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# EXISTENCE OF SOLUTIONS TO INTEGRAL TYPE BVPs FOR SECOND ORDER ODEs ON THE WHOLE LINE

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Some integral type boundary value problems (BVPs) for second order differential equations (ODEs) with one-dimensional $p$-Laplacian on the whole line are discussed. Sufficient conditions to guarantee the existence of solutions are established. The emphasis is put on the one-dimensional $p$-Laplacian term $[\rho(t)\Phi(x'(t))]'$ involving a nonnegative function $\rho$ that may satisfy $\rho(0) = 0$, and on the fact that the differential equations are defined on the whole line.

## 1. Introduction

The study of multi-point boundary-value problems for linear second order ordinary differential equations was initiated by Il'in and Moiseev [1]. Since then, more general nonlinear multi-point boundary-value problems were studied by several authors, see the text books [2,3] and the survey papers [4,5] and the references cited there.

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In [6], a class of BVPs for the second order nonlinear ordinary differential equations on the whole line was studied. Two theorems were proved. The first theorem in [6] is established by the use of the Schauder fixed point theorem and concerns the existence of solutions, while the second theorem is concerned with the existence and uniqueness of solutions and is derived by the Banach contraction principle.

In [7], Bianconi and Papalini investigate the existence of solutions of the following boundary value problem

$$\begin{cases} [\Phi(x'(t))]' + a(t,x(t))b(x(t),x'(t)) = 0, \quad t \in R, \\ \lim\limits_{t\to-\infty} x(t) =: x(-\infty) = 0, \\ \lim\limits_{t\to+\infty} x(t) =: x(+\infty) = 1, \end{cases} \tag{1}$$

where $\Phi$ is a monotone function which generalizes the one-dimensional $p-$ Laplacian operator, $a,b$ are continuous functions. Some criteria for the existence and non-existence of solutions of BVP(1) are established in [10].

In [8,12], Avramescu and Vladimirescu study the following boundary value problem

$$\begin{cases} x''(t) + 2f(t)x'(t) + x(t) + g(t,x(t)) = 0, \quad t \in R, \\ \lim\limits_{t\to\pm\infty} x(t) =: x(\pm\infty) = 0, \\ \lim\limits_{t\to\pm\infty} x'(t) =: x(\pm\infty) = 0, \end{cases} \tag{2}$$

where $f$ and $g$ are given functions. The existence of solutions of BVP(2) is obtained in [8].

In [9], Avramescu and Vladimirescu study the following boundary value problem

$$\begin{cases} x''(t) + f(t,x(t),x'(t)) = 0, \quad t \in R, \\ \lim\limits_{t\to-\infty} x(t) = \lim\limits_{t\to+\infty} x(t), \\ \lim\limits_{t\to-\infty} x'(t) = \lim\limits_{t\to+\infty} x'(t), \end{cases} \tag{3}$$

under adequate hypothesis and using the Bohnenblust-Karlin fixed point theorem, the existence of solutions of BVP(3) is established in [12].

Cabada and Cid [13] prove the solvability of the boundary value problem on the whole line

$$\begin{cases} [\Phi(x'(t))]' + f(t,x(t),x'(t)) = 0, \quad t \in R, \\ \lim\limits_{t\to-\infty} x(t) = -1, \\ \lim\limits_{t\to+\infty} x(t) = 1, \end{cases} \tag{4}$$

where $f$ is a continuous function, $\Phi : (-a,a) \to R$ is a homeomorphism with $a \in (0,+\infty)$, i.e., $\Phi$ is singular.

Calamai [10] and Cristina Marcelli, Francesca Papalini [11] discuss the solvability of the following strongly nonlinear BVP:

$$\begin{cases} [a(x(t))\Phi(x'(t))]' + f(t,x(t),x'(t)) = 0, \quad t \in R, \\ \lim\limits_{t\to-\infty} x(t) = \alpha, \\ \lim\limits_{t\to+\infty} x(t) = \beta, \end{cases} \tag{5}$$

where $\alpha < \beta$, $\Phi$ is a general increasing homeomorphism with bounded domain (singular $\Phi$-Laplacian), $a$ is a positive, continuous function and $f$ is a Carathéodory nonlinear function. Conditions for the existence and non- existence of heteroclinic solutions in terms of the behavior of $y \to f(t,x,y)$ and $y \to \Phi(y)$ as $y \to 0$, and of $t \to f(t,x,y)$ as $|t| \to +\infty$ are obtained. The approach is based on fixed point techniques suitably combined to the method of upper and lower solutions.

Motivated by mentioned papers, we consider the more general BVP for second order differential equation on the whole line coupled with the integral type BCs, i.e. the BVP

$$\begin{cases} [\rho(t)\Phi(x'(t))]' + f(t,x(t),x'(t)) = 0, \quad t \in R, \\ \alpha \lim\limits_{t\to-\infty} x(t) - \beta \lim\limits_{t\to-\infty} \Phi^{-1}(\rho(t))x'(t) = \int\limits_{-\infty}^{+\infty} g(s,x(s),x'(s))ds, \\ \gamma \lim\limits_{t\to+\infty} x(t) + \delta \lim\limits_{t\to+\infty} \Phi^{-1}(\rho(t))x'(t) = \int\limits_{-\infty}^{+\infty} h(s,x(s),x'(s))ds, \end{cases} \tag{6}$$

where

- $\alpha \ge 0, \gamma \ge 0, \beta \ge 0, \delta \ge 0$ are constants with
$$\sigma = \alpha\delta + \alpha\gamma \int_{-\infty}^{+\infty} \frac{1}{\Phi^{-1}(\rho(s))} ds + \beta\gamma > 0,$$
- $f, g, h$ defined on $R^3$ are nonnegative Carathéodory functions,
- $\rho \in C(R,(0,\infty))$ with $\rho(t) > 0$ for all $t \in R$ and $t \neq 0$, $\rho$ satisfies
$$\int\limits_{-\infty}^{-1} \frac{1}{\Phi^{-1}(\rho(s))} ds < +\infty, \quad \int\limits_{-1}^{0} \frac{1}{\Phi^{-1}(\rho(s))} ds < +\infty,$$
$$\int\limits_{0}^{1} \frac{1}{\Phi^{-1}(\rho(s))} ds < +\infty, \quad \int\limits_{1}^{+\infty} \frac{1}{\Phi^{-1}(\rho(s))} ds < +\infty.$$
- $\Phi(s) = |s|^{p-2}s$ with $p > 1$ is called $p-$Laplacian and its inverse function is denoted by $\Phi^{-1}(s) = |s|^{q-2}s$ with $\frac{1}{p} + \frac{1}{q} = 1$.

The purpose is to establish sufficient conditions for the existence of a solution of BVP(6). The results in this paper generalize and improve some known ones since the one-dimensional $p$-Laplacian term $[\rho(t)\Phi(x'(t))]'$ involves the nonnegative function $\rho$ that may satisfy $\rho(0) = 0$.

The remainder of this paper is organized as follows: the preliminary results are given in Section 2, the main results are presented in Section 3.

## 2. Preliminary Results

In this section we present some background definitions and state the Nonlinear alternative principle. Then the preliminary results are given and proved.

