LE MATEMATICHE
Vol. LXVII (2012) – Fasc. II, pp. 107–118
doi: 10.4418/2012.67.2.9

GENERALIZED FRACTIONAL INTEGRATION OF THE $\overline{H}$ FUNCTION

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A significantly large number of earlier works on the subject of fractional calculus give interesting account of the theory and applications of fractional calculus operators in many different areas of mathematical analysis (such as ordinary and partial differential equations, integral equations, special functions, summation of series, et cetera). In the present paper, we study and develop the generalized fractional integral operators given by Saigo. First, we establish two Theorems that give the images of the product of $\overline{H}$-function and a general class of polynomials in Saigo operators. On account of the general nature of the Saigo operators, $\overline{H}$-function and a general class of polynomials a large number of new and known Images involving Riemann-Liouville and Erdélyi-Kober fractional integral operators and several special functions notably generalized Wright hypergeometric function, generalized Wright-Bessel function, the polylogarithm and Mittag-Leffler functions follow as special cases of our main findings.

1. Introduction

The fractional integral operator involving various special functions, have found significant importance and applications in various sub-field of applicable math-

Entrato in redazione: 6 gennaio 2012

AMS 2010 Subject Classification: 26A33, 33C45, 33C60, 33C70.

Keywords: Fractional integral operators by Saigo, $\overline{H}$-function; Riemann-Liouville, Erdélyi-Kober, Generalized Wright hypergeometric function, Generalized Wright-Bessel function; The polylogarithm function; Mittag-Leffler functions.

ematical analysis. Since last four decades, a number of workers like Love [19], McBride [21], Kalla [9,10], Kalla and Saxena [11, 12], Saxena et al. [31], Saigo [26, 27, 28], Kilbas [13], Kilbas and Sebastian [15] and Kiryakova [17,18], etc. have studied in depth, the properties, applications and different extensions of various hypergeometric operators of fractional integration. A detailed account of such operators along with their properties and applications can be found in the research monographs by Smako, Kilbas and Marichev [30], Miller and Ross [22], Kiryakova [17,18], Kilbas, Srivastava and Trujillo [16] and Debnath and Bhatta [3].

A useful generalization of the hypergeometric fractional integrals, including the Saigo operators [26, 27, 28], has been introduced by Marichev [20] [see details in Samko et al.[30]] and also see Kilbas and Saigo [see [14], p.258] as follows: Let α, β, η be complex numbers and $x > 0$, than the generalized fractional integral operators [the Saigo operators [26]] involving Gaussian hypergeometric function are defined by the following equations:

$$\begin{aligned}&\left(I_{0^+}^{\alpha,\beta,\eta} f\right)(x)\\ &\quad= \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{t}{x}\right) f(t)\,dt,\\ &\qquad\qquad (\Re(\alpha) > 0), \quad (1)\end{aligned}$$

and

$$\begin{aligned}&\left(I_{-}^{\alpha,\beta,\eta} f\right)(x)\\ &\quad= \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} t^{-\alpha-\beta} {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{x}{t}\right) f(t)\,dt,\\ &\qquad\qquad (\Re(\alpha) > 0), \quad (2)\end{aligned}$$

where ${}_2F_1(\cdot)$ is the Gaussian hypergeometric function defined by:

$${}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}. \quad (3)$$

When $\beta = -\alpha$, the above equations (1) and (2) reduce to the following classical Riemann-Liouville fractional integral operator (see Samko et al., [30], p.94, Eqns. (5.1), (5.3)):

$$\left(I_{0^+}^{\alpha,-\alpha,\eta} f\right)(x) = \left(I_{0^+}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\,dt, \quad (x>0) \quad (4)$$

and

$$\left(I_{-}^{\alpha,-\alpha,\eta} f\right)(x) = \left(I_{-}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} f(t)\,dt, \quad (x>0) \quad (5)$$

Again, if $\beta = 0$, the equations (1) and (2) reduce to the following Erdélyi-Kober fractional integral operator (see Samko et al., [30], p.322, Eqns. (18.5), (18.6)):

$$\left(I_{0^+}^{\alpha,0,\eta} f\right)(x) = \left(I_{\eta,\alpha}^{+} f\right)(x) = \frac{x^{-\alpha-\eta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} t^{\eta} f(t)\, dt, (x>0) \tag{6}$$

and

$$\left(I_{-}^{\alpha,0,\eta} f\right)(x) = \left(K_{\eta,\alpha}^{-} f\right)(x) = \frac{x^{\eta}}{\Gamma(\alpha)} \int_x^{\infty} (t-x)^{\alpha-1} t^{-\alpha-\eta} f(t)\, dt, (x>0) \tag{7}$$

