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CONES OF DIVISORS OF BLOW-UPS OF PROJECTIVE SPACES

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We investigate Mori dream spaces obtained by blowing-up the n-dimensional complex projective space at $n+1$, $n+2$ or $n+3$ points in very general position. Using toric techniques we study the movable cone of the blow-up of $\mathbb{P}^n$ at $n+1$ points, its decomposition into nef chambers and the action of the Weyl group on the set of chambers. Moreover, using different methods, we explicitly write down the equations of the movable cone also for $\mathbb{P}^n$ blown-up at $n+2$ points.

1. Introduction

A complex projective normal variety X is said to be a *Mori dream space* if the Cox ring

$$\mathrm{Cox}(X) := \bigoplus_{D \in \mathrm{Pic(X)}} H^0(X,D)$$

is finitely generated over $\mathbb{C}$ (see subsection 2.1). Mori dream spaces were introduced and studied by Hu and Keel in [9]: from a Mori theory point of view they are the best possible varieties we can think of. In fact, if we define the movable

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cone of a variety as the cone contained in the space of divisors $N^1(X)$ generated by the Cartier divisors without divisorial base components (see Definition 2.1 and Notation 2.2), then both the nef and the movable cone of a Mori dream space are rational polyhedral (see Theorem 2.6).

Every nef divisor on a Mori dream space is semiample, i.e. it has a multiple without base points. Moreover, we can decompose the movable cone of a Mori dream space X in the, so called, Mori chambers, identified with the nef cones of all the small $\mathbb{Q}$-factorial modifications (SQMs) of X, i.e. $\mathbb{Q}$-factorial varieties isomorphic to X in codimension 1 (see Definition 2.4). Then we can consider a one-to-one correspondence between faces of the movable cone and (divisorial or fibre type) contractions from a SQM of X to a normal $\mathbb{Q}$-factorial projective variety (see Theorem 2.6 or [9, Proposition 1.11]).

These properties are connected to the fact that a Mori dream space X is a GIT quotient of the affine variety $\mathrm{Spec}(\mathrm{Cox(X)})$ by the action of a torus and, modifying the linearization by changing the choice of a character of the torus, all the SQMs of X can be obtained in the same way (see [9, Proposition 2.9]).

Now consider the varieties $Bl_{q+n+1}((\mathbb{P}^n)^p))$ given by p copies of the projective space blown up at $q+n+1$ points in very general position, where $n \geq 2$, $p \geq 1$, $q \geq 0$. By [4, Theorem 1.3] we have a necessary and sufficient numerical condition on p,n,q for these varieties in order to be Mori dream spaces. In [13] Mukai defines an inner product on the space of divisors $N^1(Bl_{q+n+1}((\mathbb{P}^n)^p)))$ which induces a root base whose root lattice corresponds to the T-shaped root system $T_{p+1,q,n+1}$ (see subsection 2.2) and studies the action of the Weyl group on the root base.

In this paper we consider just the case of a single copy of the projective space. Note that in this case the root system and the action of the Weyl group was already studied by Dolgachev in [6].

Our aim is to describe all possible contractions from Mori dream spaces of the form $Bl_{q+n+1}(\mathbb{P}^n)$ with $q \geq 0$ by investigating the structure of the movable cone, its decomposition into nef chambers and the action of the Weyl group on the set of nef chambers.

We start from introducing preliminary definitions (section 2) and describing the classical case of Del Pezzo surfaces (see 2.4). In section 3 we consider the case $Bl_{n+1}(\mathbb{P}^n)$ for every $n \in \mathbb{N}$, $n \geq 2$, which is a toric variety.

Apart from finding explicit equations of the movable cone (Proposition 3.2), by applying the methods of toric Mori theory we are able to determine all the extremal rays of the Kleiman-Mori cone $\mathrm{NE}(X_i) \subseteq N_1(X_i)$ of all the SQMs X_i of $Bl_{n+1}(\mathbb{P}^n)$ and understand their nature, i.e. if they correspond to a divisorial, a fibre type or a small contraction (Theorem 3.10). As a corollary we obtain a description of the decomposition of the movable cone $\mathrm{Mov}(Bl_{n+1}(\mathbb{P}^n))$ into nef chambers (Corollary 3.13). Moreover, we prove that the Weyl group fixes the movable cone and permutes the nef chambers (Proposition 3.6). We also give many explicit examples. Especially, we study in details the first non-trivial case, which is $Bl_4(\mathbb{P}^3)$.

In section 4 we investigate the structure of the movable cone in a more general setting. In Corollary 4.7 we prove that for $r = n+1, n+2, n+3$

$$\mathrm{Mov}(Bl_r(\mathbb{P}^n)) = \mathrm{Eff}_{\mathbb{R}}(Bl_r(\mathbb{P}^n)) \cap \mathrm{Eff}_{\mathbb{R}}(Bl_r(\mathbb{P}^n))^{\vee},$$

where by $\mathrm{Eff}_{\mathbb{R}}$ we denote the cone of effective divisors in $N^1(Bl_r(\mathbb{P}^n))$ and we take the dual cone $\mathrm{Eff}_{\mathbb{R}}^{\vee}$ with respect to the Mukai inner product (see Definition 4.1). Thanks to this formula we are able to explicitly compute the faces of the movable cone in the cases $r = n+1$ and $r = n+2$ (see Theorem 4.9).

2. Preliminaries

We will work over the field of complex number $\mathbb{C}$. A *scheme* is a separated algebraic scheme of finite type over $\mathbb{C}$ and a *variety* is a reduced, irreducible scheme.

We denote by $\mathrm{Pic}(X)$ the Picard group of X and for $\mathbb{K} = \mathbb{Q}$, $\mathbb{R}$, we consider $\mathrm{Pic}_{\mathbb{K}}(X) = \mathrm{Pic}(X) \otimes \mathbb{K}$ to be the group of $\mathbb{K}$-Cartier divisors.

Moreover we denote by $N^1(X)_{\mathbb{K}}$ the $\mathbb{K}$-Neron-Severi group, defined as the quotient of $\mathrm{Pic}_{\mathbb{K}}(X)$ by the subgroup of numerically trivial $\mathbb{K}$-divisors. Finally we denote by $N_1(X)_{\mathbb{K}}$ the quotient group of $\mathbb{K}$-1-cycles on X by the subgroup of numerically trivial $\mathbb{K}$-1-cycles. Note that there is a natural intersection pairing between $N^1(X)_{\mathbb{K}}$ and $N_1(X)_{\mathbb{K}}$.

Definition 2.1. Let X be a normal projective variety. A divisor $D \in \mathrm{Pic}_{\mathbb{R}}(X)$ is *nef* if $(D \cdot C) \geq 0$ for every irreducible curve $C \subseteq X$. A Cartier divisor $D \in \mathrm{Pic}_{\mathbb{R}}(X)$ is *movable* if its numerical class in $N^1(X)$ lies in the closure of the cone generated by classes of Cartier divisors without divisorial base components.

Notation 2.2. We consider the following cones and semigroups.

$$\begin{aligned}
\mathrm{Nef}(X) \subseteq N^1(X)_{\mathbb{R}} &: \text{the closed cone of nef divisors}\\
\mathrm{Mov}(X) \subseteq N^1(X)_{\mathbb{R}} &: \text{the closed cone of movable divisors}\\
\mathrm{Eff}_{\mathbb{R}}(X) \subseteq N^1(X)_{\mathbb{R}} &: \text{the closure of the cone of effective divisors}\\
\mathrm{NE}(X) \subseteq N_1(X)_{\mathbb{R}} &: \text{the closure of the cone of effective 1-cycles}\\
\mathrm{Eff}(X) \subseteq \mathrm{Pic}(X) &: \text{the semigroup of effective divisors.}
\end{aligned}$$

Note that $\mathrm{Nef}(X) \subseteq \mathrm{Mov}(X) \subseteq \mathrm{Eff}_{\mathbb{R}}(X) \subseteq N^1(X)_{\mathbb{R}}$, and by Kleiman's criterion $\mathrm{Nef}(X)$ and $\mathrm{NE}(X)$ are dual to each other.

Definition 2.3. We say that a surjective morphism of projective varieties with connected fibres $f : X \to Y$ is a *divisorial contraction* if f is birational and the exceptional locus of f is a divisor.

We say that f is a *fibre-type contraction* if $dimY < dimX$.

We say that f is a small contraction if f is birational and the exceptional locus has codimension greater than 1.

Definition 2.4. Let X be a normal projective variety. A *small $\mathbb{Q}$-factorial modification (SQM)* of X is a birational map $f : X \dashrightarrow X'$ such that X' is projective and $\mathbb{Q}$-factorial and f is an isomorphism in codimension 1. We will use the term SQM also for X'.

2.1. Mori dream spaces

Definition 2.5. Let X be a normal $\mathbb{Q}$-factorial projective variety whose Picard group $\mathrm{Pic}(X)$ is a lattice. We define the *Cox ring* of X as

$$\mathrm{Cox}(X) = \bigoplus_{D \in \mathrm{Pic(X)}} H^0(X, \mathscr{O}_X(D)),$$

with multiplicative structure defined by a choice of divisors whose classes form a basis of $\mathrm{Pic}(X)$.

We say that X is a *Mori dream space* if $\mathrm{Cox}(X)$ is finitely generated.

Note that by [9] a Mori dream space X has finitely many small $\mathbb{Q}$-factorial modifications $f : X \dashrightarrow X_i$. As a SQM does not affect divisors, we can identify the Neron-Severi spaces $N^1(X)_{\mathbb{R}} = N^1(X_i)_{\mathbb{R}}$, the movable cones $\mathrm{Mov}(X) = \mathrm{Mov}(X_i)$ and the effective cones $\mathrm{Eff}_{\mathbb{R}}(X) = \mathrm{Eff}_{\mathbb{R}}(X_i)$. Moreover we identify the cone $\mathrm{Nef}(X_i) \subseteq N^1(X_i)_{\mathbb{R}}$ with its pullback $f_i^*(\mathrm{Nef}(X_i)) \subseteq N^1(X)_{\mathbb{R}}$.

Theorem 2.6. *[9, Prop.2.9+Prop 1.11] Let X be a Mori dream space and let $\{f_i : X \dashrightarrow X_i\}$ be all the SQM's of X. Then*

1. *Every nef Cartier divisor on X is semiample and* $\mathrm{Nef}(X)$ *is the affine hull of finitely many semiample line bundles, the same holds for every X_i.*

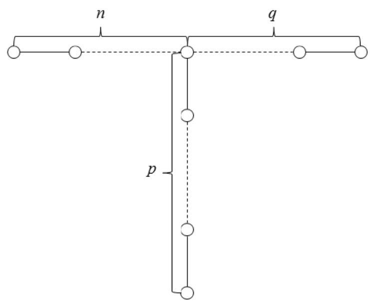
2. *The cones* $\mathrm{Nef}(X_i)$*, together with their faces, give a fan with support* $\mathrm{Mov}(X)$*. In particular* $\mathrm{Mov(X)} = \bigcup \mathrm{Nef}(X_i)$ *and for every $i \neq j$ we have that $int(\mathrm{Nef}(X_i)) \cap int(\mathrm{Nef}(X_j)) = \emptyset$.*

   The cones $\mathrm{Nef}(X_i)$ *are called nef (or Mori) chambers. Two varieties whose nef cones are two adjacent chambers are related by a flip.*

3. *The faces of* $\mathrm{Nef}(\mathrm{X_i})$ *which are contained in proper faces of* $\mathrm{Mov}(\mathrm{X})$ *are in one-to-one correspondence with the divisorial or fibre-type contractions from* X_i *to a normal projective variety. The correspondence is given by*
$$(g : X_i \to Y_g) \longmapsto g^*(\mathrm{Nef}(Y_g)) \subseteq \mathrm{Mov}(\mathrm{X}).$$
In particular the faces of $\mathrm{Mov}(\mathrm{X})$ *are in one-to-one correspondence with classes of divisorial or fibre-type contractions from SQMs of X to normal projective varieties.*

2.2. Root systems of blow-ups of projective spaces

Let $T_{n,q,p}$ be the following Dynkin diagram:



In [12] Manin describes how to associate the Weyl group of $E_6 = T_{3,2,2}$ to the configuration of the 27 lines on a nonsingular cubic surface $S \subset \mathbb{P}^3$, that is the blow-up of $\mathbb{P}^2$ at six points. Generalizing this result, in [6] Dolgachev realizes the root system $T_{n+1,q,2}$ in the cohomogolgy group of the blow-up of $\mathbb{P}^n$ in $q+n+1$ points in very general position, denoted by $Bl_{q+n+1}(\mathbb{P}^n)$. In [13] Mukai generalizes this result of Dolgachev to the root system $T_{n,q,p}$ of products of projective spaces.

In this subsection we will present the main constructions and results from [6] and [13].

