{n,0}(x)=1$, it follows that*

$$\mathcal{L}_{n,q}((t-x),x)=\left(\frac{[n]_q}{q^2[n-1]_q}-1\right)x+\frac{1}{q[n-1]_q}=\frac{q+([2]_q-q^n)x}{q^2[n-1]_q}$$

and

$$\begin{aligned}
&\mathcal{L}_{n,q}((t-x)^2,x)\\
=\ & \frac{1}{q^6[n-1]_q[n-2]_q}\Big[q^3(1+q)+\Big(q(q+1)^2[n]_q-2q^4[n-2]_q\Big)x\\
+\ & \Big([n+1]_q[n]_q-2q^4[n]_q[n-2]_q+q^6[n-1]_q[n-2]_q\Big)x^2\Big]\\
=\ & \frac{\Big([n+1]_q[n]_q-2q^4[n]_q[n-2]_q+q^6[n-1]_q[n-2]_q\Big)\varphi^2(x)}{q^6[n-1]_q[n-2]_q}\\
+\ & \Big(\frac{q(q+1)^2[n]_q-2q^4[n-2]_q-[n+1]_q[n]_q+2q^4[n]_q[n-2]_q}{q^6[n-1]_q[n-2]_q}-1\Big)x\\
+\ & \frac{q^3(1+q)}{q^6[n-1]_q[n-2]_q}.
\end{aligned}$$

It can be verified that the coefficient of x

$$\frac{q(q+1)^2[n]_q-2q^4[n-2]_q-[n+1]_q[n]_q+2q^4[n]_q[n-2]_q}{q^6[n-1]_q[n-2]_q}-1<0.$$

And

$$\begin{aligned}
&\frac{\Big([n+1]_q[n]_q-2q^4[n]_q[n-2]_q+q^6[n-1]_q[n-2]_q\Big)\varphi^2(x)}{q^6[n-1]_q[n-2]_q}\\
&=\frac{\Big(1+2q+3q^2+4q^3+3q^4+a_5q^5+.....a_{2n+1}q^{2n+1}\Big)\varphi^2(x)}{q^6[n-1]_q[n-2]_q}.
\end{aligned}$$

It is observed that $a_j\leqslant 2: j=5,6,....2n+1$. Hence, Numerator $\leqslant 4(1+q+q^2......+q^{2n+1})=4[2n+2]_q$
Now, $[2n+2]_q=(1+q^{n-2})[n-2]_q+q^{2n-4}[6]_q$ Thus

$$\begin{aligned}
&\mathcal{L}_{n,q}((t-x)^2,x)\\
&\leqslant 4\Big(\frac{(1+q^{n-2})[n-2]_q}{q^6[n-1]_q[n-2]_q}+\frac{[6]_q q^{2n-4}}{q^6[n-1]_q[n-2]_q}\Big)\varphi^2(x)+\frac{2}{q^6[n-1]_q[n-2]_q}\\
&\leqslant 4\Big(\frac{2}{q^6[n-1]_q}+\frac{6}{q^6[n-1]_q[n-2]_q}\Big)\varphi^2(x)+\frac{2}{q^6[n-1]_q[n-2]_q}\\
&\leqslant \frac{8}{q^6[n-1]_q}\Big(\varphi^2(x)+\frac{1}{[n-2]_q}\Big)=\frac{8}{q^6[n-1]_q}\delta_n^2(x).
\end{aligned}$$

Corollary 2.5. *For $q = 1$ we obtain the moments of the mixed summation-integral type operators B_n given in [11].*

Lemma 2.6. *Let $m \in N, 0 < q < 1$. There exists a constant C independent of x and n and $\hat{q} \in (0,1)$ such that*

$$\mathcal{L}_{n,q}((t-x)^4, x) \leqslant C\left(\frac{8}{q^6[n-1]_q}\delta_n^2(x)\right)^2 \ \forall q \in (0,\hat{q}].$$