**Definition 2.1.** $F$ defined on $R^3$ is called a Carathéodory function if

i) $t \to f\left(t, x, \frac{1}{\Phi^{-1}(\rho(t))}y\right)$ is measurable for any $(x, y) \in R^2$,

ii) $(x, y) \to f\left(t, x, \frac{1}{\Phi^{-1}(\rho(t))}y\right)$ is continuous for a.e. $t \in R$,

iii) for each $r > 0$, there exists nonnegative function $\phi_r \in L^1(R)$ such that $|u|, |v| \le r$ implies

$$\left| f\left(t, x, \frac{1}{\Phi^{-1}(\rho(t))}y\right) \right| \le \phi_r(t), a.e. t \in R.$$

**Lemma 2.2.** [Nonlinear alternative]. *Let $C$ be a convex subset of a normed linear space $X$, and $U$ be an open subset of $C$, with $p^* \in U$. Then, for every completely continuous map $T : \overline{U} \to C$, the following alternative holds:*

*(a) either $T$ has a fixed point in $U$ or*

*(b) there is an $x \in \partial U$, with $x = (1-\lambda)p^* + \lambda Tx$ for some $0 < \lambda < 1$.*

Let us list the assumptions:

(H) $f, g$ and $h$ are Carathéodory functions and satisfy the following:

(i) there exist nonnegative functions $a, b, c \in L^1(R)$ such that

$$\left| f\left(t, x, \frac{1}{\Phi^{-1}(\rho(t))}y\right) \right| \le a(t) + b(t)\Phi(|x|) + c(t)\Phi(|y|), t \in R \qquad (7)$$

(ii) there exist nonnegative functions $a_1, b_1, c_1 \in L^1(R)$ such that

$$\left| g\left(t, x, \frac{1}{\Phi^{-1}(\rho(t))}y\right) \right| \le a_1(t) + b_1(t)|x| + c_1(t)|y|, t \in R, \qquad (8)$$

(iii) there exist nonnegative functions $a_2, b_2, c_2$ such that

$$\left|h\left(t, x, \frac{1}{\Phi^{-1}(\rho(t))}y\right)\right| \leq a_2(t) + b_2(t)|x| + c_2(t)|y|, t \in R. \tag{9}$$

Choose

$$X = \left\{ x \in C^0(R) : \begin{array}{c} x \text{ is bounded on } R \\ t \to \Phi^{-1}(\rho(t))x'(t) \text{ is continuous} \\ \text{and there exist the limits} \\ \lim_{t\to-\infty} x(t) \\ \lim_{t\to+\infty} x(t) \\ \lim_{t\to-\infty} \Phi^{-1}(\rho(t))x'(t) \\ \lim_{t\to+\infty} \Phi^{-1}(\rho(t))x'(t) \end{array} \right\}.$$

For $x \in X$, define the norm of $x$ by

$$||x|| = \max\left\{ \sup_{t\in R} |x(t)|,\ \sup_{t\in R} \Phi^{-1}(\rho(t))|x'(t)| \right\}.$$

One can prove that $X$ is a Banach space with the norm $||x||$ for $x \in X$.

Denote

$$\sigma = \alpha\delta + \alpha\gamma \int_{-\infty}^{+\infty} \frac{1}{\Phi^{-1}(\rho(s))} ds + \beta\gamma > 0,$$

and given $x \in X$,

$$\sigma_0 = \frac{\gamma \int_{-\infty}^{+\infty} g(r, x(r), x'(r))dr - \alpha \int_{-\infty}^{+\infty} h(r, x(r), x'(r))dr}{\sigma},$$

$$\sigma_1 = -\left| \int_{-\infty}^{+\infty} f(s, x(s), x'(s))ds \right| - \Phi(\sigma_0),$$

$$\sigma_2 = \left| \int_{-\infty}^{+\infty} f(s, x(s), x'(s))ds \right| - \Phi(\sigma_0).$$

**Lemma 2.3.** *Suppose that $x \in X$ and (H) holds. Then there exists a unique*

$$A_x \in [\sigma_1, \sigma_2] \tag{10}$$

*such that*

$$\alpha\delta\Phi^{-1}(A_x)+\alpha\gamma\int_{-\infty}^{+\infty}\frac{1}{\Phi^{-1}(\rho(s))}\Phi^{-1}\left(A_x+\int_s^{+\infty}f(r,x(r),x'(r))dr\right)ds$$
$$+\beta\gamma\Phi^{-1}\left(A_x+\int_{-\infty}^{+\infty}f(r,x(r),x'(r))dr\right)$$
$$+\gamma\int_{-\infty}^{+\infty}g(r,x(r),x'(r))dr-\alpha\int_{-\infty}^{+\infty}h(r,x(r),x'(r))dr=0.$$

*Furthermore, it holds that*

$$|A_x|\leq\int_{-\infty}^{+\infty}|f(s,x(s),x'(s))|ds+\Phi(|\sigma_0|)\,. \tag{11}$$

*Proof.* Since $x\in X$, $f,g,h$ are Carathéodory functions, then

$$||x||=\max\left\{\sup_{t\in R}|x(t)|,\ \sup_{t\in R}\Phi^{-1}(\rho(t))|x'(t)|\right\}<+\infty,$$

and

$$\int_{-\infty}^{+\infty}f(r,x(r),x'(r))dr,\ \int_{-\infty}^{+\infty}g(r,x(r),x'(r))dr,\ \int_{-\infty}^{+\infty}h(r,x(r),x'(r))dr$$

converge. Let

$$G(c)=\alpha\delta\Phi^{-1}(c)$$
$$+\alpha\gamma\int_{-\infty}^{+\infty}\frac{1}{\Phi^{-1}(\rho(s))}\Phi^{-1}\left(c+\int_s^{+\infty}f(r,x(r),x'(r))dr\right)ds$$
$$+\beta\gamma\Phi^{-1}\left(c+\int_{-\infty}^{+\infty}f(r,x(r),x'(r))dr\right)$$
$$+\gamma\int_{-\infty}^{+\infty}g(r,x(r),x'(r))dr-\alpha\int_{-\infty}^{+\infty}h(r,x(r),x'(r))dr.$$

It is easy to see from $\sigma>0$ that $G(c)$ is strictly increasing on $R$.

On the other hand, we find that

$$
\begin{aligned}
G(\sigma_1) &= \alpha\delta\Phi^{-1}\left(-\left|\int_{-\infty}^{+\infty} f(r,x(r),x'(r))dr\right| - \Phi(\sigma_0)\right) \\
&+ \alpha\gamma\int_{-\infty}^{+\infty}\frac{1}{\Phi^{-1}(\rho(s))}\Phi^{-1}\left(-\left|\int_{-\infty}^{+\infty} f(r,x(r),x'(r))dr\right| - \Phi(\sigma_0)\right. \\
&\left.+ \int_{s}^{+\infty} f(r,x(r),x'(r))dr\right)ds \\
&+ \beta\gamma\Phi^{-1}\left(-\left|\int_{-\infty}^{+\infty} f(r,x(r),x'(r))dr\right| - \Phi(\sigma_0)\right. \\
&\left.+ \int_{-\infty}^{+\infty} f(r,x(r),x'(r))dr\right) \\
&+ \gamma\int_{-\infty}^{+\infty} g(r,x(r),x'(r))dr - \alpha\int_{-\infty}^{+\infty} h(r,x(r),x'(r))dr \\
&\le \alpha\delta\Phi^{-1}(-\Phi(\sigma_0)) + \alpha\gamma\int_{-\infty}^{+\infty}\frac{1}{\Phi^{-1}(\rho(s))}\Phi^{-1}(-\Phi(\sigma_0))\,ds \\
&+ \beta\gamma\Phi^{-1}(-\Phi(\sigma_0)) \\
&+ \gamma\int_{-\infty}^{+\infty} g(r,x(r),x'(r))dr - \alpha\int_{-\infty}^{+\infty} h(r,x(r),x'(r))dr = 0.
\end{aligned}
$$