Recently, Gupta et al. [6] have obtained the images of the product of two H-functions in Saigo operator given by (1) and (2) thereby generalized several important results obtained earlier by Kilbas, Kilbas and Sebastian, Saxena et al., as mentioned in the paper cited above. Similarly, Purohit et al. [24] have also obtained image formulas for the Bessel function of first kind involving Saigo-Maeda fractional integral operators [see Saigo and Maeda [[29], p.393, eqn (4.12) and (4.13)], in term of the generalized Wright function. It has recently become a subject of interest for many researchers in the field of fractional calculus and its applications. Motivated by these avenues of applications, a number of workers have made use of the fractional calculus operators to obtain the image formulas. The aim of the present paper is to obtain two results that give the images of the product of $\overline{H}$-function and a general class of polynomials in Saigo operators.

A lot of research work has recently come up on the study and development of a function that is more general than the Fox H-function, popularly known as $\overline{H}$-function. On account of the importance and considerable popularity achieved by $\overline{H}$-function, during last two decades due to its applications in various fields of science and engineering. It was introduced by Inayat-Hussain [7, 8] and now stands on a fairly firm footing through the research contributions of various authors [1, 2, 4, 5. 7, 8, and 25].

The $\overline{H}$-function is defined and represented in the following manner [4]:

$$\overline{H}_{p,q}^{m,n}[z] \equiv \overline{H}_{p,q}^{m,n}\left[z \,\middle|\, \begin{array}{l} (a_j,\alpha_j;A_j)_{1,n},\ (a_j,\alpha_j)_{n+1,p} \\ (b_j,\beta_j)_{1,m}, (b_j,\beta_j;B_j)_{m+1,q} \end{array}\right] = \frac{1}{2\pi i}\int_L z^{\xi}\overline{\phi}(\xi)\ d\xi. \quad (z\neq 0) \tag{8}$$

$$\overline{\phi}(\xi) = \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{n}\left\{\Gamma(1-a_j+\alpha_j\xi)\right\}^{A_j}}{\prod_{j=m+1}^{q}\left\{\Gamma(1-b_j+\beta_j\xi)\right\}^{B_j}\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j\xi)} \tag{9}$$

It may be noted that the $\overline{\phi}(\xi)$ contain fractional powers of some of the gamma function, m,n,p,q are integers such that $1 \le m \le q$, $0 \le n \le p$, $(\alpha_j)_{1,p}$, $(\beta_j)_{1,q}$ are positive real numbers and $(A_j)_{1,n}, (B_j)_{m+1,q}$ may take non-integer values, which we assume to be positive for standardization purpose. $(a_j)_{1,p}$and $(b_j)_{1,q}$ are complex numbers.
The nature of contour L, sufficient conditions of convergence of defining integral (8) and other details about the $\overline{H}$-functions can be seen in the papers [7, 8]. Evidently, when the exponents A_j and B_j are unity, the $\overline{H}$-function reduces to the well-known Fox H-function [32].
The behavior of the $\overline{H}$-function for small values of $|z|$ follows easily from a result given by Rathie [25]:

$$\overline{H}^{m,n}_{p,q}[z] = o(|z|^{\alpha}) \text{ where } \alpha = \min_{1\le j\le m}\{\Re(\mathrm{b_j}/\beta_j),\ |z| \to 0\} \tag{10}$$

Also, $S_n^m[x]$ occurring in the sequel denotes the general class of polynomials introduced by Srivastava ([33], p. 1, Eq. (1)):

$$S_n^m[x] = \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k} x^k,\ n = 0, 1, 2, \ldots \tag{11}$$

where m is an arbitrary positive integer and the coefficients $A_{n,k}$ $(n,k \ge 0)$ are arbitrary constants, real or complex. On suitably specializing the coefficients $A_{n,k}, S_n^m[x]$ yields a number of known polynomials as its special cases. These include, among others, the Hermite polynomials, the Jacobi polynomials, the Laguerre polynomials, the Bessel polynomials, the Gould–Hopper polynomials, the Brafman polynomials and several others [see Srivastava and Singh [34], pp. 158–161].