Proof. The proof is similar to Lemma 5 of [7]. □

3. Convergence

The operators $\mathcal{L}_{n,q}$ do not satisfy the conditions of the Bohman-Korovkin theorem in case $0 < q < 1$. To make this theorem applicable we can choose a sequence (q_n) in place of the number q such that $\lim_{n\to\infty} q_n = 1$. With this modification we obtain

Theorem 3.1. *Let (q_n), $0 < q_n < 1$ be a real sequence such that $\lim_{n\to\infty} q_n = 1$. Then, the sequence $\mathcal{L}_{n,q_n}(f,x) \rightrightarrows f$ for any $f \in C_B[0,\infty), x \in [a,b] \subset (0,\infty)$.*

Theorem 3.2. *For $f \in C_B^2[0,\infty), q \in (0,\hat{q}]$ we have*

$$\left|\mathcal{L}_{n,q}(f,x) - \sum_{k=0}^{2}\frac{\mathcal{L}_{n,q}((t-x)^k,x)}{k!}f^{(k)}(x)\right| \leqslant C\,\omega\left(f'', \sqrt{\frac{8}{q^6[n-1]_q}}\delta_n(x)\right)\left[\frac{8\delta_n^2(x)}{q^6[n-1]_q}\right].$$

Proof. From finite Taylor's expansion for the function f around x, given in [21], we obtain

$$\begin{aligned}
&\left|\mathcal{L}_{n,q}(f,x) - \sum_{k=0}^{2}\frac{\mathcal{L}_{n,q}((t-x)^k,x)}{k!}f^{(k)}(x)\right| \\
= &\left|\mathcal{L}_{n,q}((t-x)^2\mu(t-x),x)\right| \\
\leqslant &\ C\frac{1}{2!}\omega(f'',\delta)\left[\left|\mathcal{L}_{n,q}((t-x)^2,x)\right| + \delta^{-2}\left|\mathcal{L}_{n,q}((t-x)^4,x)\right|\right] \\
\leqslant &\ C\frac{1}{2!}\omega(f'',\delta)\left[\frac{8}{q^6[n-1]_q}\delta_n^2(x) + \delta^{-2}\left(\frac{8}{q^6[n-1]_q}\delta_n^2(x)\right)^2\right]
\end{aligned}$$

Choosing $\delta = \sqrt{\frac{8}{q^6[n-1]_q}\delta_n^2(x)}$ the theorem follows. □

Theorem 3.3 (Voronovskaya-type)**.** *If* $f \in C_B^2[0,\infty)$*, and* q_n *be a sequence in* $(0,1)$ *such that* $\lim_{n\to\infty} q_n = 1$*, then we have*

$$\lim_{n\to\infty}[n-1]_{q_n}\Big[\mathcal{L}_{n,q_n}(f,x)-f(x)\Big] = (1+x)f'(x)+f''(x).$$

Proof. The proof follows from Theorem 3.2 and the limits

$$\lim_{n\to\infty}[n-1]_{q_n}\mathcal{L}_{n,q_n}((t-x)^j,x), \quad j=1,2.$$

In view of the limit $\lim_{q_n\to 1}[n]_{q_n} = n$, we get

$$\begin{aligned}\lim_{n\to\infty}[n-1]_{q_n}\mathcal{L}_{n,q_n}((t-x),x) &= \lim_{n\to\infty}[n-1]_{q_n}\left[\frac{q_n+([2]_{q_n}-q_n^n)x}{q_n^2[n-1]_{q_n}}\right]\\ &= 1+x.\end{aligned}$$

We write $\mathcal{L}_{n,q_n}((t-x)^2,x) = a_0+a_1x+a_2x^2$, where a_i are the coefficients of x^i given in Cor.2.4. Then,

$$\lim_{n\to\infty}[n-1]_{q_n}\mathcal{L}_{n,q_n}((t-x)^2,x) = \lim_{n\to\infty}[n-1]_{q_n}(a_0+a_1x+a_2x^2).$$