Similarly we find that

$$
\begin{aligned}
G(\sigma_2) &= \alpha\delta\Phi^{-1}\left(\left|\int_{-\infty}^{+\infty} f(r,x(r),x'(r))dr\right| - \Phi(\sigma_0)\right) \\
&+ \alpha\gamma\int_{-\infty}^{+\infty}\frac{1}{\Phi^{-1}(\rho(s))}\Phi^{-1}\left(\left|\int_{-\infty}^{+\infty} f(r,x(r),x'(r))dr\right| - \Phi(\sigma_0)\right. \\
&\left.+ \int_{s}^{+\infty} f(r,x(r),x'(r))dr\right)ds \\
&+ \beta\gamma\Phi^{-1}\left(\left|\int_{-\infty}^{+\infty} f(r,x(r),x'(r))dr\right| - \Phi(\sigma_0) + \int_{-\infty}^{+\infty} f(r,x(r),x'(r))dr\right) \\
&+ \gamma\int_{-\infty}^{+\infty} g(r,x(r),x'(r))dr - \alpha\int_{-\infty}^{+\infty} h(r,x(r),x'(r))dr \\
&\ge \alpha\delta\Phi^{-1}(-\Phi(\sigma_0)) + \alpha\gamma\int_{-\infty}^{+\infty}\frac{1}{\Phi^{-1}(\rho(s))}\Phi^{-1}(-\Phi(\sigma_0))\,ds \\
&+ \beta\gamma\Phi^{-1}(-\Phi(\sigma_0)) + \gamma\int_{-\infty}^{+\infty} g(r,x(r),x'(r))dr - \alpha\int_{-\infty}^{+\infty} h(r,x(r),x'(r))dr \\
&= 0.
\end{aligned}
$$

Hence there exists a unique

$$A_x \in \left[ -\left| \int_{-\infty}^{+\infty} f(s,x(s),x'(s))ds \right| - \Phi(\sigma_0), \left| \int_{-\infty}^{+\infty} f(s,x(s),x'(s))ds \right| - \Phi(\sigma_0) \right] = [\sigma_1, \sigma_2]$$

such that

$$\alpha\delta\Phi^{-1}(A_x) + \alpha\gamma \int_{-\infty}^{+\infty} \frac{1}{\Phi^{-1}(\rho(s))} \Phi^{-1}\left( A_x + \int_s^{+\infty} f(r,x(r),x'(r))dr \right) ds + \beta\gamma\Phi^{-1}\left( A_x + \int_{-\infty}^{+\infty} f(r,x(r),x'(r))dr \right) + \gamma \int_{-\infty}^{+\infty} g(r,x(r),x'(r))dr - \alpha \int_{-\infty}^{+\infty} h(r,x(r),x'(r))dr = 0.$$

It is easy to see that

$$|A_x| \le \int_{-\infty}^{+\infty} |f(s,x(s),x'(s))|ds + \Phi(|\sigma_0|) \le \int_{-\infty}^{+\infty} |f(s,x(s),x'(s))|ds + \Phi\left( \frac{\gamma \int_{-\infty}^{+\infty} |g(r,x(r),x'(r))|dr + \alpha \int_{-\infty}^{+\infty} |h(r,x(r),x'(r))|dr}{\sigma} \right).$$

The proof is complete. □

Define the operator $T$ on $X$ by

$$(Tx)(t) = B_x + \int_{-\infty}^{t} \frac{1}{\Phi^{-1}(\rho(s))} \Phi^{-1}\left( A_x + \int_s^{+\infty} f(r,x(r),x'(r))dr \right) ds, \quad (12)$$

where $A_x$ satisfies

$$\alpha\delta\Phi^{-1}(A_x) + \alpha\gamma \int_{-\infty}^{+\infty} \frac{1}{\Phi^{-1}(\rho(s))} \Phi^{-1}\left( A_x + \int_s^{+\infty} f(r,x(r),x'(r))dr \right) ds + \beta\gamma\Phi^{-1}\left( A_x + \int_{-\infty}^{+\infty} f(r,x(r),x'(r))dr \right) + \gamma \int_{-\infty}^{+\infty} g(r,x(r),x'(r))dr - \alpha \int_{-\infty}^{+\infty} h(r,x(r),x'(r))dr = 0,$$

and

$$B_x = \frac{\int_{-\infty}^{+\infty} g(r,x(r),x'(t))dr + \beta\Phi^{-1}\left( A_x + \int_{-\infty}^{+\infty} f(r,x(r),x'(r))dr \right)}{\alpha} \text{ for } \alpha > 0,$$

and

$$B_x = \frac{\int_{-\infty}^{+\infty} h(r,x(r),x'(r))dr - \delta\Phi^{-1}(A_x) - \gamma\int_{-\infty}^{+\infty} \frac{1}{\Phi^{-1}(\rho(s))}\Phi^{-1}\left(A_x + \int_s^{+\infty} f(r,x(r),x'(r))dr\right)ds}{\gamma}, \gamma > 0.$$

**Lemma 2.4.** *Suppose that (H) holds. Then*

*(i) $T : X \to X$ is well defined,*

*(ii) it holds that*

$$\begin{cases} [\rho(t)\Phi((Tx)'(t))]' + f(t,x(t),x'(t)) = 0, \quad t \in R, \\ \alpha \lim_{t\to-\infty}(Tx)(t) - \beta \lim_{t\to-\infty}\Phi^{-1}(\rho(t))(Tx)'(t) \\ \qquad = \int_{-\infty}^{+\infty} g(s,x(s),x'(s))ds, \\ \gamma \lim_{t\to+\infty}(Tx)(t) + \delta \lim_{t\to+\infty}\Phi^{-1}(\rho(t))(Tx)'(t) \\ \qquad = \int_{-\infty}^{+\infty} h(s,x(s),x'(s))ds, \end{cases} \tag{13}$$

*(iii) $x \in X$ is a solution of BVP(6) if and only if $x$ is a fixed point of $T$ in $X$,*

*(iv) $T : X \to X$ is completely continuous.*

*Proof.* We consider the case $\alpha > 0$. The proof of the case $\gamma > 0$ is similar and is omitted.
(i) For $x \in X$, by Lemma 2.3, both $A_x$ and $B_x$ are unique respectively. So $Tx$ is defined. We need to prove $Tx \in X$. From (12), $Tx \in C^0(R)$ and there exist the limits

$$\lim_{t\to-\infty}(Tx)(t) = B_x,$$
$$\lim_{t\to+\infty}(Tx)(t) = B_x + \int_{-\infty}^{+\infty}\frac{1}{\Phi^{-1}(\rho(s))}\Phi^{-1}\left(A_x + \int_s^{+\infty} f(r,x(r),x'(r))dr\right)ds.$$

Furthermore,

$$\Phi^{-1}(\rho(t))(Tx)'(t) = \Phi^{-1}\left(A_x + \int_t^{+\infty} f(r,x(r),x'(r))dr\right). \tag{14}$$

It is easy to see that then $t \to \Phi^{-1}(\rho(t))(Tx)' \in C^0(R)$ and there exist the limits

$$\lim_{t\to-\infty}\Phi^{-1}(\rho(t))(Tx)'(t) = \Phi^{-1}\left(A_x + \int_{-\infty}^{+\infty} f(r,x(r),x'(r))dr\right),$$
$$\lim_{t\to+\infty}\Phi^{-1}(\rho(t))(Tx)'(t) = \Phi^{-1}(A_x).$$

It follows that $T: X \to X$ is well defined.
(ii) From (12) and (14), we get

$$[\rho(t)\Phi((Tx)'(t))]' + f(t,x(t),x'(t)) = 0,\ t \in R,$$
$$\alpha \lim_{t\to-\infty}(Tx)(t) - \beta \lim_{t\to-\infty}\Phi^{-1}(\rho(t))(Tx)'(t)$$
$$= \alpha B_x - \beta\Phi^{-1}\left(A_x + \int_{-\infty}^{+\infty} f(r,x(r),x'(r))dr\right) = \int_{-\infty}^{+\infty} g(s,x(s),x'(s))ds,$$
$$\gamma \lim_{t\to+\infty}(Tx)(t) + \delta \lim_{t\to+\infty}\Phi^{-1}(\rho(t))(Tx)'(t) = \int_{-\infty}^{+\infty} h(s,x(s),x'(s))ds.$$

(iii) It is easy to see that $x \in X$ is a solution of BVP(6) if and only if $x$ is a fixed point of $T$ in $X$.
(iv) First, we prove that the function $A_x : X \to R$ is continuous in $x$.

