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GENERALIZATION OF SOME INEQUALITIES FOR THE $(q_1,\ldots,q_s)$-GAMMA FUNCTION

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Recently were established q-analogues of some inequalities involving the gamma functions. In this paper are presented the $(q_1,\ldots,q_s)$-analogues of those inequalities.

1. Introduction

The Euler gamma function $\Gamma(x)$ is defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt,$$

and it's logarithmic derivative, the psi or digamma function, is defined for $x > 0$ by

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}.$$

Alsina and Tomás [1] have proved that

$$\frac{1}{n!} \le \frac{(\Gamma(1+x))^n}{\Gamma(1+nx)} \le 1,$$

for all $x \in [0,1]$ and for all nonnegative integers n.

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This inequality generalized to

$$\frac{1}{\Gamma(1+a)} \leq \frac{(\Gamma(1+x))^a}{\Gamma(1+ax)} \leq 1, \tag{1}$$

for all $a \geq 1$ and $x \in [0,1]$ (see[10]).

Later, Shabani [11] using the series representation of the function $\psi(x)$ and the ideas used in [10] established several double inequalities involving the gamma function. In particular, Shabani [11, Theorem 2.4] showed

$$\frac{(\Gamma(a))^c}{(\Gamma(b))^d} \leq \frac{(\Gamma(a+bx))^c}{(\Gamma(b+ax))^d} \leq \frac{(\Gamma(a+b))^c}{(\Gamma(a+b))^d}, \tag{2}$$

for all $x \in [0,1]$, $a \geq b > 0$, c,d are positive real numbers such that $bc \geq ad$, and $\psi(b+ax) > 0$.

F.H Jackson (see [3–5, 12]) defined the q-analogue of the gamma function as

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, 0 < q < 1,$$

and

$$\Gamma_q(x) = \frac{(q^{-1};q^{-1})_\infty}{(q^{-x};q^{-1})_\infty}(q-1)^{1-x}q^{\binom{x}{2}}, q > 1,$$

where $(a;q)_\infty = \prod_{j\geq 0}(1-aq^j)$.

The q-analogue of the psi function is defined for $0 < q < 1$ as the logarithmic derivative of the q-gamma function, that is,

$$\psi_q(x) = \frac{d}{dx}\log\Gamma_q(x).$$

Many properties of the q-gamma function were derived by Askey [2].

Kim (see [7, 8]), recently studied q-Bernstein type polynomials.

It is well known that $\Gamma_q(x) \to \Gamma(x)$ and $\psi_q(x) \to \psi(x)$ as $q \to 1^-$.

Kim and Adiga [6] gave the q-analogue of (1) as

$$\frac{1}{\Gamma_q(1+a)} \leq \frac{(\Gamma_q(1+x))^a}{\Gamma_q(1+ax)} \leq 1,$$

for all $0 < q < 1$, $a \geq 1$ and $x \in [0,1]$.

Later, Mansour [9] studied the q-analogue of (2) and obtained:

$$\frac{(\Gamma_q(a))^c}{(\Gamma_q(b))^d} \leq \frac{(\Gamma_q(a+bx))^c}{(\Gamma_q(b+ax))^d} \leq \frac{(\Gamma_q(a+b))^c}{(\Gamma_q(a+b))^d}, \tag{3}$$

for all $x \in [0,1]$, $0 < q < 1$, $a \geq b > 0$, c,d are positive real numbers such that $bc \geq ad$, and $\psi_q(b+ax) > 0$.