We obtain the limits $\lim_{n\to\infty}[n-1]_{q_n}a_0 = \lim_{n\to\infty}[n-1]_{q_n}\frac{q_n^3(1+q_n)}{q_n^6[n-1]_{q_n}[n-2]_{q_n}} = 0$,

$$\lim_{n\to\infty}[n-1]_{q_n}a_1x = \lim_{n\to\infty}[n-1]_{q_n}\frac{\left(q_n(q_n+1)^2[n]_{q_n}-2q_n^4[n-2]_{q_n}\right)}{q_n^6[n-1]_{q_n}[n-2]_{q_n}}$$

$$= \lim_{n\to\infty}\frac{\left(q_n(q_n+1)^2(1+q_n+q_n^2[n-2]_{q_n})-2q_n^4[n-2]_{q_n}\right)}{q_n^6[n-2]_{q_n}} = 2.$$

And simplification yields

$$\begin{aligned}&\lim_{n\to\infty}[n-1]_{q_n}a_2\\
&= \lim_{n\to\infty}[n-1]_{q_n}\frac{\left([n+1]_{q_n}[n]_{q_n}-2q_n^4[n]_{q_n}[n-2]_{q_n}+q_n^6[n-1]_{q_n}[n-2]_{q_n}\right)}{q_n^6[n-1]_{q_n}[n-2]_{q_n}}\\
&= \lim_{n\to\infty}\left[\frac{\left(([2]_{q_n}+{q_n}^2[n-1]_{q_n})([2]_{q_n}+{q_n}^2[n-2]_{q_n})\right)}{q_n^6[n-1]_{q_n}[n-2]_{q_n}}\right.\\
&+ \left.\frac{-2q_n^4(1+q_n[n-1]_{q_n})[n-2]_{q_n}+q_n^6[n-1]_{q_n}[n-2]_{q_n}}{q_n^6[n-1]_{q_n}[n-2]_{q_n}}\right]\\
&= \lim_{n\to\infty}\left[\frac{[2]_{q_n}^2+[2]_{q_n}q_n^2(1-2q_n^2)[n-2]_{q_n}-2q_n^4}{q_n^6[n-1]_{q_n}[n-2]_{q_n}}\right.\\
&+ \left.\frac{[2]_{q_n}q_n^2[n-1]_{q_n}+q_n^4(1-2q_n+q_n^2)[n-1]_{q_n}[n-2]_{q_n}}{q_n^6[n-1]_{q_n}[n-2]_{q_n}}\right] = 0.\end{aligned}$$

□

4. Local Error Estimates

Theorem 4.1. *Let $f \in C_B[0,\infty)$ and $x \in (0,\infty)$, then for $n \geqslant 2$, we have*

$$\left|\mathcal{L}_{n,q}(f,x) - f(x)\right| \leqslant 2\,\omega\left(f, \sqrt{\frac{8}{q^6[n-1]_q}}\delta_n(x)\right)$$

Proof. In view of $\mathcal{L}_{n,q}(1,x) = 1$ and Hölder's inequality, we obtain

$$\begin{aligned}\left|\mathcal{L}_{n,q}(f,x) - f(x)\right| &\leqslant \delta^{-1}\omega(f,\delta)\mathcal{L}_{n,q}\big(|u-x|,x\big) + \omega(f,\delta)\mathcal{L}_{n,q}(1,x)\\ &\leqslant \delta^{-1}\omega(f,\delta)\sqrt{\mathcal{L}_{n,q}\big((u-x)^2,x\big)} + \omega(f,\delta)\\ &\leqslant \omega(f,\delta)\Big(\delta^{-1}\sqrt{\frac{8}{q^6[n-1]_q}}\delta_n(x) + 1\Big).\end{aligned}$$

Choosing $\delta = \sqrt{\frac{8}{q^6[n-1]_q}}\delta_n(x)$ the theorem follows. □

Theorem 4.2. *Let $f \in C_B[0,\infty)$, $q \in (0,1)$ and $n \geqslant 2$. Then, for all $x \in [0,\infty)$ and $f \in C_B[0,\infty)$ we have*

$$|\mathcal{L}_{n,q}(f,x) - f(x)| \leqslant C\omega_2\left(f, \sqrt{\frac{8}{q^6[n-1]_q}}\delta_n(x)\right) + \omega\left(f, \frac{q+[n]_q x}{q^2[n-1]_q}\right),$$

where C is positive constant.