Let $\{x_n\} \in X$ with $x_n \to x_0$ as $n \to \infty$. Let $\{A_{x_n}\}(n = 1,2,\ldots,m)$ be constants decided by equation

$$\alpha\delta\Phi^{-1}(A_{x_n}) + \alpha\gamma\int_{-\infty}^{+\infty}\frac{1}{\Phi^{-1}(\rho(s))}\Phi^{-1}\left(A_{x_n} + \int_s^{+\infty} f(r,x_n(r),x_n'(r))dr\right)ds$$
$$+\beta\gamma\Phi^{-1}\left(A_{x_n} + \int_{-\infty}^{+\infty} f(r,x_n(r),x_n'(r))dr\right)$$
$$+\gamma\int_{-\infty}^{+\infty} g(r,x_n(r),x_n'(t))dr - \alpha\int_{-\infty}^{+\infty} h(r,x_n(r),x_n'(r))dr = 0,$$

corresponding to $x_n$ $(n = 0,1,2,\ldots)$. Since $x_n \to x_0$ as $n \to \infty$, there exists an $M > 0$ such that $\|x_n\| \le M(n = 0,1,2,\cdots)$. The fact $f,g,h$ are Carathéodory functions means there exists $\phi_M \in L^1(R)$ such that

$$|f(t,x_n(t),x_n'(t))| = \left|f\left(t,x_n(t),\frac{1}{\Phi^{-1}(\rho(t))}\Phi^{-1}(\rho(t))x_n'(t)\right)\right|$$
$$\le \phi_M(t), t \in R,$$
$$|g(t,x_n(t),x_n'(t))| \le \phi_M(t), t \in R,$$
$$|h(t,x_n(t),x_n'(t))| \le \phi_M(t), t \in R.$$

Then

$$\int_{-\infty}^{\infty}|f(r,x_n(r),x_n'(r))|dr \le \int_{-\infty}^{\infty}\phi_M(r)dr < \infty,$$
$$\int_{-\infty}^{\infty}|g(r,x_n(r),x_n'(r))|dr \le \int_{-\infty}^{\infty}\phi_M(r)dr < \infty,$$
$$\int_{-\infty}^{\infty}|h(r,x_n(r),x_n'(r))|dr \le \int_{-\infty}^{\infty}\phi_M(r)dr < \infty.$$

So

$$
\begin{aligned}
&A_{x_n} \in \\
&\left[-\left|\int_{-\infty}^{\infty} f(r,x_n(r),x_n'(r))dr\right| - \Phi(\sigma_0), \left|\int_{-\infty}^{\infty} f(r,x_n(r),x_n'(r))dr\right| - \Phi(\sigma_0)\right] \\
&\qquad\subseteq \left[-\int_{-\infty}^{\infty} \phi_M(r)dr - \Phi\left(\frac{\gamma+\alpha}{\sigma}\int_{-\infty}^{+\infty} \phi_M(s)ds\right),\right. \\
&\qquad\qquad \left.\int_{-\infty}^{\infty} \phi_M(r)dr + \Phi\left(\frac{\gamma+\alpha}{\sigma}\int_{-\infty}^{+\infty} \phi_M(s)ds\right)\right],
\end{aligned}
$$

which means that $\{A_{x_n}\}$ is uniformly bounded. It follows that

$$
\begin{aligned}
&\int_{-\infty}^{+\infty} \frac{1}{\Phi^{-1}(\rho(s))}\left|\Phi^{-1}\left(A_{x_n} + \int_s^{+\infty} f(r,x_n(r),{x_n}'(r))dr\right)\right| ds \\
&\le \int_{-\infty}^{+\infty} \frac{1}{\Phi^{-1}(\rho(s))} ds \Phi^{-1}\left(2\int_{-\infty}^{\infty} \phi_M(r)dr + \Phi\left(\frac{\gamma+\alpha}{\sigma}\int_{-\infty}^{+\infty} \phi_M(s)ds\right)\right).
\end{aligned}
$$

Suppose that $\{A_{x_n}\}$ does not converge to $A_{x_0}$. Then there exist two subsequences $\{A_{x_{n_k}^{(1)}}\}$ and $\{A_{x_{n_k}^{(2)}}\}$ of $\{A_{x_n}\}$ with $A_{x_{n_k}^{(1)}} \to c_1$ and $A_{x_{n_k}^{(2)}} \to c_2$ as $k \to \infty$, but $c_1 \ne c_2$. By the construction of $A_{x_n}$, $(n = 1, 2, \dots)$, we have

$$
\begin{aligned}
&\alpha\delta\Phi^{-1}(A_{x_{n_k}^{(1)}}) + \alpha\gamma\int_{-\infty}^{+\infty} \frac{1}{\Phi^{-1}(\rho(s))}\Phi^{-1}\left(A_{x_{n_k}^{(1)}}\right. \\
&\qquad\qquad \left. + \int_s^{+\infty} f(r,{x_{n_k}}^{(1)}(r),{x_{n_k}}^{(1)\prime}(r))dr\right) ds \\
&\qquad + \beta\gamma\Phi^{-1}\left(A_{x_{n_k}^{(1)}} + \int_{-\infty}^{+\infty} f(r,{x_{n_k}}^{(1)}(r),{x_{n_k}}^{(1)\prime}(r))dr\right) \\
&+ \gamma\int_{-\infty}^{+\infty} g(r,{x_{n_k}}^{(1)}(r),{x_{n_k}}^{(1)\prime}(t))dr - \alpha\int_{-\infty}^{+\infty} h(r,{x_{n_k}}^{(1)}(r),{x_{n_k}}^{(1)\prime}(t))dr = 0.
\end{aligned}
$$

Let $k \to \infty$, using Lebesgue's dominated convergence theorem, the above equality implies

$$
\begin{aligned}
&\alpha\delta\Phi^{-1}(c_1) + \alpha\gamma\int_{-\infty}^{+\infty} \frac{1}{\Phi^{-1}(\rho(s))}\Phi^{-1}\left(c_1 + \int_s^{+\infty} f(r,x_0(r),x_0'(r))dr\right) ds \\
&\qquad + \beta\gamma\Phi^{-1}\left(c_1 + \int_{-\infty}^{+\infty} f(r,x_0(r),x_0'(r))dr\right) \\
&\qquad + \gamma\int_{-\infty}^{+\infty} g(r,x_0(r),x_0'(t))dr - \alpha\int_{-\infty}^{+\infty} h(r,x_0(r),x_0'(t))dr = 0.
\end{aligned}
$$

Since $\{A_{x_0}\}$ is unique with respect to $x_0$, we get $c_1 = A_{x_0}$. Similarly, $c_2 = A_{x_0}$. Thus $c_1 = c_2$, a contradiction. So, for any $x_n \to x_0$, one has $A_{x_n} \to A_{x_0}$, which means $A_x : X \to X$ is continuous.