Next we define $(q_1,\dots,q_s)$-gamma function and $(q_1,\dots,q_s)$-psi function as

$$\Gamma_{q_1,\dots,q_s}(x)=\frac{\left(q_1q_2\cdots q_s;q_1,q_2,\dots,q_s\right)_\infty}{\left((q_1q_2\cdots q_s)^x;q_1,q_2,\dots,q_s\right)_\infty}(1-q_1q_2\cdots q_s)^{1-x}$$

and

$$\psi_{q_1,\dots,q_s}(x)=\frac{d}{dx}\log\Gamma_{q_1,\dots,q_s}(x),$$

where

$$(a;q_1,q_2,\dots,q_s)_\infty=\prod_{j_1\ge 0}\prod_{j_2\ge 0}\cdots\prod_{j_s\ge 0}(1-aq_1^{j_1}q_2^{j_2}\cdots q_s^{j_s}).$$

Clearly, when $s=1$ we get the standard q-gamma function and q-psi function. From above definitions we find that

$$\Gamma_{q_1,\dots,q_s}(1)=\Gamma_{q_1,\dots,q_s}(2)=1. \tag{4}$$

In this paper, by using similar techniques as in [9] and [11] we give the $(q_1,\dots,q_s)$-inequalities of the above results.

2. Main results

At first, we note that the $(q_1,\dots,q_s)$-analogue of the psi function has the following series representation

$$\psi_{q_1,\dots,q_s}(x)=-\log(1-q_1q_2\cdots q_s)+\sum_{i=1}^{s}\log q_i\cdot\sum_{j_1,\dots,j_s\ge 0}\frac{\prod_{i=1}^{s}q_i^{j_i+x}}{1-\prod_{i=1}^{s}q_i^{j_i+x}}. \tag{5}$$

Using this representation we will be able to give the $(q_1,\dots,q_s)$-analogue of several known results.

Theorem 2.1. *Let $x\ge 0$, $0<q_i<1$ for all $i=1,2,\dots,s$. Let a be a real number.*

1) If $a\ge 1$ then

$$\frac{1}{\Gamma_{q_1,\dots,q_s}(a+1)}\le\frac{\Gamma_{q_1,\dots,q_s}(1+x)^a}{\Gamma_{q_1,\dots,q_s}(1+ax)}\le 1.$$

2) If $a\in[0,1)$ then

$$1\le\frac{\Gamma_{q_1,\dots,q_s}(1+x)^a}{\Gamma_{q_1,\dots,q_s}(1+ax)}\le\frac{1}{\Gamma_{q_1,\dots,q_s}(a+1)}.$$

Proof. Let $f(x) = \frac{\Gamma_{q_1,\dots,q_s}(1+x)^a}{\Gamma_{q_1,\dots,q_s}(1+ax)}$ and $g(x) = \log f(x)$. Then

$$g(x) = a \log \Gamma_{q_1,\dots,q_s}(1+x) - \log \Gamma_{q_1,\dots,q_s}(1+ax),$$

which implies that

$$g'(x) = a(\psi_{q_1,\dots,q_s}(1+x) - \psi_{q_1,\dots,q_s}(1+ax)).$$

The series representation of $\psi_{q_1,\dots,q_s}(x)$, see (5), gives

$$\begin{aligned}&\psi_{q_1,\dots,q_s}(1+x) - \psi_{q_1,\dots,q_s}(1+ax)\\ &\quad = \sum_{i=1}^{s} \log q_i \cdot \sum_{j_1,\dots,j_s \geq 0} \frac{(1-(q_1q_2\cdots q_s)^{(a-1)x}) \prod_{i=1}^{s} q_i^{j_i+1+x}}{(1-\prod_{i=1}^{s} q_i^{j_i+1+x})(1-\prod_{i=1}^{s} q_i^{j_i+1+ax})}.\end{aligned}$$

1) Since $0 < q_i < 1$ we have that $\log q_i < 0$, for all $i = 1,2,\dots,s$. In addition, for $a \geq 1$ and $x \geq 0$ we have $1 \geq (q_1q_2\cdots q_s)^{(a-1)x}$. Hence, $g'(x) \leq 0$, that is, g is a decreasing function for $x \geq 0$. Therefore, f is a decreasing function for $x \geq 0$. For $x \in [0,1]$ we have $f(1) \leq f(x) \leq f(0)$, which is equivalent to

$$\frac{\Gamma_{q_1,\dots,q_s}(2)^a}{\Gamma_{q_1,\dots,q_s}(1+a)} \leq \frac{\Gamma_{q_1,\dots,q_s}(1+x)^a}{\Gamma_{q_1,\dots,q_s}(1+ax)} \leq \frac{\Gamma_{q_1,\dots,q_s}(1)^a}{\Gamma_{q_1,\dots,q_s}(1)}.$$