Proof. The proof is similar to the proof of Theorem 1 [2]. □

In the following theorem we generalize Theorem 4.2. We obtain the error estimate in terms of weighted Ditzian-Totik modulus of smoothness. For $\lambda = 0$, Theorem 4.2 is obtained as a particular case.

Theorem 4.3. *Let $f \in C_B[0,\infty)$ $q \in (0,1)$, and $0 \leqslant \lambda \leqslant 1$. Then, for $n \geqslant 2, q \in (0,\hat{q}]$ there holds*

$$|\mathcal{L}_{n,q}(f;x) - f(x)| \leqslant C\Big[\omega^2_{\varphi^\lambda}\left(f, \sqrt{\frac{8}{q^6[n-1]_q}}\frac{\delta_n(x)}{\varphi^\lambda(x)}\right) + \omega\Big(f, \frac{q+[n]_q x}{q^2[n-1]_q}\Big)\Big].$$

Proof. We introduce the auxiliary operators $L^*_{n,q}$ defined by

$$L^*_{n,q}(f,x) = \mathcal{L}_{n,q}(f,x) - f(x+z) + f(x), \tag{5}$$

where $z = \frac{q+[n]_q x}{q^2[n-1]_q}$ and $x \in [0,\infty)$. The operators $L^*_{n,q}$ are obviously linear and preserve the linear functions. Moreover, it follows from direct calculations that

$$L^*_{n,q}(t-x,x) = 0. \tag{6}$$

Now, in view of the linearity of the operators $\mathcal{L}_{n,q}$ it follows that

$$\begin{aligned} |\mathcal{L}_{n,q}(f,x)-f(x)| &\leqslant |L^*_{n,q}(f-g,x)-(f-g)(x)| \\ &+ |L^*_{n,q}(g,x)-g(x)|+\big|f(x+z)-f(x)\big|. \end{aligned}$$

Using the smoothness of g, and in view of (6), we get

$$|L^*_{n,q}(g,x)-g(x)| \leqslant \Big|\mathcal{L}_{n,q}\big(R_2(g,t,x)\Big|+\Big|\int_x^{x+z}(x+z-u)g''(u)du\Big|$$

where $R_2(g,t,x)=\int_x^t (t-u)g''(u)du$. It is known (see [4] p141) that

$$\begin{aligned} |R_2(g,t,x)| &\leqslant \frac{|t-x|}{x^\lambda}\left(\frac{1}{(1+x)^\lambda}+\frac{1}{(1+t)^\lambda}\right)\left|\int_x^t \varphi^{2\lambda}(u)\big|g''(u)\big|du\right| \\ &\leqslant \|\varphi^{2\lambda}g''\|(t-x)^2\left(\frac{1}{x^\lambda(1+x)^\lambda}+\frac{1}{x^\lambda(1+t)^\lambda}\right). \end{aligned}$$

It can be verified (cf.[6]) that $\mathcal{L}_{q,n}\big((1+t)^{-m},x\big)\leqslant C(1+x)^{-m}$. Therefore, using Lemma 2.6, we get

$$\begin{aligned} &\mathcal{L}_{n,q}\big(R_2(g,t,x)\big) \\ &\leqslant \frac{\|\varphi^{2\lambda}g''\|}{\varphi^{2\lambda}(x)}\mathcal{L}_{q,n}\Big((t-x)^2,x\Big)+\frac{\|\varphi^{2\lambda}g''\|}{x^\lambda}\mathcal{L}_{q,n}\Big(\frac{(t-x)^2}{(1+t)^\lambda},x\Big) \\ &\leqslant \frac{\|\varphi^{2\lambda}g''\|}{\varphi^{2\lambda}(x)}\mathcal{L}_{q,n}\big((t-x)^2,x\big) \\ &\qquad + \frac{\|\varphi^{2\lambda}g''\|}{x^\lambda}\sqrt{\mathcal{L}_{q,n}\big((t-x)^4,x\big)}\sqrt{\mathcal{L}_{q,n}\big((1+t)^{-2\lambda},x\big)} \\ &\leqslant \frac{8}{q^6[n-1]_q}\frac{C}{\varphi^{2\lambda}(x)}\delta_n^2(x)\|\varphi^{2\lambda}g''\|. \end{aligned}$$