Take integers q and n such that $q \geq 0$ and $n \geq 2$. Let us denote by E_1, $\ldots, E_{q+n+1}$ the exceptional divisors created by blowing up $q+n+1$ points in the projective space $\mathbb{P}^n$, and by H the class of a general hyperplane not passing through any of the $q+n$ original points. If $X = Bl_{q+n+1}(\mathbb{P}^n)$, these divisors will generate the Picard group, $\mathrm{Pic}(X) = \mathbb{Z}[H] \oplus \mathbb{Z}[E_1] \oplus \cdots \oplus \mathbb{Z}[E_{q+n+1}]$.

On the other hand we denote by l the class of a general line on X and by f_i the class of a line in the exceptional divisors E_i. The following relations define the usual intersection form: $(H \cdot l) = 1, (H \cdot f_j) = 0, (E_j \cdot f_j) = c_1(\mathscr{O}(-1)) = -1$ and $(E_j \cdot f_i) = 0$ for $i \neq j$.

Define the *root base* $B = \{\alpha_1, \ldots \alpha_{q+n+1}\}$ of the vector space $N^1(X)_{\mathbb{R}} = \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ by $\alpha_i := E_i - E_{i+1}$ for $1 \leq i \leq q+n$ and $\alpha_{q+n+1} := H - E_1 - \ldots - E_{n+1}$.
B becomes a root base with respect to the set $B^{\vee} = \{\alpha_1^{\vee}, \ldots, \alpha_{q+n+1}^{\vee}\} \subset N_1(X)$ where $\alpha_i^{\vee} := f_i - f_{i+1}$ for $1 \leq i \leq q+n$ and $\alpha_{q+n+1}^{\vee} := (n-1)l - f_1 - \ldots - f_{n+1}$. The *Weyl group* is generated by the simple reflections $s_i(x) = x + \alpha_i^{\vee}(x)\alpha_i$ for any element x in $N^1(X)_{\mathbb{R}}$.

Using the intersection form defined above we obtain that the *Cartan matrix* associated to this root base is:

$$\alpha_i^{\vee}(\alpha_j) = \alpha_j^{\vee}(\alpha_i) = \begin{cases} -2 & i = j \\ 0 & i \neq \{j-1, j+1, n+1\} \\ 1 & i = j-1 \\ 1 & i = j+1 \\ 1 & i = n+1, j = q+n+1 \end{cases}$$

We will associate the Dynkin diagram $T_{n+1,q,2}$ to the Cartan matrix, where each vertex represents a base root. The diagonal entries of the Cartan matrix correspond to the selfintersections, while the other entries, $\alpha_i^{\vee}(\alpha_j)$, correspond to the number of edges joining two vertices α_i and α_j.

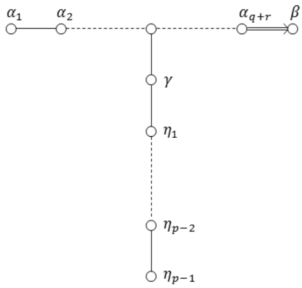
It is easy to see that for $q = 0$ one obtains the root system associated to the Dynkin diagram $\mathbb{A}_1 \times \mathbb{A}_n$, $\mathbb{A}_{n+2}$ for $q = 1$, while for $q = 2$ one gets $\mathbb{D}_{n+3}$. In the case $n = 2$ the Dynkin diagram $T_{n+1,q,2}$ corresponds to $\mathbb{E}_6$ for $q = 3$, $\mathbb{E}_7$ for $q = 4$ and $\mathbb{E}_8$ for $q = 5$. The following table presents the finite root systems associated to $Bl_{q+n+1}(\mathbb{P}^n)$ for small values of n:

0	3	4	5	6	7	8	9
$\mathbb{P}^2$	$\mathbb{A}_1 \times \mathbb{A}_2$	$\mathbb{A}_4$	$\mathbb{D}_5$	$\mathbb{E}_6$	$\mathbb{E}_7$	$\mathbb{E}_8$	infinite
$\mathbb{P}^3$		$\mathbb{A}_1 \times \mathbb{A}_3$	$\mathbb{A}_5$	$\mathbb{D}_6$	$\mathbb{E}_7$	infinite	infinite
$\mathbb{P}^4$			$\mathbb{A}_1 \times \mathbb{A}_4$	$\mathbb{A}_6$	$\mathbb{D}_7$	$\mathbb{E}_8$	infinite
$\mathbb{P}^5$				$\mathbb{A}_1 \times \mathbb{A}_5$	$\mathbb{A}_7$	$\mathbb{D}_8$	infinite

Mukai extended this result to all diagrams $T_{n,q,p}$. The Picard group of $Y = Bl_{q+n+1}((\mathbb{P}^n)^p)$ is a $\mathbb{Z}$-module of rank $p+q+n+1$ generated by the exceptional divisors of the blow up E_i and the pull back of the hyperplane class on the i-th factor H_i. The anticanonical class of Y

$$-K_Y = (n+1)(H_1 + \ldots + H_p) - (n-1)(E_1 + \ldots + E_{q+n+1}).$$

A root base for Y is defined by $\{\alpha_j = E_j - E_{j+1}\}_{1 \leq j \leq q+n}$, $\beta = E_{q+n+1}$, $\gamma = H_1 - E_1 - \ldots - E_n$, $\{\eta_i = H_i - H_{i+1}\}_{1 \leq i \leq p-1}$, and it has the associated T-shaped diagram.



As in [13] we can also define a symmetric bilinear form on $\mathrm{Pic}(Y)$ defined by: $(E_i,E_j) = -\delta_{i,j}$, $\quad (E_i,H_j) = 0$ and $\quad (H_i,H_j) = \begin{cases} n-1 & i=j \\ n & i \neq j \end{cases}$.

From now on we will denote by $X_r^n = Bl_r(\mathbb{P}^n)$ the blowing-up of $\mathbb{P}^n$ at r very general points.

2.3. Weyl group and Cremona transformations

Recall that the standard Cremona transformation along of the projective space $\mathbb{P}^n$ (along the coordinate points) is defined to be the birational map

$$[x_0,\dots,x_n] \longmapsto [x_1\cdots x_n,\dots,x_0\cdots x_{n-1}].$$

Notice that this map is given by the linear system of hypersurfaces of degree n with multiplicity $n-1$ at each of the $n+1$ coordinate points q_i. In an algebraic setting, this consists of all homogeneous polynomial of degree n for which their partial derivatives up to order $n-2$ vanish at each of the coordinate points. Now, the condition of passing through all the coordinate points annihilates all the coefficients of the x_j^n, the condition $\partial_{x_i} f(q_j) = 0$ annihilates the coefficient of $x_j^{n-1}x_i$, and so on. Therefore this linear system has dimension $n+1$, a basis for the space of sections being given by $\{\prod_{i,i\neq j} x_i\}_{j=1,\dots,n+1}$.

Also notice that on $\mathbb{P}^n$ this map contracts all the hyperplanes determined by n coordinate points to a point. For example the hyperplane passing through $q_2,\dots,q_{n+1}$ will be mapped to the point q_1.

In general we can fix $n+1$ very general points $p_1,\dots,p_{n+1}$ and consider the standard Cremona transformation along these points, defined by the composition of a projective automorphism of $\mathbb{P}^n$ that sends them to the coordinate points and the standard Cremona transformation along the coordinate points.

Hence we have that if $r \geq n+1$, and $X = X_r^n$, so that there exist $p_{n+2},\dots,p_r$ such that X is the projective space blown-up at $p_1,\dots,p_{n+1},\dots,p_r$, the Cremona transformation along $p_1,\dots,p_{n+1}$ induces an action on the Picard group of X_r^n defined by sending H to $nH-(n-1)E_1-\cdots-(n-1)E_{n+1}$ and the exceptional divisors E_j to $H-\sum_{i\neq j} E_i$ (see also [13, Proof of Theorem 1]), so that it induces an element $s_{p_1,\dots,p_{n+1}}$ of the Weyl group W of X^n_r.

Note that every element of the Weyl group W of X^n_r corresponds, in this way, to a birational map of $\mathbb{P}^n$ lying in the group generated by standard Cremona transformations and projective automorphisms of $\mathbb{P}^n$ (see for example [6, Theorem 2]).

2.4. First example: Del Pezzo surfaces

Let us consider X^2_r, a surface obtained by blowing-up $\mathbb{P}^2$ at $1\leq r\leq 8$ points in very general position. Let us summarize briefly some results on contractions from X^2_r, root systems associated to these surfaces and the structure of $\mathrm{Mov}(X^2_r)$.

Since a small $\mathbb{Q}$-factorial modification of a surface is an isomorphism, we have $\mathrm{Mov}(X^2_r)=\mathrm{Nef}(X^2_r)$, i.e. $\mathrm{Mov}(X^2_r)$ has trivial nef chamber decomposition.

A *Del Pezzo surface* is a smooth projective algebraic surface with ample anticanonical divisor class.

Proposition 2.7. *The followings hold.*

1. *Any del Pezzo surface is isomorphic to one of the followings:* $\mathbb{P}^2$, $\mathbb{P}^1\times\mathbb{P}^1$, *and* X^2_r, $1\leq r\leq 8$.

2. *Let* $X\to X'$ *be any birational morphism. If* X *is a del Pezzo surface, then* X' *is also a del Pezzo surface. Conversely, if* X' *is a del Pezzo surface, the Picard number of* X *is* ≤ 7 *and every negative self-intersection curve on* X *is a* (-1)*-curve, then* X *is also a del Pezzo surface.*

3. *Del Pezzo surfaces are Mori Dream Spaces. In particular,* $\mathrm{Nef}(X^2_r)$ *and* $\mathrm{NE}(X^2_r)$ *are rational polyhedral cones.*

Proof. For 1 and 2, see section 24 in [12]. For 3, see Corollary 2.16 in [9]. □

The following table contains the root systems corresponding to X^2_r, which were first constructed by Manin in [12].

X^2_3	X^2_4	X^2_5	X^2_6	X^2_7	X^2_8
$\mathbb{A}_1\times\mathbb{A}_2$	$\mathbb{A}_4$	$\mathbb{D}_5$	$\mathbb{E}_6$	$\mathbb{E}_7$	$\mathbb{E}_8$

By Theorem 2.6, we have a bijection between the set of contractions $\varphi: X^2_r\to Y_\varphi$ and the set of faces of $\mathrm{Nef}(X^2_r)$. In particular the face of $\mathrm{Nef}(X^2_r)$ corresponding to a contraction φ is defined by $\mathrm{Nef}(X^2_r)\cap\varphi^*N^1(Y_\varphi)=\varphi^*\,\mathrm{Nef}(Y_\varphi)$. Elementary contractions correspond to the facets.

Instead of looking at the polyhedral cone $\mathrm{Nef}(X^2_r)$, we look at the polytope Δ^2_r which we define as its compact section. By Γ^2_r we denote its dual, a compact section of $\mathrm{NE}(X^2_r)$.

The following diagram shows contractions of X^2_r, which by proposition 2.7 are all possible:

$$\begin{array}{ccccccccccc}
\mathbb{P}^1 & \leftarrow & \mathbb{P}^1\times\mathbb{P}^1 & & & & & & & & \\
& \nwarrow & & \nwarrow & & & & & & & \\
\mathbb{P}^2 & \leftarrow & X^2_1 & \leftarrow & X^2_2 & \leftarrow & X^2_3 & \leftarrow & \cdots & \leftarrow & X^2_8
\end{array}$$

In [17], Stalij investigated the combinatorial structure of the polytopes Γ^2_r. Let us summarize briefly his results.

Proposition 2.8 ([17]). *Faces of Γ^2_r of dimension $r-1$ correspond to contractions of X^2_r to $\mathbb{P}^1$ or $\mathbb{P}^2$.*

Let us mark a vertex of Δ^2_r by 0 if the dual facet of Γ^2_r corresponds to a contraction to $\mathbb{P}^2$ and by 1 if it corresponds to a contraction to $\mathbb{P}^1$.

Theorem 2.9 ([17]). *Δ^2_r is simple (dual to a simplex) at vertices marked by 0 and dual to a cross-polytope at vertices marked by 1.*

This theorem can be restated in terms of *marked polynomials*.

Definition 2.10. A marked polynomial corresponding to a polytope with faces marked with non-negative integers is a polynomial of two variables x and y such that the coefficient at x^iy^j is the number of faces of codimension i marked with j.

Note that markings of the vertices of Δ^2_r are in one-to-one correspondence with markings of the faces of Γ^2_r.

Theorem 2.11 ([17]). *The marked polynomials $P^2_r(x,y)$ for X^2_r (corresponding to Γ^2_r) satisfy*

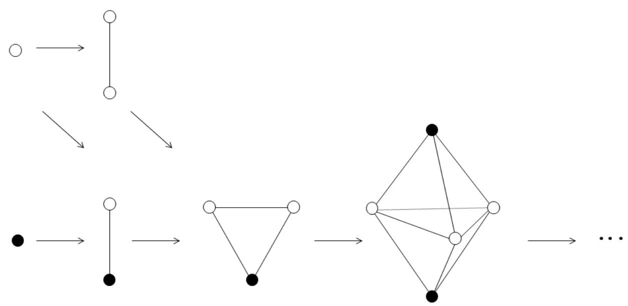
1. $\partial_x P^2_r(x,0) = \partial_x P^2_r(0,0)\cdot\partial_x P^2_{r-1}(x,0)$,
2. $2(r-1)\partial_y P^2_r(1,0) = \partial_x P^2_r(0,0)\cdot\partial_y P^2_{r-1}(1,0)$.