Using (4) the desired result follows.
2) For $a \in [0,1)$ and $x \geq 0$ we have $1 \leq (q_1q_2\cdots q_s)^{(a-1)x}$. Next we proceed in a similar way as in previous case. □

In order to establish the proof of the following results, we need the following lemmas.

Lemma 2.2. *Let $x \in [0,1]$, $0 < q_i < 1$ for all $i = 1,2,\dots,s$. Let a,b be any two positive real numbers such that $a \geq b$. Then*

$$\psi_{q_1,\dots,q_s}(a+bx) \geq \psi_{q_1,\dots,q_s}(b+ax).$$

Proof. Clearly, $a+bx, b+ax > 0$. Then from (5), we have:

$$\begin{aligned}&\psi_{q_1,\dots,q_s}(a+bx) - \psi_{q_1,\dots,q_s}(b+ax)\\ &\quad = \sum_{i=1}^{s} \log q_i \cdot \sum_{j_1,\dots,j_s \geq 0} \frac{((q_1q_2\cdots q_s)^{a-b} - (q_1q_2\cdots q_s)^{(a-b)x}) \prod_{i=1}^{s} q_i^{j_i+b+bx}}{(1-\prod_{i=1}^{s} q_i^{j_i+a+bx})(1-\prod_{i=1}^{s} q_i^{j_i+b+ax})}.\end{aligned}$$

Since $0 < q_i < 1$ we have that $\log q_i < 0$, for all $i = 1,2,\dots,s$. In addition, since $a \geq b$ and $x \in [0,1]$ we get that $(q_1q_2\cdots q_s)^{a-b} \leq (q_1q_2\cdots q_s)^{(a-b)x}$. Hence,

$$\psi_q(a+bx) - \psi_q(b+ax) \geq 0,$$

which completes the proof. □

Lemma 2.3. *Let $x \in [0,1]$, $0 < q_i < 1$ for all $i = 1,2,\dots,s$. Let a,b be any two positive real numbers such that $a \geq b$ and $\psi_{q_1,\dots,q_s}(b+ax) > 0$. Let c,d be any two positive real numbers such that $bc \geq ad$. Then*

$$bc\psi_{q_1,\dots,q_s}(a+bx) - ad\psi_{q_1,\dots,q_s}(b+ax) \geq 0.$$

Proof. Lemma 2.2 together with the inequality $\psi_{q_1,\dots,q_s}(b+ax) > 0$ gives that $\psi_{q_1,\dots,q_s}(a+bx) > 0$. Thus from Lemma 2.2 one obtains

$$bc\psi_{q_1,\dots,q_s}(a+bx) \geq ad\psi_{q_1,\dots,q_s}(a+bx) \geq ad\psi_{q_1,\dots,q_s}(b+ax),$$

as required. □

Now we present the $(q_1,\dots,q_s)$-inequality of (3) .

Theorem 2.4. *Let $x \in [0,1]$, $0 < q_i < 1$ for all $i = 1,2,\dots,s$, $a \geq b > 0, c,d$ positive real numbers with $bc \geq ad$ and $\psi_{q_1,\dots,q_s}(b+ax) > 0$. Then*

$$\frac{\Gamma_{q_1,\dots,q_s}(a)^c}{\Gamma_{q_1,\dots,q_s}(b)^d} \leq \frac{\Gamma_{q_1,\dots,q_s}(a+bx)^c}{\Gamma_{q_1,\dots,q_s}(b+ax)^d} \leq \frac{\Gamma_{q_1,\dots,q_s}(a+b)^c}{\Gamma_{q_1,\dots,q_s}(a+b)^d}.$$