Since

$$\begin{aligned} &z^2\Big(\sqrt{\frac{8}{q^6[n-1]_q}}\delta_n(x)\Big)^{-4\lambda}\varphi^{4\lambda}(x) \\ &\qquad = \varphi^{4\lambda}(x)\Big(\frac{q+[n]_qx}{q^2[n-1]_q}\Big)^2\Big(\frac{8}{q^6[n-1]_q}\Big(\varphi^2(x)+\frac{1}{[n-2]_q}\Big)\Big)^{-2} \end{aligned}$$

is bounded for all values of x, we obtain

$$\left|\int_x^{x+z}(x+z-u)g''(u)du\right| \leqslant \frac{\left(\sqrt{\frac{8}{q^6[n-1]_q}}\delta_n(x)\right)^4}{\varphi^{4\lambda}(x)}\|g''\|.$$

Collecting these estimates, we get

$$|L^*_{n,q}(g;x)-g(x)| \leqslant C\frac{\delta_n^2(x)}{\varphi^{2\lambda}(x)}\|\varphi^{2\lambda}g''\| + \frac{\left(\sqrt{\frac{8}{q^6[n-1]_q}}\delta_n(x)\right)^4}{\varphi^{4\lambda}(x)}\|g''\|.$$

Now,

$$\begin{aligned}|L^*_{n,q}(f-g,x)-(f-g)(x)| &\leqslant \mathcal{L}_{n,q}(f-g,x)| \\ &+\|f-g\| \leqslant \|f-g\|\mathcal{L}_{n,q}(1,x)+\|f-g\| \leqslant 2\|f-g\|.\end{aligned}$$

Hence, we obtain

$$\begin{aligned}&|\mathcal{L}_{n,q}(f,x)-f(x)| \\ \leqslant\ & C\left(\|f-g\| + \frac{\left(\sqrt{\frac{8}{q^6[n-1]_q}}\delta_n(x)\right)^2}{\varphi^{2\lambda}(x)}\|\varphi^{2\lambda}g''\| + \frac{\left(\sqrt{\frac{8}{q^6[n-1]_q}}\delta_n(x)\right)^4}{\varphi^{4\lambda}(x)}\|g''\|\right) \\ +\ & |f(x+z)-f(x)|.\end{aligned}$$

Finally, in view of equivalence (1) of $\overline{K}_{2,\varphi^\lambda}(f,t^2)$ and $\omega^2_{\varphi^\lambda}(f,t)$ we get

$$\begin{aligned}|\mathcal{L}_{n,q}(f,x)-f(x)| &\leqslant C\overline{K}_{2,\varphi^\lambda}\left(f, \frac{\left(\sqrt{\frac{8}{q^6[n-1]_q}}\delta_n(x)\right)^2}{\varphi^{2\lambda}(x)}\right) + \omega(f,z) \\ &\leqslant C\omega^2_{\varphi^\lambda}\left(f, \sqrt{\frac{8}{q^6[n-1]_q}}\frac{\delta_n(x)}{\varphi^\lambda(x)}\right) + \omega\left(f, \frac{q+[n]_q x}{q^2[n-1]_q}\right).\end{aligned}$$