Second, we show that $T$ is continuous on $X$. Since $A_x$ is continuous, then $B_x$ is continuous too. From the continuity of $A_x$ and $B_x$, and since $f$ is a Carathéodory function, the result follows.

Third, we show that $T$ is maps bounded subsets into bounded sets. Let $D \subseteq X$ be a given bounded set. Then, there exists $M > 0$ such that $D \subseteq \{x \in X : \|x\| \leq M\}$. Then there exists $\phi_M \in L^1(R)$ such that

$$\begin{aligned}
|f(t,x(t),x'(t))| &= \left| f\left(t, x(t), \frac{1}{\Phi^{-1}(\rho(t))}\Phi^{-1}(\rho(t))x'(t)\right)\right| \\
&\leq \phi_M(t), t \in R, \\
|g(t,x(t),x'(t))| &\leq \phi_M(t), t \in R, \\
|h(t,x(t),x'(t))| &\leq \phi_M(t), t \in R.
\end{aligned}$$

So

$$\begin{aligned}
\int_{-\infty}^{\infty} |f(r,x(r),x'(r))|dr &\leq \int_{-\infty}^{\infty} \phi_M(r)dr < \infty, \\
\int_{-\infty}^{\infty} |g(r,x(r),x'(r))|dr &\leq \int_{-\infty}^{\infty} \phi_M(r)dr < \infty, \\
\int_{-\infty}^{\infty} |h(r,x(r),x'(r))|dr &\leq \int_{-\infty}^{\infty} \phi_M(r)dr < \infty.
\end{aligned}$$

Similarly we have

$$\begin{aligned}
|A_x| &\leq \int_{-\infty}^{\infty} \phi_M(r)dr + \Phi\left(\frac{\gamma+\alpha}{\sigma}\int_{-\infty}^{+\infty}\phi_M(s)ds\right) < \infty, \\
|B_x| &= \left| \frac{\int_{-\infty}^{+\infty} g(r,x(r),x'(r))dr + \beta\Phi^{-1}\left(A_x + \int_{-\infty}^{+\infty} f(r,x(r),x'(r))dr\right)}{\alpha} \right| \\
&\leq \frac{\int_{-\infty}^{+\infty}\phi_M(r)dr + \beta\Phi^{-1}\left(2\int_{-\infty}^{\infty}\phi_M(r)dr + \Phi\left(\frac{\gamma+\alpha}{\sigma}\int_{-\infty}^{+\infty}\phi_M(s)ds\right)\right)}{\alpha}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|(Tx)(t)| &= \left| B_x + \int_{-\infty}^{t} \frac{1}{\Phi^{-1}(\rho(s))}\Phi^{-1}\left(A_x + \int_{s}^{+\infty} f(r,x(r),x'(r))dr\right)ds \right| \\
&\leq \frac{\int_{-\infty}^{+\infty}\phi_M(r)dr + \beta\Phi^{-1}\left(2\int_{-\infty}^{\infty}\phi_M(r)dr + \Phi\left(\frac{\gamma+\alpha}{\sigma}\int_{-\infty}^{+\infty}\phi_M(s)ds\right)\right)}{\alpha} \\
&+ \int_{-\infty}^{+\infty}\frac{1}{\Phi^{-1}(\rho(s))}ds\Phi^{-1}\left(2\int_{-\infty}^{\infty}\phi_M(r)dr + \Phi\left(\frac{\gamma+\alpha}{\sigma}\int_{-\infty}^{+\infty}\phi_M(s)ds\right)\right) \\
&=: M_1.
\end{aligned}$$

On the other hand, we have

$$\Phi^{-1}(\rho(t))|(Tx)'(t)| = \left|\Phi^{-1}\left(A_x + \int_t^{\infty} f(u,x(u),x'(u))du\right)\right|$$
$$\leq \Phi^{-1}\left(2\int_{-\infty}^{\infty}\phi_M(r)dr + \Phi\left(\frac{\gamma+\alpha}{\sigma}\int_{-\infty}^{+\infty}\phi_M(s)ds\right)\right) =: r_1.$$

Then

$$\|Tx\| = \max\left\{\sup_{t\in R}|(Tx)(t)|, \sup_{t\in R}\Phi^{-1}(\rho(t))|(Tx)'(t)|\right\} < \infty.$$

So, $\{TD\}$ is bounded.

Fourth, we prove that both $\{Tx : x \in D\}$ and $\{\Phi^{-1}(\rho)(Tx)' : x \in D\}$ are equi-continuous on each finite subinterval on $R$.

Let $D \subset \{x \in X : ||x|| \leq M\}$. For any $K > 0$, $t_1, t_2 \in [-K, K]$ with $t_1 \leq t_2$ and $x \in X$ , since $f, g, h$ are Carathéodory functions, then there exists $\phi_M \in L^1(R)$ such that

$$|f(t,x(t),x'(t))| \leq \phi_M(t), t \in R,$$
$$|g(t,x(t),x'(t))| \leq \phi_M(t), t \in R,$$
$$|h(t,x(t),x'(t))| \leq \phi_M(t), t \in R.$$

Then

$$\int_{-\infty}^{\infty}|f(r,x(r),x'(r))|dr \leq \int_{-\infty}^{\infty}\phi_M(r)dr < \infty,$$
$$\int_{-\infty}^{\infty}|g(r,x(r),x'(r))|dr \leq \int_{-\infty}^{\infty}\phi_M(r)dr < \infty,$$
$$\int_{-\infty}^{\infty}|h(r,x(r),x'(r))|dr \leq \int_{-\infty}^{\infty}\phi_M(r)dr < \infty.$$

Then

$$\left|A_x + \int_t^{+\infty} f(r,x(r),x'(r))dr\right| \leq 2\int_{-\infty}^{\infty}\phi_M(r)dr + \Phi\left(\tfrac{\gamma+\alpha}{\sigma}\int_{-\infty}^{+\infty}\phi_M(s)ds\right) =: r_1.$$

Since $\Phi^{-1}(s)$ is uniformly continuous on $[-r_1, r_1]$, then for each $\varepsilon > 0$ there exists $\mu > 0$ such that $|s_1 - s_2| < \mu$ with $s_1, s_2 \in [-r_1, r_1]$ implies that $|\Phi^{-1}(s_1) - \Phi^{-1}(s_2)| < \varepsilon$.

Since

$$|\rho(t_1)\Phi((Tx)'(t_1)) - \rho(t_2)\Phi((Tx)'(t_2))|$$
$$= \left|\int_{t_2}^{t_1} f(r,x(r),x'(r))dr\right|$$
$$\leq \int_{t_1}^{t_2}\phi_M(r)dr \to 0 \text{ uniformly as } t_1 \to t_2,$$

then there exists $\overline{\sigma}>0$ such that $|t_2-t_1|<\overline{\sigma}$ implies that $|\rho(t_1)\Phi((Tx)'(t_1))-\rho(t_2)\Phi((Tx)'(t_2))|<\mu$. Thus $|t_1-t_2|<\overline{\sigma}$ implies that

$$|\Phi^{-1}(\rho(t_1))(Tx)'(t_1)-\Phi^{-1}(\rho(t_2))(Tx)'(t_2)|<\varepsilon. \tag{15}$$

On the other hand, we have

$$|(Tx)(t_1)-(Tx)(t_2)|=\int_{t_1}^{t_2}\Phi^{-1}\left(\frac{1}{\rho(s)}\right)\Phi^{-1}\left(A_x+\int_s^{+\infty}f(r,x(r),x'(r))dr\right)ds$$

$$\leq\int_{t_1}^{t_2}\Phi^{-1}\left(\frac{1}{\rho(s)}\right)ds\Phi^{-1}\left(2\int_{-\infty}^{\infty}\phi_M(r)dr+\Phi\left(\frac{\gamma+\alpha}{\sigma}\int_{-\infty}^{+\infty}\phi_M(s)ds\right)\right)$$

$\to 0$ uniformly as $t_1\to t_2$.