Proof. Let $f(x) = \frac{\Gamma_{q_1,\dots,q_s}(a+bx)^c}{\Gamma_{q_1,\dots,q_s}(b+ax)^d}$ and $g(x) = \log f(x)$. Then

$$g(x) = c\log\Gamma_{q_1,\dots,q_s}(a+bx) - d\log\Gamma_{q_1,\dots,q_s}(b+ax),$$

which implies that

$$g'(x) = \frac{d}{dx}g(x) = bc\frac{\Gamma'_{q_1,\dots,q_s}(a+bx)}{\Gamma_{q_1,\dots,q_s}(a+bx)} - ad\frac{\Gamma'_{q_1,\dots,q_s}(b+ax)}{\Gamma_{q_1,\dots,q_s}(b+ax)}$$

$$= bc\psi_{q_1,\dots,q_s}(a+bx) - ad\psi_{q_1,\dots,q_s}(b+ax).$$

Thus, Lemma 2.3 gives $g'(x) \geq 0$, that is, g is an increasing function on $[0,1]$. Therefore, f is an increasing function on $[0,1]$. Hence, for all $x \in [0,1]$ we have $f(0) \leq f(x) \leq f(1)$, which is equivalent to

$$\frac{\Gamma_{q_1,\dots,q_s}(a)^c}{\Gamma_{q_1,\dots,q_s}(b)^d} \leq \frac{\Gamma_{q_1,\dots,q_s}(a+bx)^c}{\Gamma_{q_1,\dots,q_s}(b+ax)^d} \leq \frac{\Gamma_{q_1,\dots,q_s}(a+b)^c}{\Gamma_{q_1,\dots,q_s}(a+b)^d},$$

as requested. □

Using the similar arguments of proofs as in Lemmas 2.2 - 2.3 and Theorem 2.4 we obtain the following results.

Lemma 2.5. *Let $x \geq 1$, $0 < q_i < 1$ for all $i = 1,2,\ldots,s$, and a,b be any two positive real numbers with $b \geq a$. Then*

$$\psi_{q_1,\ldots,q_s}(a+bx) \geq \psi_{q_1,\ldots,q_s}(b+ax).$$

Lemma 2.6. *Let $x \geq 1$, $0 < q_i < 1$ for all $i = 1,2,\ldots,s$, a,b be any two positive real numbers with $b \geq a$ and $\psi_{q_1,\ldots,q_s}(b+ax) > 0$, and c,d be any two positive real numbers such that $bc \geq ad$. Then*

$$bc\psi_{q_1,\ldots,q_s}(a+bx) - ad\psi_{q_1,\ldots,q_s}(b+ax) \geq 0.$$

By the similar techniques as in the proof of Theorem 2.4 with using Lemmas 2.5 and 2.6 instead Lemmas 2.2 and 2.3 the following result can be proved.

Theorem 2.7. *Let $x \geq 1$, $0 < q_i < 1$ for all $i = 1,2,\ldots,s$, a,b be any two positive real numbers with $b \geq a$ and $\psi_{q_1,\ldots,q_s}(b+ax) > 0$, and c,d be any two positive real numbers such that $bc \geq ad$. Then $\frac{\Gamma_{q_1,\ldots,q_s}(a+bx)^c}{\Gamma_{q_1,\ldots,q_s}(b+ax)^d}$ is an increasing function on* $[1,+\infty)$.

In addition, with similar arguments as in the proof of Lemma 2.3 we obtain the following lemmas.