2. Preliminary lemmas

The following lemmas will be required to establish our main results.

Lemma 2.1. (Kilbas and Sebastain [15], p. 871, Eq. (15) to (18)). *Let* $\alpha, \beta, \eta \in \mathbb{C}$ *be such that* $\Re(\alpha) > 0$ *and* $\Re(\mu) > \max\{0, \Re(\beta - \eta)\}$ *then the following relation holds:*

$$\left(I_{0^+}^{\alpha,\beta,\eta} t^{\mu-1}\right)(x) = \frac{\Gamma(\mu)\Gamma(\mu+\eta-\beta)}{\Gamma(\mu+\alpha+\eta)\Gamma(\mu-\beta)} x^{\mu-\beta-1} \tag{12}$$

In particular, if $\beta = -\alpha$ *and* $\beta = 0$ *in eq.* (12)*, we have:*

$$\left(I_{0^+}^{\alpha} t^{\mu-1}\right)(x) = \frac{\Gamma(\mu)}{\Gamma(\mu+\alpha)} x^{\mu+\alpha-1}, \quad \Re(\alpha) > 0, \Re(\mu) > 0 \tag{13}$$

$$\left(I^{+}_{\eta,\alpha}t^{\mu-1}\right)(x) = \frac{\Gamma(\mu+\eta)}{\Gamma(\mu+\alpha+\eta)}x^{\mu-1}, \quad \Re(\alpha)>0, \Re(\mu)>-\Re(\eta). \tag{14}$$

Lemma 2.2. (Kilbas and Sebastain [15], p. 872, Eq. (21) to (24)). *Let $\alpha,\beta,\eta\in\mathbb{C}$ be such that $\Re(\alpha)>0$ and $\Re(\mu)<1+\min\{\Re(\beta),\Re(\eta)\}$. Then the following relation holds:*

$$\left(I_{-}^{\alpha,\beta,\eta}t^{\mu-1}\right)(x) = \frac{\Gamma(\beta-\mu+1)\Gamma(\eta-\mu+1)}{\Gamma(1-\mu)\Gamma(\alpha+\beta+\eta-\mu+1)}x^{\mu-\beta-1} \tag{15}$$

In particular, if $\beta=-\alpha$ and $\beta=0$ in eq. (15)*, we have*

$$\left(I_{-}^{\alpha}t^{\mu-1}\right)(x) = \frac{\Gamma(1-\alpha-\mu)}{\Gamma(1-\mu)}x^{\mu+\alpha-1}, \quad 1-\Re(\mu)>\Re(\alpha)>0. \tag{16}$$

$$\left(K^{-}_{\eta,\alpha}t^{\mu-1}\right)(x) = \frac{\Gamma(\eta-\mu+1)}{\Gamma(1-\mu+\alpha+\eta)}x^{\mu-1}, \quad \Re(\mu)<1+\Re(\eta). \tag{17}$$

3. Main Results

Theorem 3.1.

$$\left\{I_{0^+}^{\alpha,\beta,\eta}\left(t^{\mu-1}\prod_{j=1}^{s}S_{n_j}^{m_j}[c_j t^{\lambda_j}]\overline{\mathrm{H}}_{\mathrm{P,Q}}^{\mathrm{M,N}}\left[zt^{\nu}\,\middle|\,\begin{matrix}(\mathrm{a_j},\alpha_j;A_j)_{1,N},\,(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(\mathrm{b_j},\beta_j;B_j)_{M+1,Q}\end{matrix}\right]\right)\right\}(x) =$$

$$x^{\mu-\beta-1}\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_s=0}^{[n_s/m_s]}\frac{(-n_1)_{m_1k_1}\ldots(-n_s)_{m_sk_s}}{k_1!\ldots k_s!}A'_{n_1,m_1}\ldots A^{(s)}_{n_s,m_s}c_1^{k_1}\ldots c_s^{k_s}(x)^{\sum_{j=1}^{s}\lambda_jk_j}$$

$$\overline{H}_{P+2,Q+2}^{M,N+2}\left[zx^{\nu}\,\middle|\,\begin{matrix}\left(1-\mu-\sum_{j=1}^{s}\lambda_jk_j,\nu;1\right),\left(1-\mu-\eta+\beta-\sum_{j=1}^{s}\lambda_jk_j,\nu;1\right),(\mathrm{a_j},\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(\mathrm{b_j},\beta_j;B_j)_{M+1,Q},\left(1-\mu+\beta-\sum_{j=1}^{s}\lambda_jk_j,\nu;1\right),\left(1-\mu-\alpha-\eta-\sum_{j=1}^{s}\lambda_jk_j,\nu;1\right)\end{matrix}\right] \tag{18}$$