Using this result and knowing (by geometric arguments) that Δ^2_2 is a triangle with two vertices marked by 1 and one marked by 0, we can recover the weak combinatorial structure for $3\le r\le 7$. That is, the numbers of faces in each dimension can be computed, and the marking of the vertices can be described. The proof in [17] does not work for $r=8$.

Theorem 2.12 ([17]). *Consider the class of polytopes $\mathscr{P}$ marked with 0 and 1, closed with respect to taking faces, satisfying the equations in Theorem 2.11 (so containing Δ_r^2 for $2 \leq r \leq 8$), and such that the only two-dimensional polytope in $\mathscr{P}$ is Δ_2^2. Then the weak combinatorial structure of polytopes in $\mathscr{P}$ is uniquely determined in dimensions $3 \leq r \leq 7$. Moreover, in dimensions $r \geq 9$ there are no such polytopes.*

The polytopes Δ_r^2 are the so-called Gosset polytopes $(r-4)_{21}$. In [11], Lee constructed these polytopes in $N^1(X_r^2)$ for $3 \leq r \leq 8$ in a more direct way. Using intersection theory on surfaces and the action of the Weyl group (obtained from the associated root system) on $\mathrm{Pic}(X_r^2)$, he proved that the convex hull of divisor classes of (-1)-curves in $N^1(X_r^2)$ (they lie in the hyperplane defined by $-K_{X_r^2}.[D] = 1$ by adjuction formula) is a Gosset polytope $(r-4)_{21}$. By this result combined with the cone and contraction theorem, we can give another proof of the fact that all possible contractions of X_r^2 for $1 \leq r \leq 8$ are these in the picture below.

Since $\mathrm{Nef}(X_2^2) \subseteq N^1(X_r^2)$ and $\varphi : X \to Y$ gives $\varphi^* : N^1(Y) \to N^1(X)$, we can draw the diagram of marked polytopes, reverse to the diagram of contractions, where arrows indicate inclusions of facets. Obviously, these inclusions in most cases can be realized in a few different ways. In the picture a vertex of a polytope is black if it corresponds to a contraction to $\mathbb{P}^2$ (i.e. is marked by 0) and white if it corresponds to a contraction to $\mathbb{P}^1$ (i.e. is marked by 1).



For example, we construct X_2^2 by blowing-up two points P and Q to (-1)-curves E_P and E_Q. We have two elementary contractions to X_1^2 corresponding to blow-downs of E_P or E_Q and the third, to $\mathbb{P}^1 \times \mathbb{P}^1$, which is a blow-down of the line through P and Q. Similarly, X_3^2 contains six (-1)-curves: three exceptional divisors of blow-ups and three lines through points which are blown-up.

They give six divisorial contractions to X_2^2, corresponding to the facets of the associated polytope.

3. Toric case: $n+1$ blow-ups of $\mathbb{P}^n$

In this section we present the case of X^n_{n+1} . Since X^n_{n+1} is a toric variety, combinatorial methods can be applied to understand some part of its geometric structure. For general theory of toric varieties see [5].

3.1. Toric set-up

Let $N \simeq \mathbb{Z}^n$ be a lattice and let $M = \mathrm{Hom}(N,\mathbb{Z})$ its dual by the standard pairing $\langle\cdot,\cdot\rangle$. Take a standard basis $e_1,\dots,e_n$ of $N_{\mathbb{R}} = N\otimes\mathbb{R}$, set $e_0 = -(e_1+\dots+e_n)$ and $f_i = -e_i$. Define a rational polyhedral simplicial fan Σ_n with the set of rays $\Sigma_n(1) = \{e_0,\dots,e_n,f_0,\dots,f_n\}$ by taking the cones generated by sets of n linearly independent vectors, $n-1$ of them from $\{e_0,\dots,e_n\}$, and adding all their faces. The smooth toric variety associated to Σ_n is X^n_{n+1}. The rays e_i give a standard fan for $\mathbb{P}^n$ and the division of its cones obtained by adding the rays f_i correspond to the blowing-up at $n+1$ very general points.

Recall that a ray $v \in \Sigma_n(1)$ corresponds to a torus-invariant prime divisor $Y(v)$ on X^n_{n+1} and the classes of $Y(v)$ for all $v\in\Sigma_n$ generate $\mathrm{Pic}(X^n_{n+1})$. Denote exceptional divisors in X^n_{n+1} by $E_0,\dots,E_n$ and the pull-back of a hyperplane in $\mathbb{P}^n$ by H. Then $H,E_0,\dots,E_n$ form a basis of $\mathrm{Pic}_{\mathbb{R}}(X^n_{n+1})$. Note that $Y(e_i) = H-\sum_{j=0}^n E_j+E_i$ and $Y(f_i)=E_i$ for $0\le i\le n$. We have the exact sequence

$$0\to M\to \widehat{M}=\mathbb{Z}^{\Sigma_n(1)} \xrightarrow{\psi} \mathrm{Pic}(X^n_{n+1})\to 0$$

where if we write $\mathbb{Z}^{\Sigma_n(1)} = \mathbb{Z}\cdot\widehat{e}_0^*\oplus\dots\oplus\mathbb{Z}\cdot\widehat{e}_n^*\oplus\mathbb{Z}\cdot\widehat{f}_0^*\oplus\dots\oplus\mathbb{Z}\cdot\widehat{f}_0^*$, then the first morphism takes $u\in M$ to $\sum_{i=0}^n(<u,e_i>\widehat{e}_i^*+<u,f_i>\widehat{f}_i^*)$, and $\psi(\widehat{e}_i^*)=Y(e_i)$ and $\psi(\widehat{f}_i^*)=Y(f_i)$ for $0\le i\le n$. The dual exact sequence is

$$0\to \mathrm{Pic}(X^n_{n+1})^\vee\to\widehat{N}=\widehat{M}^\vee\xrightarrow{\pi} N\to 0.$$

Note that $\mathrm{Pic}(X^n_{n+1})^\vee$ is the free abelian group generated by 1-cycles in X, and the natural pairing induced from $\langle\cdot,\cdot\rangle$ coincides with the intersection form.

Let $\widehat{N}_{\mathbb{R}}=\widehat{N}\otimes\mathbb{R}$ and denote its basis by $\widehat{e}_0,\dots,\widehat{e}_n,\widehat{f}_0,\dots,\widehat{f}_n$ which is dual to $\widehat{e}_0^*,\dots,\widehat{e}_n^*,\widehat{f}_0^*,\dots,\widehat{f}_n^*$. An extension of ψ (and π) to $\widehat{M}_{\mathbb{R}}\to\mathrm{Pic}_{\mathbb{R}}(X^n_{n+1})$ (and $\widehat{N}_{\mathbb{R}}\to N_{\mathbb{R}}$) will be also called ψ (and π) by abuse of notation. Then $\pi=\ker(\psi)^T$. Note that $\pi(\widehat{e}_i)=e_i$ and $\pi(\widehat{f}_i)=f_i$ for $0\le i\le n$. Thus ψ and π are explicitly given by the following matrices.

$$\psi = \begin{pmatrix} 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & -1 & 1 & 0 & \cdots & 0 \\ -1 & 0 & \cdots & -1 & 0 & 1 & \cdots & 0 \\ -1 & -1 & \cdots & -1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

and

$$\pi = \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 & 1 & -1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \cdots & 0 & 1 & 0 & -1 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 1 & 1 & 0 & 0 & \cdots & -1 \end{pmatrix}.$$

Let σ^+ be the positive orthant in $\widehat{M}_{\mathbb{R}}$ and σ_i^+ for $i = 1, \dots 2n+2$ be its facets.

Proposition 3.1. *Using the notation above,*

$$\mathrm{Eff}_{\mathbb{R}}(X^n_{n+1}) = \psi(\sigma^+) \quad \textit{and} \quad \mathrm{Mov}(X^n_{n+1}) = \bigcap_{i=1}^{2n+2} \psi(\sigma_i^+).$$

Proof. The first formula is an easy consequence of Lemma 15.1.8 in [5]. For the second see [5, Proposition 15.2.4]. □

Explicitly, the rays of $\mathrm{Eff}_{\mathbb{R}}(X^n_{n+1})$ are

$$H - \sum_{i \in I} E_i \text{ for all } I \subseteq \{0, \dots, n\} \text{ with } |I| = n, \text{ and } E_i \text{ for all } i \in \{0, \dots, n\}.$$

Proposition 3.2. *Let $D = dH - \sum_{i=0}^n m_i E_i \in Pic(X^n_{n+1})$. Then $D \in Mov(X^n_{n+1})$ if and only if the following inequalities are satisfied:*

1. *$m_i \geq 0$, for all $i \in \{0, \dots, n\}$;*
2. *$(n-1)d - \sum_{i \in I} m_i \geq 0$, for all $I \subseteq \{0, \dots, n\}$, with $|I| = n$;*
3. *$d \geq m_i$, for all $i \in \{0, \dots, n\}$.*

This proposition will be proved in a more general context in Theorem 4.9. However, the proof in the toric set-up is purely combinatorial and not very difficult.

Example 3.3. The simplest case after del Pezzo surfaces is X^3_4. Ray generators of $\mathrm{Mov}(X^3_4)$ are as follows:

degree 4	$H, 3H-2E_0-2E_1-2E_2-2E_3$
degree 3	$H-E_0, H-E_1, H-E_2, H-E_3,$ $2H-2E_0-E_1-E_2-E_3, 2H-E_0-2E_1-E_2-E_3,$ $2H-E_0-E_1-2E_2-E_3, 2H-E_0-E_1-E_2-2E_3$
degree 2	$H-E_0-E_1, H-E_0-E_2, H-E_0-E_3$ $H-E_1-E_2, H-E_1-E_3, H-E_2-E_3$

The degree of divisors will be defined in Definition 4.1.

3.2. Toric Mori theory

By toric Mori theory, we have an explicit description of the Kleiman-Mori cone $\mathrm{NE}(X^n_{n+1})$ in terms of basis of $\widehat{N}_{\mathbb{R}}$. First, let us briefly recall some facts from toric Mori theory. For the reference see [14], [18] or the final chapter of [5]. By the orbit-cone correspondence (see [5, Theorem 3.2.6]), a torus-invariant curve $Y(\omega)$ on a n-dimensional $\mathbb{Q}$-factorial toric variety $X(\Sigma)$ is associated to a $(n-1)$-dimensional cone ω in Σ generated by primitive lattice elements $\varepsilon_1,\dots,\varepsilon_{n-1}$. There are two n-dimensional cones δ_{n+1} and δ_n in Σ generated by $\varepsilon_1,\dots,\varepsilon_{n-1},\varepsilon_n$ and $\varepsilon_1,\dots,\varepsilon_{n-1},\varepsilon_{n+1}$ respectively, where ε_n and ε_{n+1} are primitive on rays on opposite sides of ω. Then the linear equation

$$a_1\varepsilon_1+\ldots+a_{n+1}\varepsilon_{n+1}=0.$$

has a unique solution if we set $a_{n+1}=1$. Reorder the indices so that

$$\begin{array}{lll} a_i<0 & \text{for} & 1\le i\le\alpha, \\ a_i=0 & \text{for} & \alpha+1\le i\le\beta, \\ a_i>0 & \text{for} & \beta+1\le i\le n+1. \end{array}$$

Then by Proposition on p. 257 of [18],

$$\mathrm{NE}(X)=\sum_{\omega\in\Sigma(n-1)}\mathbb{R}_{\geq 0}[a_1\widehat{\varepsilon}_1+\ldots+a_\alpha\widehat{\varepsilon}_\alpha+a_{\beta+1}\widehat{\varepsilon}_{\beta+1}+\ldots+a_{n+1}\widehat{\varepsilon}_{n+1}],$$

where $\widehat{\varepsilon}_i$ is an element of a chosen basis of $\widehat{N}_{\mathbb{R}}$ such that $\pi(\widehat{\varepsilon}_i)=\varepsilon_i$.

The Toric Contraction Theorem I and Remark on p. 259-260 of [18] give a description of the contractions of extremal rays:

1. if $\alpha=0$, then the extremal ray gives a fibre type contraction whose generic fibre is $(n-\beta)$-dimensional,
2. if $\alpha=1$, then the extremal ray gives a divisorial contraction,
3. if $\alpha>1$, then the extremal ray gives a small contraction.