This completes the proof of the theorem. □

Let $C_{B,x^2}[0,\infty)$ be the space of the continuous and bounded functions defined on $[0,\infty)$ such that $|f(x)| \leqslant C(1+x^2)$, where C is a constant depending on f. The space $C^*_{x^2}[0,\infty)$ is defined by $\{f \in C_{B,x^2}[0,\infty) : \sup \frac{|f(x)|}{1+x^2} < \infty\}$. We define $\|f\|_{x^2}$ by $\sup \frac{|f(x)|}{1+x^2}$ in the space $C^*_{x^2}[0,\infty)$. For a positive number a, we define

$$\omega_a(f,\delta) = \sup_{|t-x|\leqslant\delta}\ \sup_{x,t\in[0,a]} |f(t)-f(x)|$$

the usual modulus of continuity of f on the closed interval $[0,a]$. It is known that for a function $f \in C_{x^2}[0,\infty)$, modulus of continuity $\omega_a(f,\delta)$ tends to zero. From similar methods in Theorem 2 and Theorem 4 of [2] we obtain the following results:

Theorem 4.4. *Let $f \in C_{x^2}[0,\infty), q = q_n \in (0,1)$ such that $q_n \to 1$ as $n \to \infty$ and ω_{a+1}be its modulus of continuity on finite interval $[0,a+1] \subset [0,\infty)$ where $a > 0$. Then, for every $n > 1$*

$$\|\mathcal{L}_{n,q_n}(f) - f\| = \frac{K}{q_n^6[n-1]_{q_n}} + 2\omega_{a+1}\left(f, \sqrt{\frac{K}{q_n^6[n-1]_{q_n}}}\right)$$

where $K = 48C(1+a^2)(1+a+a^2)$.

Theorem 4.5. *Let $q = q_n$ satisfies $0 < q_n < 1$ and let $q_n \to 1$ as $n \to \infty$. For each $f \in C_{x^2}[0,\infty)$, and $\alpha > 0$, we have*

$$\lim_{n\to\infty} \sup_{x\in[0,\infty)} \frac{|\mathcal{L}_{n,q_n}(f,x) - f(x)|}{(1+x^2)^\alpha} = 0.$$

In the end we give an error estimate for the functions in a subclass of the space $\mathrm{Lip}_M\alpha$.

Theorem 4.6. *Let $f \in C_B[0,\infty) \cap Lip_M\alpha$, $\alpha \in (0,1]$ for $x \in [0,A]$ $A > 0$. Then,*

$$|\mathcal{L}_{n,q}(f) - f| \leqslant M\left(\sqrt{\frac{8}{q^6[n-1]_q}}\delta_n(x)\right)^\alpha.$$

Proof. In view of $\mathcal{L}_{n,q}(1) = 1$, we can write

$$\begin{aligned}
|\mathcal{L}_{n,q}(f) - f| &\leqslant \sum_{k=0}^{\infty} p_{n,k}(q;x) \int_0^{\infty/A} q^k b_{n,k}(q;u)|f(u) - f(x)|\, d_q u \\
&\leqslant \sum_{k=0}^{\infty} p_{n,k}(q;x) \int_0^{\infty/A} q^k b_{n,k}(q;u)|t - x|^\alpha\, d_q u
\end{aligned}$$

Taking $\frac{1}{p} = \frac{\alpha}{2}$, $\frac{1}{q} = \frac{2-\alpha}{2}$, in Hölder's inequality, we obtain

$$\begin{aligned}
|\mathcal{L}_{n,q}(f) - f| &\leqslant \left(\mathcal{L}_{n,q}\left((t-x)^2, x\right)\right)^{1/2} \left(\mathcal{L}_{n,q}(1,x)\right)^{(2-\alpha)/2} \\
&\leqslant M\left(\frac{8}{q^6[n-1]_q}\delta_n^2(x)\right)^{\alpha/2}.
\end{aligned}$$

This completes the proof. □

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REFERENCES

[1] A. Aral - V. Gupta, *The $q-$derivative and applications to $q-$Szász Mirakyan operators*, Calcolo 43 (2006), 151–170.

[2] A. Aral - V. Gupta, *On the Durrmeyer type modification of the $q-$Baskakov type operators*, Nonlinear Anal. 72 (2010), 1171–1180.