Then there exists $\overline{\overline{\sigma}}>0$ such that $|t_1-t_2|<\overline{\overline{\sigma}}$ implies

$$|(Tx)(t_1)-(Tx)(t_2)|<\varepsilon. \tag{16}$$

Then (15) and (16) imply both $\{Tx:x\in D\}$ and $\{\Phi^{-1}(\rho)(Tx)':x\in D\}$ are equi-continuous on $[-K,K]$. So both $\{Tx:x\in D\}$ and $\{\Phi^{-1}(\rho)(Tx)':x\in D\}$ are equi-continuous on each finite subinterval on $R$.

Now, we show that both $\{Tx:x\in D\}$ and $\{\Phi^{-1}(\rho)(Tx)':x\in D\}$ are equi-convergent at $+\infty$ and $-\infty$ respectively.

Since

$$|\rho(t)\Phi((Tx)'(t))-A_x|=|\int_t^{\infty}f(r,x(r),x'(r))dr|\leq\int_t^{\infty}\phi_M(r)dr\to 0$$

uniformly as $t\to\infty$, we get similarly that

$$|\Phi^{-1}(\rho(t))(Tx)'(t)-\Phi^{-1}(A_x)|\to 0$$

uniformly as $t\to\infty$. In fact, for any $\varepsilon>0$, there exists $\mu>0$ such that $|s_1-s_2|<\mu$ implies that $|\Phi^{-1}(s_1)-\Phi^{-1}(s_2)|<\frac{\varepsilon}{2}$. So there exists $T_{1,\varepsilon}>0$ such that $t>T_{1,\varepsilon}$ implies that $|\rho(t)\Phi((Tx)'(t))-A_x|<\mu$. Hence

$$|\Phi^{-1}(\rho(t))(Tx)'(t)-\Phi^{-1}(A_x)|=|\Phi^{-1}\Big(\rho(t)\Phi((Tx)'(t))\Big)-\Phi^{-1}(A_x)|<\frac{\varepsilon}{2}$$

for all $t>T_{1,\varepsilon}$. Then

$$|\Phi^{-1}(\rho(t_1))(Tx)'(t_1)-\Phi^{-1}(\rho(t_2))(Tx)'(t_2)|<\varepsilon,\ t_1,t_2>T_{1,\varepsilon}. \tag{17}$$

On the other hand, we have

$$\begin{aligned}
&\left|(Tx)(t)-B_x-\int_{-\infty}^{+\infty}\frac{1}{\Phi^{-1}(\rho(s))}\Phi^{-1}\left(A_x+\int_s^{+\infty}f(r,x(r),x'(r))dr\right)ds\right| \\
&=\left|\int_t^{+\infty}\Phi^{-1}\left(\frac{1}{\rho(s)}\right)\Phi^{-1}\left(A_x+\int_s^{\infty}f(r,x(r),x'(r))dr\right)ds\right| \\
&\le\int_t^{+\infty}\Phi^{-1}\left(\frac{1}{\rho(s)}\right)ds\Phi^{-1}\left(2\int_{-\infty}^{\infty}\phi_M(r)dr+\Phi\left(\frac{\gamma+\alpha}{\sigma}\int_{-\infty}^{+\infty}\phi_M(s)ds\right)\right) \\
&\to 0 \text{ uniformly as } t\to+\infty.
\end{aligned}$$

Then there exists $T_{2,\varepsilon}>0$ such that

$$|(Tx)(t_1)-(Tx)(t_2)|<\varepsilon,\quad t>T_{2,\varepsilon}. \tag{18}$$

So (17) and (18) imply that both $\{\Phi^{-1}(\rho)(Tx)':x\in D\}$ and $\{Tx:x\in D\}$ are equiconvergent at $+\infty$.

Similarly we can prove that both $\{Tx:x\in D\}$ and $\{\Phi^{-1}(\rho)(Tx)':x\in D\}$ are equiconvergent at $-\infty$. The details are omitted.

Therefore, the operator $T:X\to X$ is completely continuous. The proof is complete. □

## 3. Main Theorems

Let $X$ and the operator $T$ on $X$ be defined in Section 2. Denote

$$C_{q-1}=\begin{cases}1,\ q\in(1,2],\\ 4^{q-2},\ q>2,\end{cases}$$
$$D_{q-1}=\begin{cases}1,\ q\in(1,2],\\ 2^{q-2},\ q>2.\end{cases}$$

The primary inequalities are as follows:

$$|k_1+k_2+k_3+k_4|^{q-1}\le C_{q-1}(|k_1|^{q-1}+|k_2|^{q-1}+|k_3|^{q-1}+|k_4|^{q-1}), \tag{19}$$

and

$$|k_1+k_2|^{q-1}\le D_{q-1}(|k_1|^{q-1}+|k_2|^{q-1}). \tag{20}$$

**Theorem 3.1.** *Suppose that $\alpha > 0$ and (H) holds. Then BVP(6) has at least one solution if*

$$\alpha > \int_{-\infty}^{+\infty} b_1(s)ds + C_{q-1}\left\{\left[2^{q-1}\Phi^{-1}\left(\int_{-\infty}^{+\infty} b(s)ds\right) + \frac{\gamma}{\sigma}\int_{-\infty}^{+\infty} b_1(s)ds \right.\right.$$
$$\left. + \frac{\alpha}{\sigma}\int_{-\infty}^{+\infty} b_2(s)ds\right]\left(\beta + \int_{-\infty}^{+\infty} c_1(s)ds + \alpha\int_{-\infty}^{+\infty}\frac{1}{\Phi^{-1}(\rho(s))}ds\right)$$
$$+\left[2^{q-1}\Phi^{-1}\left(\int_{-\infty}^{+\infty} c(s)ds\right) + \frac{\gamma}{\sigma}\int_{-\infty}^{+\infty} c_1(s)ds + \frac{\alpha}{\sigma}\int_{-\infty}^{+\infty} c_2(s)ds\right]$$
$$\left.\times\left(\alpha - \int_{-\infty}^{+\infty} b_1(s)ds\right)\right\}.$$

*Proof.* From Lemma 2.4, $T : X \to X$ is a completely continuous operator and $x$ is a solution of BVP(6) if and only if $x$ is a fixed point of $T$ in $X$.