Going back to our case, every $(n-1)$-dimensional cone in Σ_n is spanned by

$$e_j \text{ for all } j \in J \text{ and } f_k \text{ such that } k \notin J \qquad \text{or} \qquad e_i \text{ for all } i \in I$$

where I and J are subsets of $\{0,\dots,n\}$ such that $|I| = n-1$ and $|J| = n-2$. In the first case the two adjacent cones of maximal dimension are constructed by adding vectors e_p and e_r such that $p,r \notin J \cup \{k\}$, and in the second case by adding f_k and f_j such that $k, j \notin I$. They satisfy the following linear relations:

$$\sum_{i\in I} e_i - f_k = 0 \text{ for } k \notin I \qquad \text{and} \qquad -(\sum_{j\in J} e_i) + f_k + f_l = 0 \text{ for } k,l \notin J.$$

Note that in the first case $\alpha = 1$, so the corresponding curves give divisorial contractions $X^n_{n+1} \to X^n_n$, and for the second case, $\alpha = n-1$, so they give small contractions.

To analyze small contractions, we need to study flips. Let $R = [a_1\widehat{\varepsilon}_1 + \ldots + a_\alpha\widehat{\varepsilon}_\alpha + a_{\beta+1}\widehat{\varepsilon}_{\beta+1} + \ldots + a_{n+1}\widehat{\varepsilon}_{n+1}]$ be an extremal ray of the Kleiman-Mori cone of a variety X, obtained from an $(n-1)$-dimensional cone ω. Assume that $\alpha > 1$, so this ray gives a small contraction. Let δ_i be a cone generated by $\{\widehat{\varepsilon}_1,\ldots,\widehat{\varepsilon}_{n+1}\} \setminus \{\widehat{\varepsilon}_i\}$, and let $\delta(\omega)$ be a cone generated by $\{\widehat{\varepsilon}_1,\ldots,\widehat{\varepsilon}_{n+1}\}$. Then by Lemma on p. 259 of [18], we obtain two simplicial subdivisions

$$\delta(\omega) = \bigcup_{i=\beta+1}^{n+1} \delta_i = \bigcup_{i=1}^{\alpha} \delta_i.$$

The idea of the Toric Flip Theorem (see e.g. [18], p. 263) is that if we exchange one the first subdivision with the second one, we obtain a (simplicial) fan Σ'_n such that there is a birational map $X(\Sigma_n) \dashrightarrow X(\Sigma'_n)$ which is an isomorphism in codimension 1. This map will be called an elementary transformation with respect to R. Thus $X(\Sigma'_n)$ is a SQM of X^n_{n+1}. By Proposition 5.7. in [7], every SQM of X^n_{n+1} is obtained by a finite succession of elementary transformations.

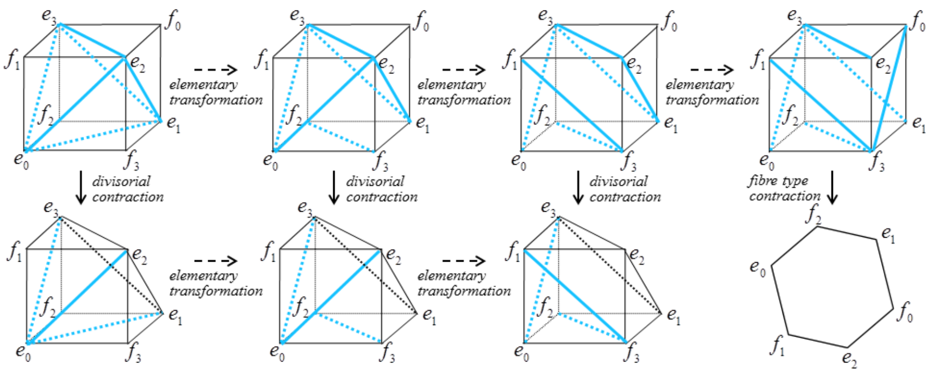
Since X^n_{n+1} is a MDS, the movable cone $\mathrm{Mov}(X^n_{n+1})$ has a closed convex chamber decomposition into nef cones of SQMs of X^n_{n+1} (which is studied in section 3.4). Note that every SQM of X^n_{n+1} is also a toric variety, because it has the same Cox ring as X^n_{n+1} and we use [9], 2.10. Interiors of nef cones are disjoint, and adjacent nef cones share some faces. Let X' and X'' be SQMs of X^n_{n+1} and $\mathrm{Nef}(X')$ and $\mathrm{Nef}(X'')$ be adjacent. Then for some Cartier divisor $D \in \mathrm{Nef}(X') \cap \mathrm{Nef}(X'')$, the linear system $|D|$ gives a small contraction, so X' and X'' are connected by an elementary transformation. The associated fans of SQMs of X^n_{n+1} are complete simplicial fans with the set of rays $\Sigma_n(1)$.

Example 3.4. Extremal rays of the Kleiman-Mori cone $\mathrm{NE}(X^3_4)$ are

1. of divisorial contraction type: $[\widehat{e_i}+\widehat{e_j}+\widehat{e_k}-\widehat{f_l}]$,

2. of small contraction type: $[-\widehat{e_i}-\widehat{e_j}+\widehat{f_k}+\widehat{f_l}]$,

where $\{i,j,k,l\}=\{0,1,2,3\}$. The polytope corresponding to the fan Σ_3 is a cube. It can be easily seen that every complete simplicial fan with the set of rays $\Sigma_3(1)$ comes from a subdivision of this cube such that all faces are divided into triangles.



The divisorial contraction of $[\widehat{e_1}+\widehat{e_2}+\widehat{e_3}-\widehat{f_0}]$ is the birational morphism contracting the exceptional divisor E_0.

There are successive elementary transformations $X^3_4 \dashrightarrow X(\Sigma'_3) \dashrightarrow X(\Sigma''_3) \dashrightarrow X(\Sigma'''_3)$. The first one is coming from the extremal ray $[-\widehat{e_0}-\widehat{e_1}+\widehat{f_2}+\widehat{f_3}]$. It is performed on the facet $e_0f_3e_1f_2$ of the cube Σ_3 and it is a change of a diagonal which divides it. The second one and the third one are coming from the extremal rays $[-\widehat{e_0}-\widehat{e_2}+\widehat{f_1}+\widehat{f_3}]$ and $[-\widehat{e_1}-\widehat{e_2}+\widehat{f_0}+\widehat{f_3}]$, respectively.

Note that not every ray of the type $[-\widehat{e_i}-\widehat{e_j}+\widehat{f_k}+\widehat{f_l}]$ is extremal. On the facet $e_2f_0e_3f_1$ of the cube from Σ'_3, it is possible change of a diagonal which divides it. To do this, $[-\widehat{e_2}-\widehat{e_3}+\widehat{f_0}+\widehat{f_1}]$ should be an extremal ray of $\mathrm{NE}(X(\Sigma'_3))$. However, there are rays $[\widehat{e_0}+\widehat{e_1}-\widehat{f_2}-\widehat{f_3}]$, $[-\widehat{e_1}-\widehat{e_3}+\widehat{f_0}+\widehat{f_2}]$ and $[-\widehat{e_0}-\widehat{e_2}+\widehat{f_1}+\widehat{f_3}]$ of $\mathrm{NE}(X(\Sigma'_3))$ such that

$$\begin{aligned}(\widehat{e_0}+\widehat{e_1}-\widehat{f_2}-\widehat{f_3})+(-\widehat{e_1}-\widehat{e_3}+\widehat{f_0}+\widehat{f_2})+(-\widehat{e_0}-\widehat{e_2}+\widehat{f_1}+\widehat{f_3})\\ =-\widehat{e_2}-\widehat{e_3}+\widehat{f_0}+\widehat{f_1},\end{aligned}$$

so $[-\widehat{e_2}-\widehat{e_3}+\widehat{f_0}+\widehat{f_1}]$ is not extremal. Observe that if we change the diagonal on the facet $e_2f_0e_3f_1$ of the cube from Σ'_3, we get a non-projective complete toric variety, which also can be found in Example 3.9.

Similarly, one can easily check that $[\widehat{e_3}+\widehat{f_3}]$ is not an extremal ray on both $\mathrm{NE}(X(\Sigma'_3))$ and $\mathrm{NE}(X(\Sigma''_3))$.

The Kleiman-Mori cone $\mathrm{NE}(X(\Sigma_3'''))$ has an extremal ray $[\widehat{e_3}+\widehat{f_3}]$, and this gives the fibre type contraction $X(\Sigma_3''') \to X_3^2$.

3.3. Weyl group action

Recall that the associated root system of X_{n+1}^n is $\mathbb{A}_n \times \mathbb{A}_1$, and its Weyl group W_n is isomorphic to $S_{n+1} \times \mathbb{Z}/2\mathbb{Z}$. Define the Weyl group action on $\widehat{M}_{\mathbb{R}}$ as follows:

$$\widehat{e}_i^* \mapsto \widehat{e}_{\sigma(i)}^* \quad \text{and} \quad \widehat{f}_i^* \mapsto \widehat{f}_{\sigma(i)}^* \quad \text{for } (\sigma,0) \in S_{n+1} \times \mathbb{Z}/2\mathbb{Z}$$

and

$$\widehat{e}_i^* \mapsto \widehat{f}_{\sigma(i)}^* \quad \text{and} \quad \widehat{f}_i^* \mapsto \widehat{e}_{\sigma(i)}^* \quad \text{for } (\sigma,1) \in S_{n+1} \times \mathbb{Z}/2\mathbb{Z}.$$

Clearly $M_{\mathbb{R}}$ is preserved, so the action descends to $\mathrm{Pic}_{\mathbb{R}}(X_{n+1}^n)$. By a straightforward computation we obtain

Lemma 3.5. *The Weyl group $S_{n+1} \times \mathbb{Z}/2\mathbb{Z}$ action on $\mathrm{Pic}_{\mathbb{R}}(X_{n+1}^n)$ is given (as in [4, Section 2]) by*

$$E_i \mapsto E_{\sigma(i)} \text{ and } H - \sum_{i=0}^{n} E_i \mapsto H - \sum_{i=0}^{n} E_i \text{ for } (\sigma,0) \in S_{n+1} \times \mathbb{Z}/2\mathbb{Z}$$

and

$$E_i \mapsto H - \sum_{i=0}^{n} E_i + E_{\sigma(i)} \text{ and } H - \sum_{i=0}^{n} E_i \mapsto -H + \sum_{i=0}^{n} E_i \text{ for } (\sigma,1) \in S_{n+1} \times \mathbb{Z}/2\mathbb{Z}.$$

We consider also the dual action on $\widehat{N}_{\mathbb{R}}$, $N_{\mathbb{R}}$ and $\mathrm{Pic}_{\mathbb{R}}(X_{n+1}^n)^\vee$.

Now we study the Weyl group action on the set of fans and the set of nef cones of SQMs of X_{n+1}^n. The Weyl group action on $N_{\mathbb{R}}$ permutes the rays of Σ_n, so W_n permutes the fans of SQMs of X_{n+1}^n.

Proposition 3.6. *The Weyl group action on $\mathrm{Pic}_{\mathbb{R}}(X_{n+1}^n)$ fixes $\mathrm{Mov}(X_{n+1}^n)$ and $\mathrm{Eff}(X_{n+1}^n)$, and permutes its nef chambers in $\mathrm{Mov}(X_{n+1}^n)$.*

Proof. The Weyl group action on $\widehat{M}_{\mathbb{R}}$ preserves the positive orthant and permutes its facets. Hence the fact that $\mathrm{Eff}(X_{n+1}^n)$ and $\mathrm{Mov}(X_{n+1}^n)$ are fixed follows from the description of these cones in Proposition 3.1.