Lemma 2.8. *Let $x \in [0,1]$, $0 < q_i < 1$ for all $i = 1,2,\ldots,s$, a,b be any two positive real numbers with $a \geq b$ and $\psi_{q_1,\ldots,q_s}(a+bx) < 0$, and c,d be any two positive real numbers such that $ad \geq bc$. Then*

$$bc\psi_{q_1,\ldots,q_s}(a+bx) - ad\psi_{q_1,\ldots,q_s}(b+ax) \geq 0.$$

Lemma 2.9. *Let $x \geq 1$, $0 < q_i < 1$ for all $i = 1,2,\ldots,s$, a,b be any two positive real numbers with $b \geq a$ and $\psi_{q_1,\ldots,q_s}(a+bx) < 0$, and c,d be any two positive real numbers such that $ad \geq bc$. Then*

$$bc\psi_{q_1,\ldots,q_s}(a+bx) - ad\psi_{q_1,\ldots,q_s}(b+ax) \geq 0.$$

Again, by similar techniques as in the proof of Theorem 2.4 and using Lemmas 2.3 and 2.8 we get the following.

Theorem 2.10. *Let $x \in [0,1]$, $0 < q_i < 1$ for all $i = 1,2,\ldots,s$, a,b be any two positive real numbers with $a \geq b$ and $\psi_{q_1,\ldots,q_s}(a+bx) < 0$, and c,d be any two positive real numbers such that $ad \geq bc$. Then $\frac{\Gamma_{q_1,\ldots,q_s}(a+bx)^c}{\Gamma_{q_1,\ldots,q_s}(b+ax)^d}$ is an increasing function on* $[0,1]$.

Finally, by similar techniques as in the proof of Theorem 2.4 and using Lemmas 2.5 and 2.9 we obtain the following.

Theorem 2.11. *Let $x \geq 1$, $0 < q_i < 1$ for all $i = 1,2,\ldots,s$, a,b be any two positive real numbers with $b \geq a$ and $\psi_{q_1,\ldots,q_s}(a+bx) < 0$, and c,d be any two positive real numbers such that $ad \geq bc$. Then $\frac{\Gamma_{q_1,\ldots,q_s}(a+bx)^c}{\Gamma_{q_1,\ldots,q_s}(b+ax)^d}$ is an increasing function on $[1,+\infty)$.*

3. Further results

In this section we present several generalization of the above results.

3.1. The case $q_1,q_2,\ldots,q_s > 1$

On the case $q_1,q_2,\ldots,q_s > 1$ we define the $(q_1,\ldots,q_s)$-analogue of gamma function as

$$\Gamma_{q_1,\ldots,q_s}(x) = \frac{((q_1q_2\cdots q_s)^{-1};q_1^{-1},\ldots,q_s^{-1})_\infty}{((q_1q_2\cdots q_s)^{-x};q_1^{-1},\ldots,q_s^{-1})_\infty}(q_1q_2\cdots q_s-1)^{1-x}(q_1q_2\cdots q_s)^{\binom{x}{2}},$$

for all $q_1,q_2,\ldots,q_s > 1$. Note that

$$\Gamma_{q_1,\ldots,q_s}(1) = \Gamma_{q_1,\ldots,q_s}(2) = 1. \tag{6}$$

In this case the $(q_1,\ldots,q_s)$-analogue of the psi function

$$\psi_{q_1,\ldots,q_s}(x) = \frac{d}{dx}\log\Gamma_{q_1,\ldots,q_s}(x)$$

has the following series representation

$$\begin{aligned}\psi_{q_1,\ldots,q_s}(x) &= (x-\frac{1}{2})\sum_{i=1}^{s}\log q_i - \log(q_1q_2\cdots q_s-1)\\ &\quad+\sum_{i=1}^{s}\log q_i\cdot\sum_{j_1,\ldots,j_s\geq 0}\frac{1}{1-\prod_{i=1}^{s}q_i^{j_i+x}},\end{aligned} \tag{7}$$

for all $q_1,q_2,\ldots,q_s > 1$. Using this representation we will be able to give the $(q_1,\ldots,q_s)$-analogue of our results.