The sufficient conditions of validity of (18) *are as follows:*

(i) $\alpha,\beta,\eta,\mu,a,b,z\in\mathbb{C}$ *and* $\lambda_j,\nu>0 \quad \forall j\in\{1,...,s\}$

(ii) $|\arg z|<\frac{1}{2}\Omega\pi$ *and* $\Omega>0$ *where*
$\Omega=\sum_{j=1}^{M}\beta_j+\sum_{j=1}^{N}A_j\alpha_j-\sum_{j=M+1}^{Q}B_j\beta_j-\sum_{j=N+1}^{P}\alpha_j; \quad \forall i\in\{1,...,r\}$

(iii) $\Re(\alpha)>0$ *and* $\Re(\mu)+\nu\min_{1\le j\le M}\Re\left(\frac{b_j}{\beta_j}\right)>\max\{0,\Re(\beta-\eta)\}$

Proof. In order to prove (18), we first express the product of a general class of polynomials occurring on its left-hand side in the series form given by (11), replace the $\overline{H}$-function occurring therein by its well-known Mellin–Barnes contour integral given by (8), interchange the order of summations, ξ-integrals and taking the fractional integral operator inside (which is permissible under the conditions stated), it takes the following form (say I) after a little simplification:

$$I=\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_s=0}^{[n_s/m_s]}\frac{(-n_1)_{m_1k_1}\dots(-n_s)_{m_sk_s}}{k_1!\dots k_s!}A'_{n_1,m_1}\dots A^{(s)}_{n_s,m_s}$$
$$c_1^{k_1}\dots c_s^{k_s}\left\{\frac{1}{(2\pi i)}\int_L\overline{\phi}(\xi)\,z^{\xi}\left(I_{0^+}^{\alpha,\beta,\eta}t^{\mu+\sum\limits_{j=1}^{s}\lambda_jk_j+\nu\xi-1}\right)(x)\,d\xi\right\}$$

Finally, applying Lemma 2.1 and re-interpreting the Mellin-Barnes contour integral thus obtained in terms of the $\overline{H}$-function defined by (8), we arrive at the right hand side of (18) after a little simplification. □

If we put $\beta=-\alpha$ in Image 1, we arrive at the following new and interesting corollary concerning Riemann-Liouville fractional integral operator defined by (4) and using (13).

Corollary 3.2.

$$\left\{I_{0^+}^{\alpha}\left(t^{\mu-1}\prod_{j=1}^{s}S_{n_j}^{m_j}[c_jt^{\lambda_j}]\overline{\mathrm{H}}_{\mathrm{P,Q}}^{\mathrm{M,N}}\left[zt^{\nu}\left|\begin{array}{l}(\mathrm{a_j},\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(\mathrm{b_j},\beta_j;B_j)_{M+1,Q}\end{array}\right.\right]\right)\right\}(x)$$
$$=x^{\mu-\beta-1}\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_s=0}^{[n_s/m_s]}\frac{(-n_1)_{m_1k_1}\dots(-n_s)_{m_sk_s}}{k_1!\dots k_s!}A'_{n_1,m_1}\dots A^{(s)}_{n_s,m_s}c_1^{k_1}\dots c_s^{k_s}(x)^{\sum_{j=1}^{s}\lambda_jk_j}$$
$$\overline{H}_{P+1,Q+1}^{M,N+1}\left[zx^{\nu}\left|\begin{array}{l}\left(1-\mu-\sum_{j=1}^{s}\lambda_jk_j,\nu;1\right),(\mathrm{a_j},\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(\mathrm{b_j},\beta_j;B_j)_{M+1,Q},\left(1-\mu-\alpha-\sum_{j=1}^{s}\lambda_jk_j,\nu;1\right)\end{array}\right.\right]\quad(19)$$

where the conditions of existence of the above corollary follow easily with the help of (18).
Again, if we put $\beta=0$ in Image 1, we get the following result which is also believed to be new and pertains to $Erd\acute{e}lyi-Kober$ fractional integral operators defined by (6) and using (14).