Let us consider the Weyl group action on the nef cones of SQMs of X_{n+1}^n. Since the nef cone in $N^1(X_{n+1}^n) = \mathrm{Pic}_{\mathbb{R}}(X_{n+1}^n)$ is dual to Kleiman-Mori cone in $N^1(X_{n+1}^n) = \mathrm{Pic}_{\mathbb{R}}(X_{n+1}^n)^\vee$, we only have to consider the Weyl group action on Kleiman-Mori cones. By the toric Mori cone theorem, any extremal ray of Kleiman-Mori cone is of the form

$$[a_1\widehat{\varepsilon}_1 + \ldots + a_\alpha\widehat{\varepsilon}_\alpha + a_{\beta+1}\widehat{\varepsilon}_{\beta+1} + \ldots + a_{n+1}\widehat{\varepsilon}_{n+1}]$$

in $\widehat{N}_{\mathbb{R}}$. Let Σ'_n and Σ''_n be fans of SQMs of X^n_{n+1}. Suppose that for some $w \in W_n$,

$$w : \Sigma'_n \to \Sigma''_n$$

in $N_{\mathbb{R}}$. Since $w : N_{\mathbb{R}} \to N_{\mathbb{R}}$ is a linear automorphism,

$$w(a_1\varepsilon_1 + \ldots + a_{n+1}\varepsilon_n) = a_1 w(\varepsilon_1) + \ldots + a_{n+1} w(\varepsilon_n).$$

Thus, $[a_1\widehat{\varepsilon}_1 + \ldots + a_\alpha\widehat{\varepsilon}_\alpha + a_{\beta+1}\widehat{\varepsilon}_{\beta+1} + \ldots + a_{n+1}\widehat{\varepsilon}_{n+1}]$ is an extremal ray of $\mathrm{NE}(X(\Sigma'_n))$ if and only if

$$[a_1 w(\widehat{\varepsilon}_1) + \ldots + a_\alpha w(\widehat{\varepsilon}_\alpha) + a_{\beta+1} w(\widehat{\varepsilon}_{\beta+1}) + \ldots + a_{n+1} w(\widehat{\varepsilon}_{n+1})]$$

is an extremal ray of $\mathrm{NE}(X(\Sigma''_n))$. It means that there is the isomorphism

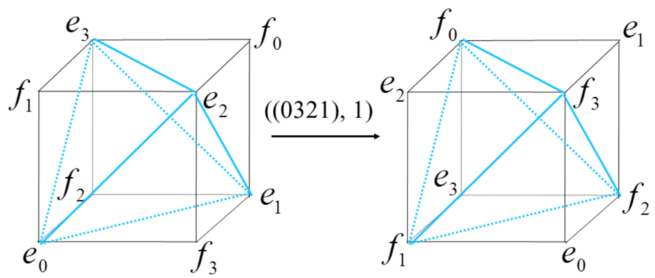
$$w : \mathrm{NE}(X(\Sigma'_n)) \to \mathrm{NE}(X(\Sigma''_n)).$$

Hence the Weyl group permutes nef chambers in $\mathrm{Mov}(X^n_{n+1})$. $\square$

We can think that the Weyl group W_n acts on the set

$$\{\text{nef chambers in } \mathrm{Mov}(X^n_{n+1})\} = \{\text{fans of SQMs of } X^n_{n+1}\}.$$

Example 3.7. Recall that the polytope corresponding to the fan Σ_3 is a cube. Let $((0321),1) \in S_4 \times \mathbb{Z}/2\mathbb{Z}$, then $w : \Sigma_3 \mapsto \Sigma'_3$ where the toric variety $X(\Sigma'_3)$ is obtained by blowing-up $X(\{f_0, f_1, f_2, f_3,$ and its faces$\})$ at 4 very general points.



It is natural to ask what orbits of Weyl group action on nef chambers are. The following proposition gives the answer.

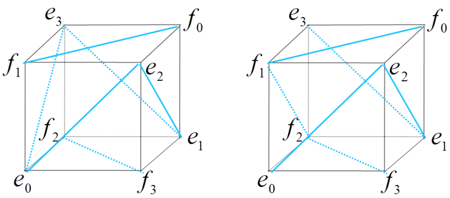
Proposition 3.8. *Isomorphism classes of SQMs of X^n_{n+1} in the category of toric varieties (i.e. isomorphism classes of fans of SQMs) form the orbits of the Weyl group action.*

Proof. Let Σ'_n and Σ''_n be fans of SQMs of X^n_{n+1} and $X(\Sigma'_n) \simeq X(\Sigma''_n)$ as toric varieties. Then there is a linear automorphism $\theta : N_{\mathbb{R}} \to N_{\mathbb{R}}$ compatible with Σ'_n and Σ''_n. The automorphism θ permutes rays of Σ'_n and Σ''_n, and because of the linear relations between the rays there is $w \in W_n$ such that $w : N_{\mathbb{R}} \to N_{\mathbb{R}}$ coincides with θ. $\square$

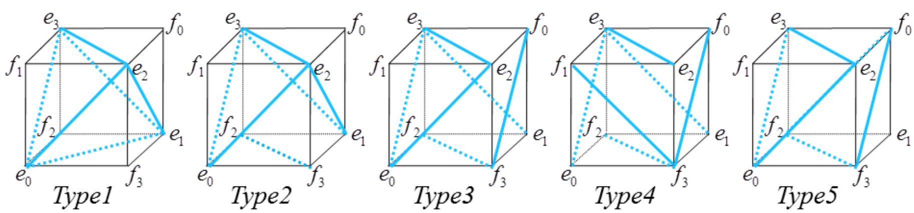
Let Σ'_n and Σ''_n be fans of SQMs of X^n_{n+1}. If $X(\Sigma'_n) \simeq X(\Sigma''_n)$ as abstract varieties, then we say that Σ'_n and Σ''_n are the same type. By the proposition, we have

$$\text{the number of Weyl group orbits} = \text{the number of types of fans of SQMs.}$$

Example 3.9. An elementary transformation between SQMs of X^3_4 can be performed on each face of a cube and it is a change of a diagonal which divides it. Therefore we have 64 candidates for *SQMs* of X^3_4. However, not all subdivisions of the cube lead to a projective variety. In the following picture one can easily check that both have nef cones which are not full dimension, so that the corresponding toric varieties are not projective.



Note that they cannot be obtained by successive elementary transformations from X^3_4. The number of elements of the Weyl group orbit of the left one is 6, and the number of elements of Weyl group orbit of the right one is 12, so we have 46 SQMs of X^3_4. They can be divided into 5 types of triangulations, which correspond to isomorphism classes of the models, as in the picture.



We give the numbers of elements in orbits corresponding to the triangulation types 1-5.

Type1	Type2	Type3	Type4	Type5
2	12	24	4	4

3.4. SQMs and flopping classes

It seems quite difficult to generalize the argument in Example 3.9 to higher dimensional cases. The set of simplicial fans with rays $\Sigma_n(1)$ for $n > 3$ is too complicated to write down, so we try to attack the problem in a different method.

Theorem 3.10. *An extremal ray of the Kleiman-Mori cone of some SQM of X^n_{n+1} is one of the following:*

1. *divisorial type: $[\sum_{i\in\{0,\dots,n\}\setminus\{k\}} \widehat{e_i} - \widehat{f_k}]$ or $[\sum_{i\in\{0,\dots,n\}\setminus\{k\}} \widehat{f_i} - \widehat{e_k}]$ for $k \in \{0,\dots,n\}$*
2. *fibre type: $[\widehat{e_i} + \widehat{f_i}]$ for $i \in \{0,\dots,n\}$*
3. *small type: $\pm[\sum_{i\in I} \widehat{e_i} - \sum_{j\in J} \widehat{f_j}]$ where $I \cup J = \{0,\dots,n\}$, I and J are disjoint, and $|I|, |J| \geq 2$.*

Moreover, they all appear on some SQM of X^n_{n+1}.

Proof. By toric Mori cone theorem, every extremal ray of the Kleiman-Mori cone of a SQM of X^n_{n+1} is obtained by an $(n-1)$-dimensional common face of two adjacent n-dimensional cones with rays in $\Sigma_n(1) = \{e_0,\dots,e_n, f_0,\dots,f_n\}$. Let σ_0, σ_1 be two such cones with the sets of generators $\{\varepsilon_1,\dots\varepsilon_{n-1},\varepsilon_n\}$ and $\{\varepsilon_1,\dots\varepsilon_{n-1},\varepsilon_{n+1}\}$. Assume that the cone generated by $\varepsilon_1,\dots\varepsilon_{n-1}$ gives a curve in an extremal ray. Set $a_{n+1} = 1$ and take a unique rational solution a_i of the linear equation

$$a_1\varepsilon_1 + \dots + a_{n+1}\varepsilon_{n+1} = 0.$$

First consider the case when the set $\{\varepsilon_1,\dots,\varepsilon_{n+1}\}$ contains the rays e_i and f_i for some i. This is possible only if $\{e_i, f_i\} = \{\varepsilon_n, \varepsilon_{n+1}\}$, because σ_0 and σ_1 are strictly convex. Then $a_0 = \dots = a_{n-1} = 0$ and $a_n = 1$, so the considered ray gives a fibre type contraction by Remark on p. 260 of [18].

Now assume that the set $\{\varepsilon_1,\dots,\varepsilon_{n+1}\}$ does not contain any pair e_i, f_i. Permuting the rays with the Weyl group action (which clearly does not change the types of extremal rays listed in the theorem) we can consider only the case when $\varepsilon_{n+1} = e_0$. Then $\{\varepsilon_1,\dots,\varepsilon_n\} \subset \{e_1,\dots,e_n, f_1,\dots,f_n\}$ and, as $\varepsilon_1 + \dots + \varepsilon_n = -e_0 = e_1 + \dots + e_n$, from each pair e_i, f_i exactly one is in $\{\varepsilon_1,\dots,\varepsilon_n\}$. Obviously e_i appear with coefficient 1 and f_i with -1. Also, it cannot happen that $\{\varepsilon_1,\dots,\varepsilon_{n+1}\} = \{e_0,\dots,e_n\}$, because a cone spanned by any n of these vectors contains some vector f_i in the interior. Thus we obtain a sum of the type 1 or 3 from the list above and, by Remark on p. 260 of [18], the corresponding ray gives a divisorial or a small contraction respectively.

Rays of the first type appear on X^n_{n+1}. In Examples 3.11 and 3.12 the constructions of SQMs of X^n_{n+1} having rays of types 2 and 3 are explained. In case of types 1 and 2 it is sufficient to show only one example, because using the Weyl group action we obtain all other possible configurations within each type. In case of type 3 by Proposition 5.2 of [16] and linear relations between rays we only have to prove the existence of SQMs of X^n_{n+1} with a ray of the form $[-\sum_{i=k}^n \widehat{e}_i + \sum_{j=0}^{k-1} \widehat{f}_j]$ for all $k \in \{2,\ldots,n-1\}$. Then we also obtain the remaining configuration acting with the Weyl group.

□

Example 3.11. Consider the n-dimensional cones in Σ_n generated by $e_2,\ldots,e_n$, f_1 and $e_2,\ldots,e_n,f_0$. The common face is a cone in Σ_n generated by $e_2,\ldots,e_n$. The associated extremal ray $[-\widehat{e}_2 - \ldots - \widehat{e}_n + \widehat{f}_1 + \widehat{f}_0]$ gives an elementary transformation $X(\Sigma_n) \dashrightarrow X(\Sigma_n^{(1)})$. Now there is an extremal ray of $\mathrm{NE}(X(\Sigma_n^{(1)}))$

$$[-\widehat{e}_1 - \widehat{e}_3 - \ldots - \widehat{e}_n + \widehat{f}_2 + \widehat{f}_0]$$

and an associated elementary transformation $X(\Sigma_n^{(1)}) \dashrightarrow X(\Sigma_n^{(2)})$. Repeat this process. Then we obtain the fan $\Sigma_n^{(n)}$, and a SQM $X(\Sigma_n^{(n)})$ of X^n_{n+1}. Using Theorem 4.10 in [15], it is easy to check that $[\widehat{e}_0 + \widehat{f}_0]$ in $\mathrm{NE}(X(\Sigma_n^{(n)}))$ is a contractible class, because there is only one primitive collection containing e_0 or f_0. By Lemma 1 in [2], $[\widehat{e}_0 + \widehat{f}_0]$ is an extremal ray of $\mathrm{NE}(X(\Sigma_n^{(n)}))$.

Example 3.12. We use notations as in the previous example. We assume that $n \geq 4$. In $\Sigma_n^{(2)}$, we can find n-dimensional cones generated by $f_0,e_3,\ldots,e_n,f_1$ and $f_0,e_3,\ldots,e_n,f_2$, and the common face is an $(n-1)$-dimensional cone generated by $f_0,e_3,\ldots,e_n$. Thus the associated ray is $[-\widehat{e}_3 - \ldots - \widehat{e}_n + \widehat{f}_0 + \widehat{f}_1 + \widehat{f}_2]$. In the similar way, we can obtain desired rays.

Corollary 3.13. Nef *chamber decomposition of* $\mathrm{Mov}(X^n_{n+1})$ *is determined by hyperplanes*

$$\psi(\pm(\sum_{i\in I}\widehat{e}_i - \sum_{j\in J}\widehat{f}_j)^\perp) = \pm\psi(\sum_{i\in I}\widehat{e}_i - \sum_{j\in J}\widehat{f}_j)^\perp$$

where $I \cup J = \{0,\ldots,n\}$*, I and J are disjoint, and* $|I|,|J| \geq 2$.

Proof. ¿From the Mori theory, the common face of adjacent nef chambers consists of divisors giving small contractions. Thus the result follows from Theorem 3.10. □

We have described flopping classes giving nef chamber decomposition of $\mathrm{Mov}(X^n_{n+1})$, but we still do not know many things for general n such as (1) which fans with the set of rays $\Sigma_n(1)$ represents nef chambers, (2) how many orbits of Weyl group action there are, and (3) how many elements are in each orbit?

3.5. Contractions and marked polynomials

We have studied the inner structure (nef chamber decomposition) of $\mathrm{Mov}(X^n_{n+1})$. In Proposition 3.2 and Theorem 4.9 we give an explicit formula for the rays of the movable cone (in the cases of $n+1$ and $n+2$ points blown-up). In this section we describe the structure of $\mathrm{Mov}(X^n_{n+1})$ using the marked polynomials (see Definition 2.10). As seen in del Pezzo surfaces case, the marked polynomials of X^n_{n+1} for $\mathrm{Mov}(X^n_{n+1})$ plays an important role in understanding outer structure of $\mathrm{Mov}(X^n_{n+1})$ and contractions of X^n_{n+1}. The idea of describing contractions in the form of marked polynomials comes from [1].