To apply Lemma 2.2, choose $p^* = 0$. Let $\lambda \in (0,1)$, consider the equation $x = (1-\lambda)p^* + \lambda Tx$. Then

$$x(t) = \lambda\left(B_x + \int_{-\infty}^{t}\frac{1}{\Phi^{-1}(\rho(s))}\Phi^{-1}\left(A_x + \int_s^{+\infty} f(r,x(r),x'(r))dr\right)ds\right). \tag{21}$$

Hence Lemma 2.4 (ii) implies that

$$\begin{cases} [\rho(t)\Phi(x'(t))]' + \lambda f(t,x(t),x'(t)) = 0, \quad t \in R, \\ \alpha \lim\limits_{t\to-\infty} x(t) - \beta \lim\limits_{t\to-\infty}\Phi^{-1}(\rho(t))x'(t) = \lambda \int\limits_{-\infty}^{+\infty} g(s,x(s),x'(s))ds, \\ \gamma \lim\limits_{t\to+\infty} x(t) + \delta \lim\limits_{t\to+\infty}\Phi^{-1}(\rho(t))x'(t) = \lambda \int\limits_{-\infty}^{+\infty} h(s,x(s),x'(s))ds. \end{cases} \tag{22}$$

Then use (H), we get

$$|x(t)| = \left|\int_{-\infty}^{t} x'(s)ds + \lim_{t\to-\infty} x(t)\right| \le \left|\int_{-\infty}^{t}\frac{1}{\Phi^{-1}(\rho(s))}\Phi^{-1}(\rho(s))x'(s)ds\right|$$
$$+\left|\frac{\lambda\int_{-\infty}^{+\infty} g(s,x(s),x'(s))ds + \beta\lim_{t\to-\infty}\Phi^{-1}(\rho(t))x'(t)}{\alpha}\right|$$

$$\leq \int_{-\infty}^{+\infty} \frac{1}{\Phi^{-1}(\rho(s))} ds \sup_{t\in R} \Phi^{-1}(\rho(t))|x'(t)|$$
$$+ \frac{\int_{-\infty}^{+\infty} |g(s,x(s),x'(s))|ds + \beta \sup_{t\in R} \Phi^{-1}(\rho(t))|x'(t)|}{\alpha}$$
$$\leq \left(\frac{\beta}{\alpha} + \int_{-\infty}^{+\infty} \frac{1}{\Phi^{-1}(\rho(s))} ds\right) \sup_{t\in R} \Phi^{-1}(\rho(t))|x'(t)|$$
$$+ \frac{\int_{-\infty}^{+\infty} [a_1(s) + b_1(s)|x(s)| + c_1(s)\Phi^{-1}(\rho(s))|x'(s)|]ds}{\alpha}$$
$$\leq \left(\frac{\beta}{\alpha} + \int_{-\infty}^{+\infty} \frac{1}{\Phi^{-1}(\rho(s))} ds\right) \sup_{t\in R} \Phi^{-1}(\rho(t))|x'(t)|$$
$$+ \frac{\int_{-\infty}^{+\infty} a_1(s)ds + \int_{-\infty}^{+\infty} b_1(s)ds \sup_{t\in R} |x(t)|}{\alpha}$$
$$+ \frac{\int_{-\infty}^{+\infty} c_1(s)ds \sup_{t\in R} \Phi^{-1}(\rho(t))|x'(t)|}{\alpha}.$$

It follows that

$$\sup_{t\in R} |x(t)| \leq \left(\frac{\beta}{\alpha} + \int_{-\infty}^{+\infty} \frac{1}{\Phi^{-1}(\rho(s))} ds\right) \sup_{t\in R} \Phi^{-1}(\rho(t))|x'(t)|$$
$$+ \frac{\int_{-\infty}^{+\infty} a_1(s)ds + \int_{-\infty}^{+\infty} b_1(s)ds \sup_{t\in R} |x(t)|}{\alpha}$$
$$+ \frac{\int_{-\infty}^{+\infty} c_1(s)ds \sup_{t\in R} \Phi^{-1}(\rho(t))|x'(t)|}{\alpha}.$$

Since the assumption in Theorem implies that

$$\alpha > \int_{-\infty}^{+\infty} b_1(s)ds,$$

then

$$\sup_{t\in R} |x(t)| \leq \frac{\alpha\left(\frac{\beta + \int_{-\infty}^{+\infty} c_1(s)ds}{\alpha} + \int_{-\infty}^{+\infty} \frac{1}{\Phi^{-1}(\rho(s))} ds\right) \sup_{t\in R} \Phi^{-1}(\rho(t))|x'(t)| + \int_{-\infty}^{+\infty} a_1(s)ds}{\alpha - \int_{-\infty}^{+\infty} b_1(s)ds}. \tag{23}$$

Now, we consider $\sup_{t\in R}\Phi^{-1}(\rho(t))|x'(t)|$. By Lemma 2.3, one has

$$\begin{aligned}
\Phi^{-1}(\rho(t))|x'(t)| &= \left|\Phi^{-1}\left(A_x+\int_t^{+\infty} f(r,x(r),x'(r))dr\right)\right| \\
&\le \Phi^{-1}\left(2\int_{-\infty}^{+\infty}|f(r,x(r),x'(r))|dr\right. \\
&\left.+\Phi\left(\frac{\gamma\int_{-\infty}^{+\infty}|g(r,x(r),x'(r))|dr+\alpha\int_{-\infty}^{+\infty}|h(r,x(r),x'(r))|dr}{\sigma}\right)\right) \\
&\le \Phi^{-1}\left(2\int_{-\infty}^{+\infty}\left[a(s)+b(s)\Phi(|x(s)|)+c(s)\Phi(\Phi^{-1}(\rho(s))|x'(s)|)\right]dr\right. \\
&\left.+\Phi\left(\frac{\gamma\int_{-\infty}^{+\infty}|g(r,x(r),x'(r))|dr+\alpha\int_{-\infty}^{+\infty}|h(r,x(r),x'(r))|dr}{\sigma}\right)\right) \\
&\le \Phi^{-1}\left(2\int_{-\infty}^{+\infty}a(s)ds+2\int_{-\infty}^{+\infty}c(s)ds\sup_{t\in R}\rho(t)\Phi(|x'(t)|)\right. \\
&+2\int_{-\infty}^{+\infty}b(s)ds\sup_{t\in R}\Phi(|x(t)|) \\
&\left.+\Phi\left(\frac{\gamma\int_{-\infty}^{+\infty}|g(r,x(r),x'(r))|dr+\alpha\int_{-\infty}^{+\infty}|h(r,x(r),x'(r))|dr}{\sigma}\right)\right).
\end{aligned}$$

Using (18), we have

$$\begin{aligned}
\sup_{t\in R}\Phi^{-1}(\rho(t))|x'(t)| &\le 2^{q-1}C_{q-1}\Phi^{-1}\left(\int_{-\infty}^{+\infty}a(s)ds\right) \\
&+2^{q-1}C_{q-1}\Phi^{-1}\left(\int_{-\infty}^{+\infty}c(s)ds\right)\sup_{t\in R}\Phi^{-1}(\rho(t))|x'(t)| \\
&+2^{q-1}C_{q-1}\Phi^{-1}\left(\int_{-\infty}^{+\infty}b(s)ds\right)\sup_{t\in R}|x(t)| \\
&+C_{q-1}\left(\frac{\gamma\int_{-\infty}^{+\infty}|g(r,x(r),x'(r))|dr+\alpha\int_{-\infty}^{+\infty}|h(r,x(r),x'(r))|dr}{\sigma}\right) \\
&\le 2^{q-1}C_{q-1}\Phi^{-1}\left(\int_{-\infty}^{+\infty}a(s)ds\right) \\
&+2^{q-1}C_{q-1}\Phi^{-1}\left(\int_{-\infty}^{+\infty}c(s)ds\right)\sup_{t\in R}\Phi^{-1}(\rho(t))|x'(t)| \\
&+2^{q-1}C_{q-1}\Phi^{-1}\left(\int_{-\infty}^{+\infty}b(s)ds\right)\sup_{t\in R}|x(t)|
\end{aligned}$$

$$+C_{q-1}\frac{\gamma}{\sigma}\int_{-\infty}^{+\infty}[a_1(s)+b_1(s)|x(s)|+c_1(s)\Phi^{-1}(\rho(s))|x'(s)|]ds$$
$$+C_{q-1}\frac{\alpha}{\sigma}\int_{-\infty}^{+\infty}[a_2(s)+b_2(s)|x(s)|+c_2(s)\Phi^{-1}(\rho(s))|x'(s)|]ds$$

