LE MATEMATICHE
Vol. LXVII (2012) – Fasc. II, pp. 39–43
doi: 10.4418/2012.67.2.4

CURVES ON A QUADRIC SURFACE OVER $\overline{\mathbb{F}}_p$

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Here (following a paper by Giuffrida, Maggioni and Re) we study the existence of curves $C \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(a,b)|$ defined over small fields and with $h^0(C,\mathcal{O}_C(x,y))\cdot h^1(C,\mathcal{O}_C(x,y)) = 0$ for all integers x,y such that $x \geq a$ and $y \leq -2$.

1. Introduction

Let $\overline{\mathbb{F}}_p$ be the algebraic closure of the finite field $\mathbb{F}_p$.

In this paper we prove the following result.

Theorem 1.1. *Fix a prime p and integers $a \geq 1$, $b \geq 1$.*
Let $C \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(a,b)|(\overline{\mathbb{F}}_p)$ be an integral curve. Then there are infinitely many integers $n > 0$ such that $\mathcal{O}_C(nb,-na) \cong \mathcal{O}_C$ and hence $h^0(C,\mathcal{O}_C(nb,-na)) = 1$ and $h^1(C,\mathcal{O}_C(nb,-na)) = (a-1)(b-1)$.

In [3], §2, it is proved that a statement like the opposite of Theorem 1.1 (i.e. $h^0(C,\mathcal{O}_C(x,y))\cdot h^1(C,\mathcal{O}_C(x,y)) = 0$ for all $x \geq a$ and $y \leq -2$) is equivalent to the case $n = m = 1$ of [3], Problem 1.3. Hence the case $n = m = 1$ of [3], Problem 1.3, has a negative answer over the algebraic closure of a finite field. In characteristic zero [3], Problem 1.3, is equivalent to two other very interesting problems ([3], Problems 1.2 and 1.3). In positive characteristic one has to use

Entrato in redazione: 14 novembre 2011
AMS 2010 Subject Classification: 14N05, 14H50, 14J26.
Keywords: Quadric surface, Bihomogeneous polynomial, Curve in a quadric surface.
The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

divided power instead of differential operators in the formulation of [3], Problem 1.2. We do not claim that Theorem 1.1 is related to either Problem 1.1 or Problem 1.2 of [3].

Conjecture 1.2. Fix integers $a \geq 1$ and $b \geq 1$. Is there a smooth curve $C \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(a,b)|$ defined over $\mathbb{Q}$ and such that for all $x \geq a$, $y \leq -2$ either $h^0(C,\mathcal{O}_C(x,y)) = 0$ (case $ay+bx \leq ab-a-b$) or $h^1(C,\mathcal{O}_C(x,y)) = 0$ (case $ay+bx \geq ab-a-b$)?

In section 2 we prove the case $a = b = 2$ of the conjecture (see Proposition 2.2), the case $a = 1$ or $b = 1$ being trivial (see Proposition 2.3).

Theorem 1.3. *Fix integers $a > 0$, $b > 0$, $x \geq a$ and $y \leq -2$. Then there is an integer $p_0 = p_0(a,b,x,y)$ such that for all primes $p \geq p_0$ there is a smooth $C \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(a,b)|$ defined over $\mathbb{F}_p$ and such that either $h^0(C,\mathcal{O}_C(x,y)) = 0$ (case $ay+bx \leq ab-a-b$) or $h^1(C,\mathcal{O}_C(x,y)) = 0$ (case $ay+bx \geq ab-a-b$).*

Question 1.4. Fix a prime p and integers a,b,x,y such that $x \geq a > 0$, $b > 0$ and $y \leq -2$. Which is the minimal p-power q (assuming it exists) such that there a smooth $C \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(a,b)|$ defined over $\mathbb{F}_q$ and such that either $h^0(C,\mathcal{O}_C(x,y)) = 0$ (case $ay+bx \leq ab-a-b$) or $h^1(C,\mathcal{O}_C(x,y)) = 0$ (case $ay+bx \geq ab-a-b$)?

The trouble in Theorem 1.1, Conjecture 1.2 and in all in related questions comes from the fact that we consider problems which are not open, i.e. the set of their solutions does not form an open subset in a certain parameter space: the set of their solutions is only a countable intersection of Zariski open subsets, each of them corresponding to a pair $(x,y) \in \mathbb{Z}^2$. If we fix (x,y), then the statement of [3], Theorem 2.1, is strong enough to get the following result.