Corollary 3.3.

$$\left\{I_{\eta,\alpha}^{+}\left(t^{\mu-1}\prod_{j=1}^{s}S_{n_j}^{m_j}[c_jt^{\lambda_j}]\overline{H}_{\mathrm{P,Q}}^{\mathrm{M,N}}\left[zt^{\nu}\left|\begin{array}{l}(\mathrm{a_j},\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(\mathrm{b_j},\beta_j;B_j)_{M+1,Q}\end{array}\right.\right]\right)\right\}(x)$$
$$=x^{\mu-1}\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_s=0}^{[n_s/m_s]}\frac{(-n_1)_{m_1k_1}\dots(-n_s)_{m_sk_s}}{k_1!\dots k_s!}A'_{n_1,m_1}\dots A^{(s)}_{n_s,m_s}c_1^{k_1}\dots c_s^{k_s}(x)^{\sum_{j=1}^{s}\lambda_jk_j}$$
$$\overline{H}_{P+1,Q+1}^{M,N+1}\left[zx^{\nu}\left|\begin{array}{l}(1-\mu-\eta-\sum_{j=1}^{s}\lambda_jk_j,\nu;1),(\mathrm{a_j},\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(\mathrm{b_j},\beta_j;B_j)_{M+1,Q},(1-\mu-\alpha-\eta-\sum_{j=1}^{s}\lambda_jk_j,\nu;1)\end{array}\right.\right]\quad(20)$$

The sufficient conditions of validity of (20) *are as follows:*

$$\Re(\alpha) > 0 \quad \textit{and} \quad \Re(\mu) + \nu \min_{1\le j\le M}\{\Re(\frac{b_j}{\beta_j})\} > -\Re(\eta)$$

and the conditions (i) and (ii) of Theorem 3.1 are also satisfied.

Theorem 3.4.

$$\left\{I_{-}^{\alpha,\beta,\eta}\left(t^{\mu-1}\prod_{j=1}^{s} S_{n_j}^{m_j}[c_j t^{\lambda_j}]\overline{\mathrm{H}}_{\mathrm{P,Q}}^{\mathrm{M,N}}\left[zt^{\nu}\left|\begin{smallmatrix}(\mathrm{a_j},\alpha_j;A_j)_{1,N}(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(\mathrm{b_j},\beta_j;B_j)_{M+1,Q}\end{smallmatrix}\right.\right]\right)\right\}(x) =$$

$$x^{\mu-\beta-1}\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_s=0}^{[n_s/m_s]}\tfrac{(-n_1)_{m_1k_1}\cdots(-n_s)_{m_sk_s}}{k_1!\ldots k_s!}A'_{n_1,m_1}\ldots A^{(s)}_{n_s,m_s}c_1^{k_1}\ldots c_s^{k_s}(x)^{\sum_{j=1}^{s}\lambda_jk_j}$$

$$\overline{H}_{P+2,Q+2}^{M,N+2}\left[zx^{\nu}\left|\begin{smallmatrix}\left(\mu-\beta+\sum_{j=1}^{s}\lambda_jk_j,\nu;1\right),\left(\mu-\eta+\sum_{j=1}^{s}\lambda_jk_j,\nu;1\right),(\mathrm{a_j},\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(\mathrm{b_j},\beta_j;B_j)_{M+1,Q},\left(\mu+\sum_{j=1}^{s}\lambda_jk_j,\nu;1\right),\left(\mu-\alpha-\beta-\eta+\sum_{j=1}^{s}\lambda_jk_j,\nu;1\right)\end{smallmatrix}\right.\right] \qquad (21)$$

The sufficient conditions of validity of (21) *are*

$$\Re(\alpha) > 0 \quad \textit{and} \quad \Re(\mu) - \nu \min_{1\le j\le M}\{\Re(\frac{b_j}{\beta_j})\} < 1 + \min\{\Re(\beta),\Re(\eta)\}$$

and the conditions (i) and (ii) in Theorem 3.1 are also satisfied.

Proof. We easily obtain the Theorem 3.4 after a little simplification, making use of similar lines as adopted in (18) and using Lemma 2.2. □

If we put $\beta = -\alpha$ and $\beta = 0$ in Image 2 and using (16) and (17), in succession we shall easily arrive at the corresponding corollaries concerning Riemann-Liouville and $Erd\acute{e}lyi - Kober$ fractional integral operators respectively.