Theorem 3.1. *Let $x \geq 0$, $q_i > 1$ for all $i = 1,2,\ldots,s$. Let a be a real number.*

1) If $a \in [0,1]$ then

$$1 \leq \frac{\Gamma_{q_1,\ldots,q_s}(1+x)^a}{\Gamma_{q_1,\ldots,q_s}(1+ax)} \leq \frac{1}{\Gamma_{q_1,\ldots,q_s}(a+1)}.$$

2) If $a > 1$ then

$$\frac{1}{\Gamma_{q_1,\ldots,q_s}(a+1)} \leq \frac{\Gamma_{q_1,\ldots,q_s}(1+x)^a}{\Gamma_{q_1,\ldots,q_s}(1+ax)} \leq 1.$$

Proof. Let $f(x)=\frac{\Gamma_{q_1,\dots,q_s}(1+x)^a}{\Gamma_{q_1,\dots,q_s}(1+ax)}$ and $g(x)=\log f(x)$. Then

$$g(x)=a\log\Gamma_{q_1,\dots,q_s}(1+x)-\log\Gamma_{q_1,\dots,q_s}(1+ax),$$

which implies that

$$g'(x)=a(\psi_{q_1,\dots,q_s}(1+x)-\psi_{q_1,\dots,q_s}(1+ax)).$$

The series representation of $\psi_{q_1,\dots,q_s}(x)$, see (7), gives

$$\begin{aligned}&\psi_{q_1,\dots,q_s}(1+x)-\psi_{q_1,\dots,q_s}(1+ax)\\&=\sum_{i=1}^{s}\log q_i\left(x(1-a)+\sum_{j_1,\dots,j_s\geq 0}\frac{(1-(q_1q_2\cdots q_s)^{(a-1)x})\prod_{i=1}^{s}q_i^{j_i+1+x}}{(1-\prod_{i=1}^{s}q_i^{j_i+1+x})(1-\prod_{i=1}^{s}q_i^{j_i+1+ax})}\right).\end{aligned}$$

1) Since $q_i>1$ we have that $\log q_i>0$, for all $i=1,2,\dots,s$. In addition, since $a\leq 1$ and $x\geq 0$ we get that $g'(x)\geq 0$, that is, g is an increasing function on $x\geq 0$. Therefore, f is an increasing function on $x>0$. Hence, for all $x\geq 0$ we have $f(0)\leq f(x)\leq f(1)$, which is equivalent to

$$\frac{\Gamma_{q_1,\dots,q_s}(1)^a}{\Gamma_{q_1,\dots,q_s}(1)}\leq\frac{\Gamma_{q_1,\dots,q_s}(1+x)^a}{\Gamma_{q_1,\dots,q_s}(1+ax)}\leq\frac{\Gamma_{q_1,\dots,q_s}(2)^a}{\Gamma_{q_1,\dots,q_s}(1+a)}.$$

Using (6) the desired result follows.

2) In a similar way as in previous case. □

As we can see from the proof of Theorem 2.1 and Theorem 3.1 that all the results in the previous section can be extend to the case $q_1,q_2,\dots,q_s>1$ by using a simple modification of the proofs.

3.2. The case $q_1,q_2,\dots,q_k\in(0,1)$ and $q_{k+1},q_{k+2},\dots,q_s>1$

In this case we define the $(q_1,\dots,q_s)$-analogue of gamma function as

$$\begin{aligned}\Gamma_{q_1,\dots,q_s}(x)=&\frac{(q_1q_2\cdots q_k;q_1,q_2,\dots,q_k)_\infty}{((q_1q_2\cdots q_k)^x;q_1,q_2,\dots,q_k)_\infty}(1-q_1q_2\cdots q_k)^{1-x}\\&\cdot\frac{((q_{k+1}q_{k+2}\cdots q_s)^{-1};q_{k+1}^{-1},\dots,q_s^{-1})_\infty}{((q_{k+1}q_{k+2}\cdots q_s)^{-x};q_{k+1}^{-1},\dots,q_s^{-1})_\infty}\\&\cdot(q_{k+1}q_{k+2}\cdots q_s-1)^{1-x}(q_{k+1}q_{k+2}\cdots q_s)^{\binom{x}{2}},\end{aligned}$$

i.e.