Proposition 1.5. *Fix integers $a \geq 1$, $b \geq 1$, $x \geq a$ and $y \leq -2$. Then there exists a smooth curve $C \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(a,b)|$ defined over $\mathbb{Q}$ and such that either $h^0(C,\mathcal{O}_C(x,y)) = 0$ (case $ay+bx \leq ab-a-b$) or $h^1(C,\mathcal{O}_C(x,y)) = 0$ (case $ay+bx \geq ab-a-b$).*

Now we fix the integers $a \geq 2$, $b \geq 2$ and fix any field K such that $\sharp(K) \geq \max\{a-1,b-1\}$ (for instance, any infinite field is allowed). Fix a distinct points $P_1,\dots,P_a \in \mathbb{P}^1$ and b distinct points $Q_1,\dots,Q_b \in \mathbb{P}^1$. Set $L_i := \mathbb{P}^1 \times \{P_i\} \subset \mathbb{P}^1\times\mathbb{P}^1$, $1 \leq i \leq a$, and $M_j := \{Q_j\}\times\mathbb{P}^1 \subset \mathbb{P}^1\times\mathbb{P}^1$, $1 \leq j \leq b$. Each L_i (resp. M_j) is a divisor of type $(1,0)$ (resp. $(0,1)$) of $\mathbb{P}^1\times\mathbb{P}^1$ defined over K. Set $\Sigma(a,b) := \{L_i \cap M_j\}_{1\leq i\leq a, 1\leq j\leq b}$. Notice that $\Sigma(a,b) \subseteq \mathbb{P}^1\times\mathbb{P}^1(K)$, $\sharp(\Sigma(a,b)) = ab$ and that $\Sigma(a,b)$ is the complete intersection of a curve of type $(a,0)$ and a curve of type $(b,0)$. Consider the following condition $\$(\Sigma(a,b))$ that a field K

may have: there is an integral curve $C \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(a,b)|(K)$ such that $\Sigma(a,b) \subset C_{reg}$. If K has property $\$(\Sigma(a,b))$, let $\Sigma(a,b)_K$ denote the set of all curves $C \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(a,b)|(K)$ such that $\Sigma(a,b) \subset C_{reg}$.

Proposition 1.6. *Fix integers $a \geq 2$, $b \geq 2$ and a field K satisfying $\$(\Sigma(a,b))$. Fix any $C \in \Sigma(a,b)_K$ and any integer $m > 0$. We have $h^0(C,\mathcal{O}_C(x,y)) = 0$ if $a \leq x \leq -1+ma$ and $y \leq b-1-mb$, while and $h^1(C,\mathcal{O}_C(u,v)) = 0$ if $u \geq m(a+1)+1$ and $v \geq 1-mb$. For every $t \in \{0,\dots,b-1\}$ we have $h^0(C,\mathcal{O}_C(ma,-mb+t)) = t+1$ and $h^1(C,\mathcal{O}_C(ma,-mb+t)) = ab-a-b+t-at > 0$.*

Over $\mathbb{C}$ the same result is proved in [3], Lemma 2.5 and Corollary 2.6. Since the proofs in [3] are easily adapted to the general case, we omit the proof of Proposition 1.6.

2. Proofs and related results

Set $Q := \mathbb{P}^1 \times \mathbb{P}^1$.

Proof of Proposition 1.5. Let $\mathcal{S}$ be the set of all smooth complex curves $C \in |\mathcal{O}_Q(a,b)|$ such that either $h^0(C,\mathcal{O}_C(x,y)) = 0$ (case $ay+bx \leq ab-a-b$) or $h^1(C,\mathcal{O}_C(x,y)) = 0$ (case $ay+bx \geq ab-a-b$). By [3], Theorem 2.1 , $\mathcal{S}$ is a non-empty Zariski open subset of the projective space $|\mathcal{O}_Q(a,b)|(\mathbb{C})$. Since $|\mathcal{O}_Q(a,b)|(\mathbb{Q})$ is Zariski dense in $|\mathcal{O}_Q(a,b)|(\mathbb{C})$, there is $C \in \mathcal{S}$ defined over $\mathbb{Q}$. $\square$

Proof of Theorem 1.1. Let $f \in H^0(Q,\mathcal{O}_Q(a,b))$ be an equation of C. Since f is defined over $\overline{\mathbb{F}}_p$, it is defined over a finite extension, $\mathbb{F}_q$, of $\mathbb{F}_p$. Hence C and $\mathcal{O}_C(1,-1)$ are defined over $\mathbb{F}_q$. We have $\deg(\mathcal{O}_C(b,-a)) = ab-ba = 0$. Since $\mathrm{Pic}^0(C)$ is an algebraic group, $\mathrm{Pic}^0(C)(\mathbb{F}_q)$ is a finite abelian group. Hence $\mathcal{O}_C(b,-a)$ has finite order, i.e. there is an integer $c > 0$ such that $\mathcal{O}_C(cb,-ca) \cong \mathcal{O}_C$. Hence $h^0(C,\mathcal{O}_C(nb,-na)) = 1$ and $h^1(C,\mathcal{O}_C(nb,-na)) = p_a(C) = (a-1)(b-1)$ for any $n = uc$, $u \in \mathbb{Z}$. $\square$