[3] M. M. Derriennic, *Modified Bernstein polynomials and Jacobi polynomials in $q-$calculus*, Rend. Circ. Mat. Palermo Serie II (Suppl.76) (2005), 269–290.

[4] Z. Ditzian - V. Totik, *Moduli of Smoothness*, Springer, New York, 1987.

[5] J. L. Durrmeyer, *Une formule d'inversion, de la transformée de Laplace: Application à la théorie des Moments*, Thèse de 3e Cycle, Faculté des Sciences de l'université de Paris, 1967.

[6] A. R. Gairola - P. N. Agrawal, *Direct and inverse theorems for the Bézier variant of certain summation-integral type operators*, Turk. J. Math. 33 (2009), 1–14.

[7] A. R. Gairola - P. N. Agrawal - G. Dobhal - K. K. Singh, *Moments of a $q-$Baskakov-beta operators in case* $0 < q < 1$, in communication.

[8] V. Gupta - W. Heping, *The rate of convergence of $q-$Durrmeyer operators for* $0 < q < 1$, Math. Meth. Appl. Sci. 31 (2008), 1946–1955.

[9] V. Gupta, *Some approximation properties of $q-$Durrmeyer operators*, Appl. Math. Comput. 197 (1) (2008), 172–178.

[10] V. Gupta, *On certain $q-$Durrmeyer type operators*, Appl. Math. Comput. 209 (2009), 415–420.

[11] V. Gupta - A. Ahmad, *Simultaneous approximation by modified beta operators*, Istanbul Univ. Fen. Fak. Mat. Der. 54 (1995), 11–22.

[12] V. Gupta - T. Kimb, *On the rate of approximation by $q-$modified beta operators*, J. Math. Anal. Appl. 377 (2011), 471–480.

[13] F. H. Jackson, *On a $q-$definite integrals*, Quarterly J. Pure Appl. Math. 41 (1910), 193–203.

[14] V. G. Kac - P. Cheung, *Quantum Calculus*, Universitext, Springer-Verlag, New York, 2002.

[15] L. V. Kantorovich, *Sur certain développements suivant les polynômes de la forms de S. Bernstein*, Dokl. Akad. Nauk. SSSR. 1-2 (1930), 563–568, 595–600.

[16] T. H. Koornwinder, *$q-$Special Functions, a tutorial*, in: M. Gerstenhaber, J. Stasheff (eds) Deformation and Quantum groups with Applications of Mathematical Physics, Contemp. Math. 134 (1992), Amer. Math. Soc. 1992.

[17] A. Il'inskiĭ - S. Ostrovska, *Convergence of generalized Bernstein polynomials*, J. Approx. Theory 116 (1) (2002), 100–112.

[18] S. Ostrovska, *$q-$Bernstein polynomials and their iterates*, J. Approx. Theory 123 (2) (2003), 232–255.

[19] G. M. Phillips, *Bernstein polynomials based on the $q-$integers*, Annals of Numerical Mathematics 4 (1997), 511–518.

[20] G. M. Phillips, *Interpolation and Approximation by Polynomials*, CMS Books in Mathematics, vol. 14. Springer, Berlin, 2003.

[21] O. T. Pop, *About some linear and positive operators defined by infinite sum*, Demonstratio Math. 39 (2) (2006), 377–388.

[22] H. Wang, *Voronovskaya-type formulas and saturation of convergence for $q-$Bernstein polynomials for* $0 < q < 1$, J. Approx. Theory 145 (2007) 182–195.

[23] H. Wang, *Properties of convergence for $q-$Bernstein polynomials*, J. Math. Anal. Appl. 340 (2) (2008), 1096–1108.

[24] H. Wang - F. Meng, *The rate of convergence of $q-$Bernstein polynomials for* $0 < q < 1$, J. Approx. Theory 136 (2) (2005), 151–158.

[25] H. Wang - X. Wu, *Saturation of convergence of $q-$Bernstein polynomials in the case* $q > 1$, J. Math. Anal. Appl. 337 (1) (2008), 744–750 .

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