Theorem 3.14. *The marked polynomial of X^n_{n+1} for* $\mathrm{Mov}(X^n_{n+1})$ *is*

$$P^n_{n+1}(x,y) = \sum_{m=0}^{n-2} \binom{n+1}{m} B_{n-m}(x) x^m y^m + \binom{n+1}{n-1} x^{n+1} y^{n-1}$$

where

$$B_k(x) = 1 + 2(k+1)x + (k+1)^2x^2 + 2\sum_{i=3}^{k+1} \binom{k+1}{i} x^i.$$

Proof. We only have to count the numbers of contractions from X^n_{n+1}. For convenience, we denote the class of all SQMs of X^m_k by $\mathfrak{X}^m_k$. By Proposition 1.11 in [9] and duality between nef cone and Kleiman-Mori cone, every contraction from $\mathfrak{X}^m_k$ dropping the Picard number by one is coming from some extremal ray of the Kleiman-Mori cone of some SQM of X^m_k.
First, consider divisorial contractions from $\mathfrak{X}^n_{n+1}$. By Theorem 3.10, $[\sum_{i\neq k} \widehat{e_i} - \widehat{f_k}]$ or $[\sum_{i\neq k} \widehat{f_i} - \widehat{e_k}]$ for $k \in \{0,\ldots,n\}$ gives $\mathfrak{X}^n_{n+1} \to \mathfrak{X}^n_n$. It can be understood as eliminating 1 ray from either $\{e_0,\cdots,e_n\}$ or $\{f_0,\cdots,f_n\}$ in $\Sigma_n(1)$. More generally, a birational contraction $\mathfrak{X}^n_{n+1} \to \mathfrak{X}^n_{n+1-i}$ for $1 \le i \le n+1$ can be understood as taking off i rays from either $\{e_0,\cdots,e_n\}$ or $\{f_0,\cdots,f_n\}$ in $\Sigma_n(1)$. The number of type (1) contractions associated to codim i faces of $Mov(X^n_{n+1})$ is $2\binom{n+1}{i}$, which is the number of choices of i rays from either $\{e_0,\cdots,e_n\}$ or $\{f_0,\cdots,f_n\}$. Note that there is one exceptional case. In $\mathfrak{X}^n_n$, there exists $\mathfrak{X}^n_n \to \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ (n-times). The contraction $\mathfrak{X}^n_{n+1} \to \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ (n-times) can be thought of as eliminating a pair (e_i, f_i) of rays. The associated faces of $Mov(X^n_{n+1})$ are of codim 2, and the number is $n+1$. Thus the divisorial part of the marked polynomial is

$$\begin{aligned} B_n(x) &= 1 + 2\textstyle\sum_{i=1}^{n+1} \binom{n+1}{i} x^i + (n+1)x^2 \\ &= 1 + 2(n+1)x + (n+1)^2x^2 + 2\textstyle\sum_{i=3}^{n+1} \binom{n+1}{i} x^i. \end{aligned}$$

Now consider fibre type contractions from $\mathfrak{X}^n_{n+1}$. Note that every contraction $\mathfrak{X}^n_{n+1} \to \mathfrak{X}^k_l$ for $k, l-1 < n$ factors through a contraction $\mathfrak{X}^n_{n+1} \to \mathfrak{X}^k_{k+1}$. The case of $\mathfrak{X}^n_{n+1} \to \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ is similar. Thus we only have to count the number of contractions $\mathfrak{X}^n_{n+1} \to \mathfrak{X}^k_{k+1}$ for $1 < k < n$. This number is equal to the number of choices of $n-k$ pairs (e_i, f_i), so $\binom{n+1}{n-k}$. Hence the part of the marked polynomial associated to a contraction $\mathfrak{X}^n_{n+1} \to \mathfrak{X}^k_{k+1}$ is

$$\binom{n+1}{n-k} B_k x^{n-k} y^{n-k}$$

for $1 < k < n$. Note that this part of the marked polynomial also counts contractions $\mathfrak{X}^n_{n+1} \to \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ (k times).

Finally the number of the contractions $\mathfrak{X}^n_{n+1} \to \mathbb{P}^1$ is $\binom{n+1}{n-1}$. Thus we obtain the marked polynomial $P^n_{n+1}(x,y)$ of X^n_{n+1}.

□

Example 3.15. We can count the number of contractions as in the proof of the theorem, so we can obtain the marked polynomial of X^3_4 for $\mathrm{Mov}(X^3_4)$

$$P^3_4(x,y) = (1+8x+16x^2+8x^3+2x^4)+4(1+6x+9x^2+2x^3)xy+6x^4y^2.$$

Thus we have the numbers of faces of $\mathrm{Mov}(X^3_4)$. This number can be directly checked by computer program (for example Macaulay2) using the explicit ray generators given in Example 3.3.

4-dim faces	3-dim faces	2-dim faces	1-dim faces
$8+4$	$16+24$	$8+36$	$2+8+6$

4. The structure of the movable cone

In this section we study the shape of the movable cone of X^n_r for $n \geq 2$ and $r = n+1,\, n+2,\, n+3$. Note that in this case X^n_r is a Mori dream space by [4, Theorem 1.3].

In Theorem 4.7 we show that the movable cone is the intersection of the cone of effective divisors with its dual (Definition 4.1). Moreover we will also give explicit equations of the movable cone in the standard basis of the N^1 when $r = n+1, n+2$ (see Theorem 4.9).

Note that throughout the section we will not distinguish between classes of linear equivalence of divisors and the divisors themselves. We hope that no confusion will arise.

Let $n \in \mathbb{N}$, $n \geq 2$ and let $X = X^n_r$ be the blowing-up of $\mathbb{P}^n$ at r very general points, with $r \geq n+1$. Let us consider the standard basis of the Picard group $\mathrm{Pic}(X)$ given by $\{H, E_1, \ldots, E_r\}$, where we denote by H the pullback of an hyperplane section and the E_i's are the exceptional divisors. Consider the symmetric bilinear form $(\cdot,\cdot)$ on $\mathrm{Pic}(X)$, already defined in a more general context in subsection 2.2:

$$(H,E_j)=0, \quad (H,H)=n-1, \quad (E_i,E_j)=-\delta_{i,j}.$$

Definition 4.1. (cf. [4, Def. 2.5]) If $D\in\mathrm{Pic}(X)$ and K_X is class of the canonical divisor in $\mathrm{Pic}(X)$, we define the *degree* of D as

$$\deg(D)=\frac{1}{n-1}(D,-K_X).$$

Moreover we denote by $\mathrm{Eff}_{\mathbb{R}}(X)^\vee$ the dual of the cone of classes of effective $\mathbb{R}$-divisors with respect to the simmetric bilinear form $(\cdot,\cdot)$. In other words

$$\mathrm{Eff}_{\mathbb{R}}(X)^\vee=\{[D]\in N^1(X)\ :\ (D,F)\geq 0 \text{ for all } [F]\in\mathrm{Eff}_{\mathbb{R}}(X)\}.$$

4.1. Some general considerations about linear systems on very general blowing-ups of $\mathbb{P}^n$

The aim of this subsection is to prove lemma 4.3 and lemma 4.4, that we will use to prove the main results of the section.

Lemma 4.2. *Let $n,r\geq 2$ be natural numbers. Consider r very general points $p_1,\dots,p_r\in\mathbb{P}^n$.*

For every $(d,m_1,\dots,m_r)\in\mathbb{N}^{r+1}$ let us denote by $\mathscr{L}_d(p_1^{m_1},\dots,p_r^{m_r})$ the linear series of hypersurfaces of $\mathbb{P}^r$ of degree d having multiplicity at least m_i in p_i for every $i=1,\dots,r$. If $\mathscr{L}_d(p_1^{m_1},\dots,p_r^{m_r})\neq\emptyset$ then the general hypersurface in $\mathscr{L}_d(p_1^{m_1},\dots,p_r^{m_r})$ has multiplicity exactly m_i in p_i for every i.

Proof. Let us fix $k\in\{1,\dots,r\}$ and $(d,m_1,\dots,m_r)$ such that $\mathscr{L}_d(q_1^{m_1},\dots,q_r^{m_r})\neq\emptyset$ if the q_i's are general. Note that by semicontinuity this property does not depend on the choice of the points.

We will prove that there exists an open subset $U_{d,m_1,\dots,m_r,k}\subseteq(\mathbb{P}^n)^r$ such that the general hypersurface in $\mathscr{L}_d(p_1^{m_1},\dots,p_r^{m_r})$ has multiplicity exactly m_k in p_k if $(p_1,\dots,p_r)\in U_{d,m_1,\dots,m_r,k}$.

This will imply that if $(p_1,\dots,p_r)\in U:=\bigcap U_{d,m_1,\dots,m_r,k}$ – where the intersection is taken on $k\in\{1,\dots,r\}$ and $(d,m_1,\dots,m_r)\in\mathbb{N}^{r+1}$ such that, for general q_i's, $\mathscr{L}_d(q_1^{m_1},\dots,q_r^{m_r})\neq\emptyset$ – then whenever $\mathscr{L}_d(p_1^{m_1},\dots,p_r^{m_r})\neq\emptyset$ we have that the general hypersurface in $\mathscr{L}_d(p_1^{m_1},\dots,p_r^{m_r})$ has multiplicity exactly m_i in p_i for every i, so that the theorem holds. For simplicity we put $k=r$.

Up to restricting to an affine open subset $\mathbb{A}^n\subseteq\mathbb{P}^n$ we have that r points have the form $p_1=(x_{1,1},\dots,x_{1,n}),\dots,p_r=(x_{r,1},\dots,x_{r,n})$, where the $x_{i,j}$ are affine coordinates.

At the same time the restriction to $\mathbb{A}^n$ of an hypersurface of degree d will be defined by a polinomial of degree d, say $f(y_1,\dots,y_n)=\sum a_I y^I$, where $y=(y_1,\dots,y_n)$ and I is a multi-index such that $|I|\le d$.

Let us consider the ring of polynomials

$$R:=\mathbb{C}[\{x_{i,j}\}_{\substack{0\le i\le r\\ 0\le j\le n}},\{a_I\}_{|I|\le d}]$$

and let $\mathbb{A}^N = Spec(R)$.

Let $a\subseteq R$ be the ideal generated by the partial derivatives

$$\partial^{y^L} f(x_{1,1},\dots,x_{1,n}),\ \text{ for every multi-index } L \text{ such that } 0\le|L|\le m_1-1,$$

$$\dots\dots$$

$$\partial^{y^L} f(x_{r-1,1},\dots,x_{r-1,n}),\ \text{for every multi-index } L \text{ such that } 0\le|L|\le m_{r-1}-1,$$

$$\partial^{y^L} f(x_{r,1},\dots,x_{r,n}),\ \text{for every multi-index } L \text{ such that } 0\le|L|\le m_r-1,$$

and let $c\subseteq R$ be the ideal generated by the set

$$\partial^{y^L} f(x_{r,1},\dots,x_{r,n}),\quad \text{for every multi-index } L \text{ such that } 0\le|L|\le m_r.$$

With this notation we have that a hypersurface of degree d defined by a polinomial $f(y_1,\dots,y_n)=\sum a_I y^I$ lies in $\mathscr{L}_d(p_1^{m_1},\dots,p_r^{m_r})$ if and only if $(\{x_{i,j}\},\{a_I\})\in Z:=\mathscr{Z}(a)\subseteq\mathbb{A}^N$.

Denote by $W:=\mathscr{Z}(c)\subseteq\mathbb{A}^N$. Then an hypersurface in $\mathscr{L}_d(p_1^{m_1},\dots,p_r^{m_r})$ has multiplicity at least m_r+1 in p_r if and only if the corresponding polynomial $f\in W$.

Let $\mathbb{A}^M=(\{x_{i,j}\})=(\mathbb{A}^n)^r$ be the linear subspace of $\mathbb{A}^N$ parametrizing r-uples of points in $\mathbb{A}^n$ and let $\phi:\mathbb{A}^N\to\mathbb{A}^M$ be the natural projection.

We are reduced to prove that $Y:=\phi(Z\cap(\mathbb{A}^N\setminus W))$ contains an open subset of $\mathbb{A}^M$.

On the other hand for every choice of

$$p_1=(x_{1,1},\dots,x_{1,n}),\dots,p_{r-1}=(x_{r-1,1},\dots,x_{r-1,n}),$$

by applying [3, prop. 2.3] to the linear series $\mathscr{L}_d(p_1^{m_1},\dots,p_{r-1}^{m_{r-1}})$ we know that there exists an open subset $U_r\subseteq\mathbb{A}^n=\{(x_{r,1},\dots,x_{r,n})\}$ such that the set $(p_1,\dots,p_r-1)\times U_r$ is contained in Y. This implies that Y is not contained in any proper closed subset of $\mathbb{A}^M$, so that $\overline{Y}^{\mathbb{A}^M}=\mathbb{A}^M$.