Corollary 3.5.

$$\left\{I_{-}^{\alpha}\left(t^{\mu-1}\prod_{j=1}^{s} S_{n_j}^{m_j}[c_j t^{\lambda_j}]\overline{\mathrm{H}}_{\mathrm{P,Q}}^{\mathrm{M,N}}\left[zt^{\nu}\left|\begin{smallmatrix}(\mathrm{a_j},\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,N},(\mathrm{b_j},\beta_j;B_j)_{M+1,Q}\end{smallmatrix}\right.\right]\right)\right\}(x)$$

$$= x^{\mu+\alpha-1}\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_s=0}^{[n_s/m_s]}\tfrac{(-n_1)_{m_1k_1}\cdots(-n_s)_{m_sk_s}}{k_1!\ldots k_s!}A'_{n_1,m_1}\ldots A^{(s)}_{n_s,m_s}c_1^{k_1}\ldots c_s^{k_s}(x)^{\sum_{j=1}^{s}\lambda_jk_j}$$

$$\overline{H}_{P+1,Q+1}^{M,N+1}\left[zx^{\nu}\left|\begin{smallmatrix}\left(\alpha+\mu+\sum_{j=1}^{s}\lambda_jk_j,\nu;1\right),(\mathrm{a_j},\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(\mathrm{b_j},\beta_j;B_j)_{M+1,Q},\left(\mu+\sum_{j=1}^{s}\lambda_jk_j,\nu;1\right)\end{smallmatrix}\right.\right] \qquad (22)$$

where conditions of validity are same as (20).

Corollary 3.6.

$$\left\{K^{-}_{\eta,\alpha}\left(t^{\mu-1}\prod_{j=1}^{s}S_{n_j}^{m_j}[c_jt^{\lambda_j}]\overline{\mathrm{H}}^{\mathrm{M,N}}_{\mathrm{P,Q}}\left[zt^{\nu}\,\middle|\,\begin{matrix}(\mathrm{a_j},\alpha_j;A_j)_{1,N},\,(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(\mathrm{b_j},\beta_j;B_j)_{M+1,Q}\end{matrix}\right]\right)\right\}(x)$$

$$=x^{\mu-1}\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_s=0}^{[n_s/m_s]}\frac{(-n_1)_{m_1k_1}\ldots(-n_s)_{m_sk_s}}{k_1!\ldots k_s!}A'_{n_1,m_1}\ldots A^{(s)}_{n_s,m_s}c_1^{k_1}\ldots c_s^{k_s}(x)^{\sum_{j=1}^{s}\lambda_jk_j}$$

$$\overline{H}^{M,N+1}_{P+1,Q+1}\left[zx^{\nu}\,\middle|\,\begin{matrix}\left(\mu-\eta+\sum_{j=1}^{s}\lambda_jk_j,\nu;1\right),(\mathrm{a_j},\alpha_j;A_j)_{1,N},\,(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(\mathrm{b_j},\beta_j;B_j)_{M+1,Q},\left(\mu-\alpha-\eta+\sum_{j=1}^{s}\lambda_jk_j,\nu;1\right)\end{matrix}\right] \quad (23)$$

The conditions of validity of the above results follow easily from the conditions given with Theorem 3.4.

4. Special cases and applications

The generalized fractional integral operators of Theorems 3.1 and 3.4 established here are unified in nature and act as key formulae. Thus the general class of polynomials involved in Theorem 3.1 and 3.4 reduce to a large spectrum of polynomials listed by Srivastava and Singh [34, pp. 158–161], and so from these two results we can further obtain various fractional integral results involving a number of simpler polynomials. Again, the $\overline{H}$-function occurring in these results can be suitably specialized to a remarkably wide variety of useful functions (or product of several such functions) which are expressible in terms of generalized Wright hypergeometric function, generalized Mittag-Leffler function and Bessel functions of one variable. For example:

(i) If we take $\mathrm{M}=1$, $\mathrm{N}=\mathrm{P}$ and $\mathrm{Q}=\mathrm{Q}+1$ in (18), the $\overline{H}$-function occurring therein breaks up into the generalized Wright hypergeometric function ${}_P\overline{\Psi}_Q(.)$ given by [4]. Then, Theorem 3.1 takes the following form after a little simplification which is also believed to be new:

$$\left\{I^{\alpha,\beta,\eta}_{0^+}\left(t^{\mu-1}\prod_{j=1}^{s}S_{n_j}^{m_j}[c_jt^{\lambda_j}]_P\overline{\Psi}_Q\left[-zt^{\nu}\,\middle|\,\begin{matrix}(1-\mathrm{a_j},\alpha_j;A_j)_{1,P}\\(1-\mathrm{b_j},\beta_j;B_j)_{1,Q}\end{matrix}\right]\right)\right\}(x)$$

$$=x^{\mu-\beta-1}\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_s=0}^{[n_s/m_s]}\frac{(-n_1)_{m_1k_1}\ldots(-n_s)_{m_sk_s}}{k_1!\ldots k_s!}A'_{n_1,m_1}\ldots A^{(s)}_{n_s,m_s}c_1^{k_1}\ldots c_s^{k_s}(x)^{\sum_{j=1}^{s}\lambda_jk_j}$$

$$\overline{H}^{M,N+2}_{P+2,Q+2}\left[zx^{\nu}\,\middle|\,\begin{matrix}\left(1-\mu-\sum_{j=1}^{s}\lambda_jk_j,\nu;1\right),\left(1-\mu-\eta+\beta-\sum_{j=1}^{s}\lambda_jk_j,\nu;1\right),(1-\mathrm{a_j},\alpha_j;A_j)_{1,P}\\(1-\mathrm{b_j},\beta_j;B_j)_{1,Q},\left(1-\mu+\beta-\sum_{j=1}^{s}\lambda_jk_j,\nu;1\right),\left(1-\mu-\alpha-\eta-\sum_{j=1}^{s}\lambda_jk_j,\nu;1\right)\end{matrix}\right] \quad (24)$$

The conditions of validity of the above result easily follow from (18).

If we put $\beta=-\alpha, S_{n_j}^{m_j}=1$ and $A_j, B_j=1$ and make suitable adjustment in the parameters in the equation (24), we arrive at the known result [7, p. 117, Eq. (11)].
If we put $S_{n_j}^{m_j}=1$ and $A_j, B_j=1$ and make suitable adjustment in the parameters in the equation (24), we arrive at the known result [13, p.210, Eq. (27)].

(ii) If we take M = 1, N = P = 0 and Q = 2 in (18) and the $\overline{H}$-function occurring therein breaks up into the generalized Wright-Bessel function $\overline{J}_b^{\beta,B}(z)$ given by [5]. Then, the Theorem 3.1 takes the following form after a little simplification which is also believed to be new:

$$\left\{I_{0^+}^{\alpha,\beta,\eta}\left(t^{\mu-1}\prod_{j=1}^{s}S_{n_j}^{m_j}[c_jt^{\lambda_j}]\overline{J}_b^{\beta,B}[-zt^{\nu}]\right)\right\}(x)$$

$$=x^{\mu-\beta-1}\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_s=0}^{[n_s/m_s]}\frac{(-n_1)_{m_1k_1}\dots(-n_s)_{m_sk_s}}{k_1!\dots k_s!}A'_{n_1,m_1}\dots A^{(s)}_{n_s,m_s}c_1^{k_1}\dots c_s^{k_s}(x)^{\sum_{j=1}^{s}\lambda_jk_j}$$

$$\overline{H}_{2,4}^{1,2}\left[zx^{\nu}\,\middle|\,\begin{matrix}\left(1-\mu-\sum_{j=1}^{s}\lambda_jk_j,\nu;1\right),\left(1-\mu-\eta+\beta-\sum_{j=1}^{s}\lambda_jk_j,\nu;1\right)\\(0,1),(-\mathrm{b},\beta;B),\left(1-\mu+\beta-\sum_{j=1}^{s}\lambda_jk_j,\nu;1\right),\left(1-\mu-\alpha-\eta-\sum_{j=1}^{s}\lambda_jk_j,\nu;1\right)\end{matrix}\right]\tag{25}$$

The conditions of validity of the above result easily follow from (18).