$$\Gamma_{q_1,\dots,q_k,q_{k+1},\dots,q_s}(x)=\Gamma_{q_1,\dots,q_k}(x)\cdot\Gamma_{q_{k+1},\dots,q_s}(x)$$

In this case the $(q_1,\dots,q_s)$-analogue of the psi function

$$\psi_{q_1,\dots,q_s}(x)=\frac{d}{dx}\log\Gamma_{q_1,\dots,q_s}(x)$$

has the following series representation

$$\begin{aligned}\psi_{q_1,\dots,q_s}(x)=&-\log(1-q_1q_2\cdots q_k)+\sum_{i=1}^{k}\log q_i\cdot\sum_{j_1,\dots,j_s\geq 0}\frac{\prod_{i=1}^{k}q_i^{j_i+x}}{1-\prod_{i=1}^{k}q_i^{j_i+x}}\\&+(x-\frac{1}{2})\sum_{i=k+1}^{s}\log q_i-\log(q_{k+1}\cdots q_s-1)\\&+\sum_{i=k+1}^{s}\log q_i\cdot\sum_{j_{k+1},\dots,j_s\geq 0}\frac{1}{1-\prod_{i=k+1}^{s}q_i^{j_i+x}}.\end{aligned}$$

Now we are able to give the $(q_1,q_2,\dots,q_s)$-analogue of previous results:

Theorem 3.2. *Let $x\geq 0$, $q_1,\dots,q_k\in(0,1)$, $q_{k+1},\dots,q_s>1$. Let a be a real number.*

(1) If $a\geq 1$ then

$$\frac{1}{\Gamma_{q_1,\dots,q_s}(a+1)}\leq\frac{\Gamma_{q_1,\dots,q_s}(1+x)^a}{\Gamma_{q_1,\dots,q_s}(1+ax)}\leq 1.$$

(2) If $a\in[0,1)$ then

$$1\leq\frac{\Gamma_{q_1,\dots,q_s}(1+x)^a}{\Gamma_{q_1,\dots,q_s}(1+ax)}\leq\frac{1}{\Gamma_{q_1,\dots,q_s}(a+1)}.$$

Proof. Let $f(x)=\frac{\Gamma_{q_1,\dots,q_s}(1+x)^a}{\Gamma_{q_1,\dots,q_s}(1+ax)}$ and $g(x)=\log f(x)$. Then

$$g(x)=a\log\Gamma_{q_1,\dots,q_s}(1+x)-\log\Gamma_{q_1,\dots,q_s}(1+ax),$$

which implies that

$$g'(x)=a(\psi_{q_1,\dots,q_s}(1+x)-\psi_{q_1,\dots,q_s}(1+ax)).$$

Using the series representation one obtains:

$$\begin{aligned}&\psi_{q_1,\dots,q_s}(1+x)-\psi_{q_1,\dots,q_s}(1+ax)\\&=\sum_{i=1}^{k}\log q_i\cdot\sum_{j_1,\dots,j_k\geq 0}\frac{(1-(q_1\cdots q_k)^{(a-1)x})\prod_{i=1}^{k}q_i^{j_i+1+x}}{(1-\prod_{i=1}^{k}q_i^{j_i+1+x})(1-\prod_{i=1}^{k}q_i^{j_i+1+ax})}+\\&\sum_{i=k+1}^{s}\log q_i\left(x(1-a)+\sum_{j_{k+1},\dots,j_s\geq 0}\frac{(1-(q_{k+1}\cdots q_s)^{(a-1)x})\prod_{i=k+1}^{s}q_i^{j_i+1+x}}{(1-\prod_{i=k+1}^{s}q_i^{j_i+1+x})(1-\prod_{i=k+1}^{s}q_i^{j_i+1+ax})}\right)\end{aligned}$$

Since $0 < q_i < 1$ we have $\log q_i < 0$, for all $i = 1, 2, \ldots, k$. In addition, since $a \geq 1$ and $x \geq 0$ we obtain $(q_1 \cdots q_s)^{(a-1)x} \leq 1$. So the first member of previous sum is negative.