Proof of Theorem 1.3. Take C as in the statement of Proposition 1.5 and let $f \in H^0(Q,\mathcal{O}_Q(a,b))$ be a bihomogeneous polynomial with C as its zero-locus. By assumption f has rational coefficients. Clearing denominators we may assume the f has integer coefficients. Hence we may take f modulo any prime p (call it f_p). For large p the curve $C_p := \{f_p = 0\}$ is smooth. Hence there is a finite set S of primes such that the polynomial f induces a flat family $\{C_\alpha\}_{\alpha\in\mathrm{Spec}\,(\mathbb{Z})\setminus S}$ of smooth projective curves over $\mathrm{Spec}(\mathbb{Z}) \setminus S$, each of them being a divisor of Q with bidegree (a,b). Hence we get a line bundle $\{\mathcal{O}_{C_\alpha}(x,y)\}_{\alpha\in\mathrm{Spec}\,(\mathbb{Z})\setminus S}$ over the flat family $\{C_\alpha\}_{\alpha\in\mathrm{Spec}\,(\mathbb{Z})\setminus S}$. Since a free module is flat, the sheaf $\{\mathcal{O}_{C_\alpha}(x,y)\}_{\alpha\in\mathrm{Spec}\,(\mathbb{Z})\setminus S}$ is flat over the family $\{C_\alpha\}_{\alpha\in\mathrm{Spec}\,(\mathbb{Z})\setminus S}$. The fiber of this flat family of curves over the generic point $(0) \in \mathrm{Spec}(\mathbb{Z}) \setminus S$ is the curve C. Since $h^0(C, \mathcal{O}_C(x,y)) \cdot h^1(C, \mathcal{O}_C(x,y)) = 0$, the semicontinuity theorem for cohomology ([1], Theorem III.12.8) gives $h^0(C, \mathcal{O}_{C_p}(x,y)) \cdot h^1(C, \mathcal{O}_{C_p}(x,y)) = 0$ for all primes $p \gg 0$. $\square$

Remark 2.1. The case $a = b = 2$ of Theorem 1.1 is true for a non-integral curve C for any base field ([3], first part of Remark 2.2).

Proposition 2.2. *There is a smooth $C \in |\mathcal{O}_Q(2,2)|$ defined over $\mathbb{Q}$ and such that $h^0(C, \mathcal{O}_C(x,y)) \cdot h^1(C, \mathcal{O}_C(x,y)) = 0$ for all $(x,y) \in \mathbb{Z}^2 \setminus \{(0,0)\}$. Moreover, we may take as C any embedding $E \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ of bidegree $(2,2)$ defined over $\mathbb{Q}$ of any elliptic curve defined over $\mathbb{Q}$ and with rank > 0.*

Proof. Let E be a smooth elliptic curve defined over $\mathbb{Q}$ and such that the abelian group $\mathrm{Pic}^0(E)(\mathbb{Q})$ has rank > 0. Fix any $R \in \mathrm{Pic}^2(E)(\mathbb{Q})$ and call $u : E \to \mathbb{P}^1$ the degree 2 morphism induced by the complete linear system $|R|$. Fix any non-torsion $M \in \mathrm{Pic}^0(E)(\mathbb{Q})$. Let $v : E \to \mathbb{P}^1$ be the degree 2 morphism induced by $|R \otimes M|$. Since $M \neq \mathcal{O}_E$, the morphism $j = (u,v) : E \to Q$ is an embedding. Since M is not torsion, no line bundle $\mathcal{O}_{j(E)}(n,-n)$, $n \in \mathbb{Z} \setminus \{0\}$, is trivial. Hence we may take $C = j(E)$. $\square$

Proposition 2.3. *Fix a field K and assume either $a = 1$ or $b = 1$. There is a smooth curve $C \in |\mathcal{O}_Q(a,b)|$ defined over K. Fix any such C. Then for every $(x,y) \in \mathbb{Z}^2$ either $h^0(C, \mathcal{O}_C(x,y)) = 0$ (case $ay + bx < 0$) or $h^1(C, \mathcal{O}_C(x,y)) = 0$ (case $ay + bx \geq -1$).*

Proof. Since either $a = 1$ or $b = 1$, any smooth $C \in |\mathcal{O}_Q(a,b)|$ is geometrically integral and rational. For every $R \in \mathrm{Pic}(\mathbb{P}^1)$ either $h^0(\mathbb{P}^1, R) = 0$ (case $\deg(R) < 0$) or $h^1(\mathbb{P}^1, R) = 0$ (case $\deg(R) \geq -1$). Hence it is sufficient to check the existence of anembedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ with bidegree (a,b) defined over K. It is sufficient to do it when $a = 1$. If $b = 0$, then take the an isomorphism $\mathbb{P}^1 \to \mathbb{P}^1 \times \{o\} \subset Q$, $o \in \mathbb{P}^1(K)$. If $b > 0$, then take the identity map for the first component and any degree b surjection $\mathbb{P}^1 \to \mathbb{P}^1$ defined over K for the other surjection. $\square$

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