By the proof of [8, Theorem 3.16] we have that Y contains an open subset of its closure in a $\mathbb{P}^M$ containing $\mathbb{A}^M$. Hence it contains an open subset of $\overline{Y}^{\mathbb{A}^M}=\mathbb{A}^M$, so that we are done.

$\square$

Lemma 4.3. *Let $n \geq 2$ and let $X = X^n_r$, with $r \geq n+1$. Let $D = dH - \sum_{j=1}^r m_j E_j$ be an effective Cartier divisor and let $i \in \{1, \ldots, r\}$. Then*

$$m_i < 0 \Longleftrightarrow E_i \subseteq Bs(|D|).$$

Proof.

$(\Rightarrow)$ $0 > m_i = (D, E_i) = (D \cdot e_i)$, where e_i is a line in the exceptional divisor E_i. This implies that $e_i \subseteq Bs(|D|)$, so that $E_i \subseteq Bs(|D|)$, because we can cover E_i with all the deformations of e_i.

$(\Leftarrow)$ Suppose, by contradiction, that $m_i \geq 0$. We can write

$$D = \sum dH - \sum_{j \in J_1} m_j E_j + \sum_{j \in J_2} n_j E_j,$$

where $J_2 = \{j \in \{1, \ldots, r\} \,:\, m_j < 0\}$, $J_1 = \{1, \ldots, r\} \setminus J_2$, $n_j = -m_j \geq 0$ for all $j \in J_2$, and, by assumption, $i \in J_1$.

Let us put

$$D' = \sum dH - \sum_{j \in J_1} m_j E_j.$$

Then $E_i \subseteq Bs(|D|)$ implies that $E_i \subseteq Bs(|D'|)$, i.e. $|D'| = E_i + |D' - E_i|$, so that $\dim |D'| = \dim |D' - E_i|$. This means that for every hypersurface $L \subseteq \mathbb{P}^n$ of degree d such that $\mathrm{mult}_{p_j} L \geq m_j$ for every $j \in J_1$, we have that $\mathrm{mult}_{p_i} L \geq m_i + 1$. This is a contradiction by lemma 4.2. □

Lemma 4.4. *Let $n \geq 2$ and let $X = X^n_r$, with $r \geq n+1$. Let $D = dH - \sum_{j=1}^r m_j E_j$ be an effective Cartier divisor, let $I \subseteq \{1, \ldots, r\}$ be a subset with $|I| = n$. Then $(n-1)d - \sum_{j \in I} m_j \geq 0$ if and only if the base locus of $|D|$ does not contain the strict transform of the hyperplane through all the points in the set $\{p_j \,:\, j \in I\}$.*

Proof. Let $I \subseteq \{1, \ldots, r\}$ be such that $|I| = n$. Let us denote by H_I the strict transform of the hyperplane through the points $\{p_j \,:\, j \in I\}$, so that $H_I \in |H - \sum_{j \in I} E_j|$. Let s_I be the standard Cremona transformation based on the the points $\{p_j \,:\, j \in I\}$ and on p_{j_0}, where we choose $j_0 \notin I$. Then s_I induces an action on $Pic(X)$ that lies in the Weyl group of X (see subsection 2.3). With a slight abuse of notation we denote it again by s_I.

Then we know that $Bs(|D|) \supseteq H_I$ if and only if $Bs(|s_I(D)|) \supseteq s_I(H_I)$. But

$$s_I(D) = (dn - \sum_{j \in I \cup \{j_0\}} m_j)H - \sum_{j \in I \cup \{j_0\}} \big((n-1)d - \sum_{\substack{k \in I \cup \{j_0\} \\ k \neq j}} m_k\big)E_j - \sum_{j \notin I \cup \{j_0\}} m_j E_j,$$

and $s_I(H_I) = E_{j_0}$.

Then, by Lemma 4.3, we have that $Bs(|s_I(D)|) \supseteq s_I(H_I) = E_{j_0}$ if and only if $(n-1)d - \sum_{k \in I} m_k < 0$ and the claim is proved. □

4.2. Case $r = n+1, n+2, n+3$

Lemma 4.5. *Let $n \geq 2$ and let $X = X^n_r$, with $r = n+1$ $n+2$, $n+3$. Then* $\mathrm{Mov}(X) \subseteq \mathrm{Eff}_{\mathbb{R}}(X) \cap \mathrm{Eff}_{\mathbb{R}}(X)^{\vee}$.

Proof. Let W be the Weyl group of X and let $E_r \in \mathrm{Pic}(X)$ be the class of an irreducible exceptional divisor. By [4, Theorem 2.7] we know that $\mathrm{Eff}_{\mathbb{R}}(X)$ is generated as a cone by the divisors in the orbit $W \cdot E_r$, so that $D \in \mathrm{Eff}_{\mathbb{R}}(X)^{\vee}$ if and only if $(D, w(E_r)) \geq 0$ for all $w \in W$.

Now take a Cartier divisor $D \in \mathrm{Mov}(X)$. Then $D \in \mathrm{Eff}_{\mathbb{R}}(X)$ because D is effective. Suppose, by contradiction, that $D \notin \mathrm{Eff}_{\mathbb{R}}(X)^{\vee}$. Then there exists $w \in W$ such that $(D, w(E_r)) < 0$. But $(D, w(E_r)) = (w^{-1}(D), E_r)$ because the Weyl group is a group of isometries with respect to the pairing $(\cdot, \cdot)$. By definition $(w^{-1}(D), E_r) = (w^{-1}(D) \cdot e_r)$, where e_r is a general line in the exceptional divisor E_r. Then we have that $(w^{-1}(D) \cdot e_r) < 0$, which implies that a general line in E_r is contained in the base locus of $w^{-1}(D)$, so that $E_r \subseteq Bs(|(w^{-1}(D)|)$, and $w^{-1}(D)$ is not movable. But D is movable and the Weyl group fixes the movable cone. Hence $w^{-1}(D)$ is also movable and we get a contradiction. □

We will use the following theorem to show the reverse inclusion of lemma 4.5. Note that the same result immediately follows from [4, Theorem 2.7] by using some representation theory (classification of minuscule representations, cf. [4, Remark 2.8]). On the other hand in our proof we just use elementary geometric facts.

Theorem 4.6. *Let $n \geq 2$ and let $X = X^n_r$, with $r = n+1$, $n+2$, $n+3$. Let W be the Weyl group of X and let E_r be an exceptional divisor. Then the Weyl orbit*

$$W \cdot E_r = \{D \in Eff(X) \,:\, deg(D) = 1\}.$$

Proof. Note that $\deg E_r = 1$ and the degree is W-invariant, so that every divisor in the orbit $W \cdot E_r$ has degree 1. Let us show that every divisor of degree 1 is in the orbit of E_r.

We denote by H the pullback of an hyperplane section on X, by $E_1, \ldots, E_r$ the irreducible exceptional divisors.

Claim. Suppose $D = dH - \sum_{j=1}^{r} m_j E_j \in \mathrm{Pic}(X)$ is effective. Then there exists $w \in W$ such that $w(D) = fH - \sum_{j=1}^{r} q_j E_j$, verifies $q_1 \geq q_2 \geq \cdots \geq q_r$ and $f(n-1) - \sum_{j=1}^{n+1} q_j \geq 0$.

Proof of the claim. First of all, we may assume, after reordering, that $m_1 \geq m_2 \geq \ldots \geq m_r$. If $d(n-1) - \sum_{j=1}^{n+1} m_j \geq 0$, we are done.

If this is not the case, let us denote by s_0 the standard Cremona transformation based on $E_1,\dots,E_{n+1}$, so that

$$s_0(D) = (nd - \sum_{j=1}^{n+1} m_j)H - \sum_{i=1}^{n+1} \big((n-1)d - \sum_{\substack{k\leq n+1\\ k\neq j}} m_k\big)E_j - \sum_{j\geq n+2} m_j E_j.$$

Note that, by hypothesis, $nd - \sum_{j=1}^{n+1} m_j < d$ so that $\mathrm{ord}_H s_0(D) < ord_H D$.

Now we reorder the $E_j's$ with respect to the new coefficients and we repeat the procedure until we obtain the required inequality.

Note that we will achieve it after a finite number of steps, because otherwise we would construct an element $s \in W$ such that $\mathrm{ord}_H s(D) < 0$, so that $s(D)$ would not be effective. But this is not possible because D is effective and the Weyl group fixes the effective cone. This proves the claim.

Now let $D \in \mathrm{Pic}(X)$ be an effective divisor of degree 1. Then there exists $w \in W$ such that $w(D) = fH - \sum q_j E_j$ satisfies the following inequalities:

1. $f \geq 0$;
2. $f \geq q_1 \geq q_2 \geq \cdots \geq q_r$;
3. $f(n-1) - \sum_{j=1}^{n+1} q_j \geq 0$;
4. $f(n+1) = 1 + \sum_{j=1}^{r} q_j$.

In fact 1, 2 and 3 follow by the claim and by the effectiveness of $w(D)$. As for the forth inequality, it is verified because $\deg w(D) = \deg D$, so that

$$1 = \deg w(D) = \frac{1}{n-1}(w(D), -K_X) = f(n+1) - \sum_{j=1}^{r} q_j.$$

If $r = n+1$, from these conditions we obtain that $2f \leq 1$, so that $f = 0$. Then $\sum_{j=1}^{r} q_j = -1$ and $0 \geq q_1 \geq q_2 \geq \cdots \geq q_r$. Hence $w(D) = E_r$, so that $D \in W \cdot E_r$.

If $r = n+2$, we get that $2f \leq 1 + q_{n+2} \leq 1 + f$, so that $f \leq 1$. Moreover if $f = 1$, then $q_{n+2} = 1$, so that $q_1 = \cdots = q_{n+2} = 1$, and $w(D)$ is not effective. Hence we must have $f = 0$, so that, as in the previous case, we get that $w(D) = E_{n+2} = E_r$.

If $r = n+3$, then we get $2f \leq q_{n+2} + q_{n+3} + 1$. Then, if $f > 0$ we have that $q_1 = \cdots = q_{n+2} = f$, and $q_{n+2} \in \{f, f+1\}$. In any case we get that $w(D)$ is not effective. Thus we must have $f = 0$, so that, as before, $w(D) = E_{n+3} = E_r$, and we are done. $\square$

Theorem 4.7. *Let $n \geq 2$ and let $X = X_r^n$, with $r = n+1,\ n+2,\ n+3$. Then* $\mathrm{Mov}(X) = \mathrm{Eff}_{\mathbb{R}}(X) \cap \mathrm{Eff}_{\mathbb{R}}(X)^{\vee}$.

Proof. By Theorem 4.5 we know that one inclusion is always verified. Then it suffices to prove that $\mathrm{Eff}_{\mathbb{R}}(X) \cap \mathrm{Eff}_{\mathbb{R}}(X)^{\vee} \subseteq \mathrm{Mov}(X)$.

By [4, Theorem 2.7] we have that $\mathrm{Eff}_{\mathbb{R}}(X)$ is generated as a cone by the divisors in the Weyl orbit of an exceptional divisor E_r and $\mathrm{Eff}(X)$ is generated as a semigroup by the effective divisors of degree 1. Hence, by Theorem 4.6, $\mathrm{Eff}(X)$ is generated, as a semigroup, by the divisors in $W \cdot E_r$.

Moreover, note that if F is an irreducible fixed divisor, then F cannot be written as a sum of two Cartier divisors in $\mathrm{Eff}(X)$, so that we must have $F \in W \cdot E_r$.

Now let $D \in \mathrm{Eff}_{\mathbb{R}}(X) \cap \mathrm{Eff}_{\mathbb{R}}(X)^{\vee}$ be a Cartier divisor. Then $(D, w(E_r)) \geq 0$ for every $w \in W$, so that $(w(D), E_r) \geq 0$ for every $w \in W$.

Hence, as D is effective and W fixes the effective cone, thanks to Lemma 4.3 we get that, for every $w \in W$, $E_r \not\subseteq Bs(|w(D)|)$, which in turn implies that $w^{-1}(E_r) \not\subseteq Bs(|D|)$.

This shows that the linear series $|D|$ does not contain in its base locus any divisor in the orbit $W \cdot E_r$. Therefore it is movable, because we have proved that every irreducible fixed divisor is in the W-orbit of E_r. □

4.3. Case $r = n+1, n+2$

Lemma 4.8. *Let $n \geq 2$ and let $X = X^n_r$, with $r \in \{n+1, n+2\}$. Let*

- $\mathscr{E} = \{E_i \,:\, i \in \{1, \dots, r\}\}$,
- $\mathscr{H} = \{H - \sum_{j \in I} E_j \,:\, I \subseteq \{1, \dots, r\}, |I| = n\}$.