Since $q_j > 1$ we have $\log q_j > 0$, for all $j = k+1, \ldots, s$. In addition, for $a \geq 1$ and $x \geq 0$ we obtain $(q_{k+1} \cdots q_s)^{(a-1)x} \geq 1$. So the second member of previous sum is also negative.

Hence, $g'(x) \leq 0$, that is, g is a decreasing function for $x \geq 0$. Therefore, f is a decreasing function for $x \geq 0$. So, for $x \in [0,1]$ we have $f(1) \leq f(x) \leq f(0)$, which is equivalent to

$$\frac{\Gamma_{q_1,\ldots,q_s}(2)^a}{\Gamma_{q_1,\ldots,q_s}(1+a)} \leq \frac{\Gamma_{q_1,\ldots,q_s}(1+x)^a}{\Gamma_{q_1,\ldots,q_s}(1+ax)} \leq \frac{\Gamma_{q_1,\ldots,q_s}(1)^a}{\Gamma_{q_1,\ldots,q_s}(1)}.$$

Using (6) the desired result follows.

In a similar way, one can easily prove the other case. □

3.3. Increasing functions

In a similar way as in Theorem 2.1, Theorem 3.1 and Theorem 3.2 one can prove the following generalized theorems.

Theorem 3.3. *Let $x \in [0,1]$. Let f be an increasing, positive, differentiable function. Let $0 < q_i < 1$ for all $i = 1, 2, \ldots, s$. Let a be a real number.*

(1) If $a \geq 1$ then

$$\frac{1}{\Gamma_{q_1,\ldots,q_s}(a+1)} \leq \frac{\Gamma_{q_1,\ldots,q_s}(1+f(x))^a}{\Gamma_{q_1,\ldots,q_s}(1+af(x))} \leq 1.$$

(2) If $a \in [0,1)$ then

$$1 \leq \frac{\Gamma_{q_1,\ldots,q_s}(1+f(x))^a}{\Gamma_{q_1,\ldots,q_s}(1+af(x))} \frac{1}{\Gamma_{q_1,\ldots,q_s}(a+1)}.$$

Theorem 3.4. *Let $x \geq 0$. Let f be an increasing, positive, differentiable function. Let $q_i > 1$ for all $i = 1, 2, \ldots, s$. Let a be a real number.*

(1) If $a \in [0,1]$, then

$$1 \leq \frac{\Gamma_{q_1,\ldots,q_s}(1+f(x))^a}{\Gamma_{q_1,\ldots,q_s}(1+af(x))} \leq \frac{1}{\Gamma_{q_1,\ldots,q_s}(a+1)}.$$

(2) If $a > 1$ then

$$\frac{1}{\Gamma_{q_1,\ldots,q_s}(a+1)} \leq \frac{\Gamma_{q_1,\ldots,q_s}(1+f(x))^a}{\Gamma_{q_1,\ldots,q_s}(1+af(x))} \leq 1.$$

Theorem 3.5. *Let $x \geq 0$. Let f be an increasing, positive, differentiable function. Let $q_1, \ldots, q_k \in (0,1)$, $q_{k+1}, \ldots, q_s > 1$.*

(1) If $a \geq 1$ then

$$\frac{1}{\Gamma_{q_1,\ldots,q_s}(a+1)} \leq \frac{\Gamma_{q_1,\ldots,q_s}(1+f(x))^a}{\Gamma_{q_1,\ldots,q_s}(1+af(x))} \leq 1.$$

(2) If $a \in [0,1)$ then

$$1 \leq \frac{\Gamma_{q_1,\ldots,q_s}(1+f(x))^a}{\Gamma_{q_1,\ldots,q_s}(1+af(x))} \leq \frac{1}{\Gamma_{q_1,\ldots,q_s}(a+1)}.$$

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