(iii) If we take M = 1, N = P = 2 and Q = 2 in (18), the $\overline{H}$-function occurring therein breaks up into the polylogarithm of order P, $F(z,P)$ given by [5]. Then, the equation (18) takes the following form after a little simplification which is also believed to be new:

$$\left\{I_{0^+}^{\alpha,\beta,\eta}\left(t^{\mu-1}\prod_{j=1}^{s}S_{n_j}^{m_j}[c_jt^{\lambda_j}]F[-zt^{\nu},P]\right)\right\}(x)$$

$$=x^{\mu-\beta-1}\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_s=0}^{[n_s/m_s]}\frac{(-n_1)_{m_1k_1}\dots(-n_s)_{m_sk_s}}{k_1!\dots k_s!}A'_{n_1,m_1}\dots A^{(s)}_{n_s,m_s}c_1^{k_1}\dots c_s^{k_s}(x)^{\sum_{j=1}^{s}\lambda_jk_j}$$

$$\overline{H}_{4,4}^{1,4}\left[-zx^{\nu}\,\middle|\,\begin{matrix}\left(1-\mu-\sum_{j=1}^{s}\lambda_jk_j,\nu;1\right),\left(1-\mu-\eta+\beta-\sum_{j=1}^{s}\lambda_jk_j,\nu;1\right),(1,1;1),(1,1;P)\\(1,1),(0,1;P),\left(1-\mu+\beta-\sum_{j=1}^{s}\lambda_jk_j,\nu;1\right),\left(1-\mu-\alpha-\eta-\sum_{j=1}^{s}\lambda_jk_j,\nu;1\right)\end{matrix}\right]\tag{26}$$

The conditions of validity of the above result easily follow from (18).

(iv) If we take $z,\nu=1$ in the equation (18), the $\overline{H}$-function occurring therein reduces to the generalized Mittag-Leffler function [23, p. 19, Eq. (2.6.11)] and we easily get after little simplification the following new and interesting result.

$$\left\{I_{0^+}^{\alpha,\beta,\eta}\left(t^{\mu-1}\prod_{j=1}^{s}S_{n_j}^{m_j}[c_jt^{\lambda_j}]E_{M,N}^{\rho}[t]\right)\right\}(x)$$

$$= x^{\mu-\beta-1} \sum_{k_1=0}^{[n_1/m_1]} \cdots \sum_{k_s=0}^{[n_s/m_s]} \frac{(-n_1)_{m_1 k_1} \cdots (-n_s)_{m_s k_s}}{k_1! \ldots k_s!} A'_{n_1,m_1} \ldots A^{(s)}_{n_s,m_s} c_1^{k_1} \ldots c_s^{k_s} (x)^{\sum_{j=1}^{s} \lambda_j k_j}$$

$$\overline{H}^{M,N+2}_{P+2,Q+2}\left[x \left| \begin{array}{l} \left(1-\mu-\sum_{j=1}^{s}\lambda_j k_j,\nu;1\right),\left(1-\mu-\eta+\beta-\sum_{j=1}^{s}\lambda_j k_j,\nu;1\right),(a_j,\alpha_j)_{1,P} \\ (b_j,\beta_j)_{1,Q},\left(1-\mu+\beta-\sum_{j=1}^{s}\lambda_j k_j,\nu;1\right),\left(1-\mu-\alpha-\eta-\sum_{j=1}^{s}\lambda_j k_j,\nu;1\right) \end{array} \right. \right] \quad (27)$$

The conditions of validity of the above result can be easily followed directly from those given with (18).

If we put $S_{n_j}^{m_j} = 1$ and make suitable adjustment in the parameters in the equation (27), we arrive at the known result [13, p.210, Eq. (29)].

If we put $\beta = -\alpha$ and $S_{n_j}^{m_j} = 1$ and make suitable adjustment in the parameters in the equation (27), we arrive at the known result [6, p. 168, Eq. 2.1].

(v) If we take $\beta = -\alpha$ and $S_{n_j}^{m_j} = 1$, $z = \frac{1}{4}$, $\nu = 2$ and reduce the $\overline{H}$-function to the Bessel function of first kind in the equation (18), we also get known result [15, p.873, Eq.(25)to (29)].

A number of other special cases of Theorem 3.1 and 3.4 can also be obtained but we do not mention them here on account of lack of space.

5. Conclusion

In this paper, we have presented the Images of generalized fractional integral operators given by Saigo. The Images have been developed in terms of the product of $\overline{H}$-function and a general class of polynomials in a compact and elegant form with the help of Saigo operators. Most of the results obtained here, besides being of very general character, have been put in a compact form, avoiding the occurrence of infinite series and thus making them useful in applications. The result obtained in the present paper provides an extension of the results given by Kilbas, Kilbas and Sebastain, Saxena et al. and Gupta et al., as mentioned earlier.

Acknowledgements

The author is thankful to the referee for the valuable comments and suggestions, which have led the paper to the present form.

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