Then $\mathrm{Eff}(X) \subseteq \mathrm{Pic}(X)$ *is generated as a semigroup by the classes of divisors in* $\mathscr{E}$ *and* $\mathscr{H}$.

Proof. Let us denote by $\{p_1, \dots, p_r\}$ the very general points blown-up on $\mathbb{P}^n$.

By [4, Theorem 2.7] we have that the semigroup of effective divisors is generated by effective divisors of degree 1. Let $D = dH - \sum m_j E_j$ be a class in $\mathrm{Pic}(X)$. Then D has degree 1 if and only if

$$1 = \frac{1}{n-1}(D, -K_X) = d(n+1) - \sum_{j=1}^{r} m_j.$$

Note that the effectiveness of D implies that $d \geq 0$.

If $d = 0$, then $m_j \leq 0$ for every j because D is effective and $\sum_{j=1}^{r} m_j = 1$ because of the above formula. Then in this case $D \in \mathscr{E}$.

If $d > 0$ then the effectiveness of D implies that $m_j \leq d$ for every $j \in \{1, \dots, r\}$.

If $r = n+1$, then the above formula implies that

$$D = mH + \sum_{j \neq i} mE_j - (m-1)E_i,$$

for some $i \in \{1, \ldots, n+1\}$ and for some $m \in \mathbb{N}$.

If $m = 1$ then $D \in \mathscr{H}$. Suppose then $m \geq 2$ and define $D' = mH + \sum_{j \neq i} mE_j$. Then by Lemma 4.4 we have that $Bs(|D'|)$ contains H_i with multiplicity m, so that D' is the fixed divisor given by the strict transform of the hyperplane through $p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_{n+1}$, with multiplicity m. Thus, thanks to the generality of the point, this hyperplane does not contain p_i, so that, equivalently, $D' - E_i$ is not effective. But $m \geq 2$ implies that $D \leq D' - E_i$, whence D is also not effective. Therefore we must have $m = 1$ and the theorem follows.

Let $r = n+2$. Then we can consider a rational normal curve of degree n, say $C \subseteq \mathbb{P}^n$, such that C passes through $p_1, \ldots, p_{n+2}$ and $C \nsubseteq \mathrm{Supp}(\mu_*(D))$. In fact we know that for every choice of $n+3$ points in $\mathbb{P}^n$ there exists a rational normal curve of degree n passing through all of them. Then we have that the strict transform of C is not contained in $\mathrm{Supp}(D)$.

This implies that $(D \cdot (nl - e_1 - \cdots - e_{n+2})) \geq 0$, where we denote by l the pullback of a line in X and by e_j a general line in the exceptional divisor E_j, for every $j \in \{1, \ldots, n+2\}$. Thus

$$0 \leq ((dH - \sum_{j=1}^{n+2} m_j E_j) \cdot (nl - \sum_{j=1}^{n+2} e_j)) = nd - \sum_{j=1}^{n+2} m_j.$$

Then, using the fact that D has degree 1, we obtain that $1 \geq d(n+1) - dn = d$. Therefore $D \in \mathscr{E}$ if $d = 0$ and $D \in \mathscr{H}$ if $d = 1$, and we are done. $\square$

Thanks to corollary 4.7 we can use the previous lemma to give explicit equations of the movable cone of X^n_{n+1} and X^n_{n+2}. This generalizes Prop. 3.2.

Theorem 4.9. *Let $n \geq 2$ and let $X = X^n_r$, with $r \geq n+1$. Let us denote by $(d, m_1, \ldots, m_r)$ the coordinates in $N^1(X)$ with respect to the standard basis. Consider the following inequalities:*

1. $m_j \geq 0$, *for all* $j \in \{1, \ldots, r\}$;
2. $(n-1)d - \sum_{j \in I} m_j \geq 0$, *for all* $I \subseteq \{1, \ldots, r\}$, *with* $|I| = n$;
3. $d \geq m_j$, *for all* $j \in \{1, \ldots, r\}$;
4. $nd - \sum_{j=1}^{r} m_j \geq 0$.

Let $M_1(X) \subseteq N^1(X)$ be the cone defined by inequalities 1,2,3, and let $M_2(X) \subseteq N^1(X)$ be the cone defined by inequalities 1,2,3,4. Then, $M_1(X) = M_2(X) = \mathrm{Mov}(X)$ if $r = n+1$ and $M_2(X) = \mathrm{Mov}(X)$ if $r = n+2$.

Proof. By Corollary 4.7 we know that $\mathrm{Mov}(X) = \mathrm{Eff}_{\mathbb{R}}(X) \cap \mathrm{Eff}_{\mathbb{R}}(X)^{\vee}$.

Let $r \in \{n+1, n+2\}$ and let

$$D = dH - \sum_{j=1}^{r} m_j E_j$$

be a movable Cartier divisor. Then $d \geq m_j$, for all $j \in \{1, ..., r\}$ because D is effective.

Moreover, as $D \in Eff_{\mathbb{R}}(X)^{\vee}$, we have that for every $j \in \{1, \ldots, r\}$

$$0 \leq (D, E_j) = m_j,$$

and, for every $I \subseteq \{1, \ldots, r\}$, with $|I| = n$,

$$0 \leq (D, H - \sum_{j \in I} E_j) = (n-1)d - \sum_{j \in I} m_j.$$

Thus D satisfies inequalities 1,2,3. Hence $\mathrm{Mov}(X) \subseteq M_1(X)$.

Now let $D \in M_2(X)$. Then D is effective by [4, Lemma 4.24]. By Lemma 4.8 we know that the divisors in $\mathscr{H}$ and $\mathscr{E}$ generate $\mathrm{Eff}(X)$ as a semigroup, so that they also generate $\mathrm{Eff}_{\mathbb{R}}(X)$ as a cone. By the given inequalities we have that the intersection of D with all the divisors in $\mathscr{H}$ and $\mathscr{E}$ is non negative. Thus $D \in \mathrm{Eff}_{\mathbb{R}}(X)^{\vee}$, so that $D \in \mathrm{Mov}(X)$. Hence $M_2(X) \subseteq \mathrm{Mov}(X)$.

Suppose now $r = n+1$ and let $D = dH - \sum_{j=1}^{n+1} m_j E_j$ be a Cartier divisor in $M_1(X)$. Then

$$nd - \sum_{j=1}^{n+1} m_j = (n-1)d - \sum_{j \neq 1} m_j E_j + d - m_1 \geq (n-1)d - \sum_{j \neq 1} m_j E_j \geq 0.$$

Thus $D \in M_2(X)$. Hence we get that $M_1(X) = M_2(X) = \mathrm{Mov}(X)$ if $r = n+1$.

Finally take $r = n+2$. We have to prove that $\mathrm{Mov}(X) \subseteq M_2(X)$. Note that we have already proved that $\mathrm{Mov}(X) \subseteq M_1(X)$, so that it just remains to show that movable divisors satisfy inequality 4.

Let $D \in \mathrm{Mov}(X)$ be a Cartier divisor. Then $D \in \mathrm{Eff}(X)$. Hence, by Lemma 4.8, we can write

$$D = \sum_{i=1}^{r} a_i E_i + \sum_{|I|=n} b_I (H - \sum_{j \in I} E_j),$$

with $a_i, b_I \in \mathbb{N}$. Then

$$d = \mathrm{ord}_H D = \sum_{|I|=n} b_I;$$

$$\sum_{j=1}^{r} m_j = -\sum_{j=1}^{r} \mathrm{ord}_{E_j} D = -a_i + \sum_{|I|=n} b_I n \leq \sum_{|I|=n} b_I n = dn.$$

Thus inequality 4 is verified and the theorem is proved. □

Remark 4.10. Note that in Theorem 3.14 by toric methods we computed the marked polynomial $P^n_{n+1}(x,y)$. From it we can easily compute the number of facets of the movable cone that correspond either to fiber type contractions, i.e. the coefficient of xy, or to divisorial contractions, i.e. the coefficient of x. Thus, for every $n>2$ and $r=n+1$, the number of divisorial contractions is $2(n+1)$, while the fiber type contractions are $n+1$. Hence the toric count arising by theorem 3.14, $3(n+1)$, matches the number of facets of the movable cone, that are given explicitly in theorem 4.9 by the boundary hyperplanes associated to conditions marked with $1,2$ and 3 in the theorem.

Contractions given by divisors in the boundary of $\mathrm{Eff}(X)$ i.e. associated to conditions of type 3 (and 4, for $r=n+2$) are of fiber type (they are projections to lower dimensional MDS) while divisors in the interior of the effective cone, i.e. such that equality is attained in one condition of type 1 and 2, are associated to divisorial contractions.

We also remark that in dimension 2 conditions of type 3 (and 4) follow from those of type 1 and 2 and in fact that are no fiber type elementary contractions (see section 2.4).

4.4. Case $r=n+3$

In the following proposition we give generators of $\mathrm{Eff}(X^n_{n+3})$ as a semigroup. Unfortunately we are not able to use them to give explicit equations of the movable cone in this case.

Proposition 4.11. *Let $n\geq 2$ and let $X=X^n_{n+3}$.*

Then $\mathrm{Eff}(X)\subseteq \mathrm{Pic}(X)$ *is generated as a semigroup by the classes of divisors in*

$$\mathscr{A}=\{dH-\sum_{j\in I}dE_j-\sum_{j\notin I}(d-1)E_j\ :\ |I|=n+2-2d,\ 0\leq d\leq\frac{n+2}{2}\}.$$

Proof. By Lemma 4.6 we have that $W\cdot E_1=\{D\in\mathrm{Eff}(X):\deg D=1\}$. We will show that $\mathscr{A}=W\cdot E_1=\{D\in\mathrm{Eff}(X):\deg D=1\}$, so that the lemma follows by [4, Theorem 2.7].

We begin by showing that $\{D\in\mathrm{Eff}(X):\deg D=1\}\subseteq\mathscr{A}$:

Write $D=dH-\sum m_jE_j$. Up to a permutation of the indices we can suppose that $m_1\geq\ldots\geq m_3$. Note that

$$1=\deg(D)=d(n+1)-\sum_{j=1}^{r}m_j.$$

Moreover, as in the proof of Lemma 4.8 we know that D intersects non negatively the strict transform of a rational normal curve through $p_1,\ldots,p_{n+2}$,

so that
$$nd - \sum_{j=1}^{n+2} m_j \geq 0.$$

Then $0 \leq nd - \sum_{j=1}^{n+2} m_j = 1 + m_{n+3} - d$, so that $m_{n+3} \geq d-1$. This implies that $m_j \in \{d-1, d\}$ for every $j = 1, \ldots, n+3$. In other words there exists $k \in \{0, \ldots, n+3\}$ such that
$$D = dH - \sum_{j \leq k} dE_j - \sum_{j=k+1}^{n+3} (d-1)E_j.$$

Hence we have that $1 = \deg(D) = d(n+1) - kd - \big((n+3)-k\big)(d-1)$, so that $k = n+2-2d$.

On the other hand we must have $k \geq 0$, which implies that $d \leq \frac{n+2}{2}$. Therefore $D \in \mathscr{A}$. In order to conclude we prove that $\mathscr{A} \subseteq W \cdot E_1$.

As W contains all permutations of exceptional divisors it suffices to consider divisors of the form
$$A_d := dH - \sum_{j \leq n+2-2d} dE_j - \sum_{j=n+3-2d}^{n+3} (d-1)E_j,$$

with $0 \leq d \leq \frac{n+2}{2}$. We will show by induction on d that all these divisors are in the orbit of E_1.

If $d = 0$, then $A_0 = E_{n+3} \subseteq W \cdot E_1$ because W contains all permutations of exceptional divisor.

Now suppose $1 \leq d \leq \frac{n+2}{2}$ and suppose, by induction, that $A_{d-1} \in W \cdot E_1$. Then
$$L := (d-1)H - \sum_{j=1}^{2d-1} (d-2)E_j - \sum_{j=2d}^{n+3} (d-1)E_j \in W \cdot E_1.$$

Let s_0 be the standard Cremona transformation based on $E_1, \ldots, E_{n+1}$. We will show that A_d can be obtained from $s_0(L)$ by a permutation of the exceptional divisors. Note that $d \leq \frac{n+2}{2}$ implies that $2d-1 \leq n+1$. Then
$$\mathrm{ord}_H s_0(L) = (d-1)n - (d-2)(2d-1) - (d-1)(n+1-2d+1) = d.$$

If $1 \leq j \leq 2d-1$, then
$$\mathrm{ord}_{E_j} s_0(L) = (n-1)(d-1) - (d-2)(2d-2) - (d-1)(n+1-2d+1) = d-1.$$

If $2d \leq j \leq n+1$, then
$$\mathrm{ord}_{E_j} s_0(L) = (n-1)(d-1) - (d-2)(2d-1) - (d-1)(n+1-2d) = d.$$

Finally, if $n+2 \leq j \leq n+3$, then $\mathrm{ord}_{E_j} s_0(L) = d$.

Therefore we get A_d by applying a suitable permutation to $s_0(L)$. $\square$

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