It follows from (22) that

$$\sup_{t\in R}\Phi^{-1}(\rho(t))|x'(t)|\le 2^{q-1}C_{q-1}\Phi^{-1}\left(\int_{-\infty}^{+\infty}a(s)ds\right)$$
$$+2^{q-1}C_{q-1}\Phi^{-1}\left(\int_{-\infty}^{+\infty}c(s)ds\right)\sup_{t\in R}\Phi^{-1}(\rho(t))|x'(t)|$$
$$+2^{q-1}C_{q-1}\Phi^{-1}\left(\int_{-\infty}^{+\infty}b(s)ds\right)\sup_{t\in R}|x(t)|$$
$$+C_{q-1}\frac{\gamma}{\sigma}\int_{-\infty}^{+\infty}a_1(s)ds+C_{q-1}\frac{\gamma}{\sigma}\int_{-\infty}^{+\infty}b_1(s)ds\sup_{t\in R}|x(t)|$$
$$+C_{q-1}\frac{\gamma}{\sigma}\int_{-\infty}^{+\infty}c_1(s)ds\sup_{t\in R}\Phi^{-1}(\rho(t))|x'(t)|$$
$$+C_{q-1}\frac{\alpha}{\sigma}\int_{-\infty}^{+\infty}a_2(s)ds+C_{q-1}\frac{\alpha}{\sigma}\int_{-\infty}^{+\infty}b_2(s)ds\sup_{t\in R}|x(t)|$$
$$+C_{q-1}\frac{\alpha}{\sigma}\int_{-\infty}^{+\infty}c_2(s)ds\sup_{t\in R}\Phi^{-1}(\rho(t))|x'(t)|$$
$$=2^{q-1}C_{q-1}\Phi^{-1}\left(\int_{-\infty}^{+\infty}a(s)ds\right)+C_{q-1}\frac{\gamma}{\sigma}\int_{-\infty}^{+\infty}a_1(s)ds+C_{q-1}\frac{\alpha}{\sigma}\int_{-\infty}^{+\infty}a_2(s)ds$$
$$+C_{q-1}\left[2^{q-1}\Phi^{-1}\left(\int_{-\infty}^{+\infty}b(s)ds\right)+\frac{\gamma}{\sigma}\int_{-\infty}^{+\infty}b_1(s)ds\right.$$
$$\left.+\frac{\alpha}{\sigma}\int_{-\infty}^{+\infty}b_2(s)ds\right]\sup_{t\in R}|x(t)|$$
$$+C_{q-1}\left[2^{q-1}\Phi^{-1}\left(\int_{-\infty}^{+\infty}c(s)ds\right)+\frac{\gamma}{\sigma}\int_{-\infty}^{+\infty}c_1(s)ds\right.$$
$$\left.+\frac{\alpha}{\sigma}\int_{-\infty}^{+\infty}c_2(s)ds\right]\sup_{t\in R}\Phi^{-1}(\rho(t))|x'(t)|$$
$$\le 2^{q-1}C_{q-1}\Phi^{-1}\left(\int_{-\infty}^{+\infty}a(s)ds\right)+C_{q-1}\frac{\gamma}{\sigma}\int_{-\infty}^{+\infty}a_1(s)ds+C_{q-1}\frac{\alpha}{\sigma}\int_{-\infty}^{+\infty}a_2(s)ds$$
$$+\frac{\int_{-\infty}^{+\infty}a_1(s)ds}{\alpha-\int_{-\infty}^{+\infty}b_1(s)ds}$$
$$+C_{q-1}\left\{\left[2^{q-1}\Phi^{-1}\left(\int_{-\infty}^{+\infty}b(s)ds\right)+\frac{\gamma}{\sigma}\int_{-\infty}^{+\infty}b_1(s)ds+\frac{\alpha}{\sigma}\int_{-\infty}^{+\infty}b_2(s)ds\right]\times\right.$$
$$\frac{\beta+\int_{-\infty}^{+\infty}c_1(s)ds+\alpha\int_{-\infty}^{+\infty}\frac{1}{\Phi^{-1}(\rho(s))}ds}{\alpha-\int_{-\infty}^{+\infty}b_1(s)ds}+2^{q-1}\Phi^{-1}\left(\int_{-\infty}^{+\infty}c(s)ds\right)$$
$$\left.+\frac{\gamma}{\sigma}\int_{-\infty}^{+\infty}c_1(s)ds+\frac{\alpha}{\sigma}\int_{-\infty}^{+\infty}c_2(s)ds\right\}\sup_{t\in R}\Phi^{-1}(\rho(t))|x'(t)|.$$

It follows that

$$\left(\alpha-\int_{-\infty}^{+\infty}b_1(s)ds\right)\sup_{t\in R}\Phi^{-1}(\rho(t))|x'(t)|$$

$$\leq\Big[2^{q-1}C_{q-1}\Phi^{-1}\left(\int_{-\infty}^{+\infty}a(s)ds\right)+C_{q-1}\frac{\gamma}{\sigma}\int_{-\infty}^{+\infty}a_1(s)ds$$

$$+C_{q-1}\frac{\alpha}{\sigma}\int_{-\infty}^{+\infty}a_2(s)ds+\frac{\int_{-\infty}^{+\infty}a_1(s)ds}{\alpha-\int_{-\infty}^{+\infty}b_1(s)ds}\Big]\left(\alpha-\int_{-\infty}^{+\infty}b_1(s)ds\right)$$

$$+C_{q-1}\Big\{\Big[2^{q-1}\Phi^{-1}\left(\int_{-\infty}^{+\infty}b(s)ds\right)+\frac{\gamma}{\sigma}\int_{-\infty}^{+\infty}b_1(s)ds+\frac{\alpha}{\sigma}\int_{-\infty}^{+\infty}b_2(s)ds\Big]\times$$

$$\left(\beta+\int_{-\infty}^{+\infty}c_1(s)ds+\alpha\int_{-\infty}^{+\infty}\frac{1}{\Phi^{-1}(\rho(s))}ds\right)$$

$$+\Big[2^{q-1}\Phi^{-1}\left(\int_{-\infty}^{+\infty}c(s)ds\right)+\frac{\gamma}{\sigma}\int_{-\infty}^{+\infty}c_1(s)ds+\frac{\alpha}{\sigma}\int_{-\infty}^{+\infty}c_2(s)ds\Big]\times$$

$$\left(\alpha-\int_{-\infty}^{+\infty}b_1(s)ds\right)\Big\}\sup_{t\in R}\Phi^{-1}(\rho(t))|x'(t)|.$$

This inequality in Theorem and (22) imply that there exists a constant $M_0>0$ such that

$$||x||=\max\left\{\sup_{t\in R}|x(t)|,\ \sup_{t\in R}\Phi^{-1}(\rho(t))|x'(t)|\right\}\leq M_0. \tag{24}$$

It follows that (24) holds for all $x\in X$ satisfying $x=(1-\lambda)p^*+\lambda Tx$ with $\lambda\in(0,1)$.

Choose $U=\{x\in X:||x||<M_0+1\}$ and $C=\overline{U}$. Taking $p^*=0$ in Lemma 2.2, for any $x\in\partial U$, $x=(1-\lambda)p^*+\lambda Tx$ $(0<\lambda<1)$ does not hold. Thus Lemma 2.2 implies that the operator $T$ has at least one fixed point in $U$. So BVP(6) has at least one solution. The proof is complete. □

**Theorem 3.2.** *Suppose that $\gamma>0$ and (H) holds. Then BVP(6) has at least one solution if*

$$\gamma>\int_{-\infty}^{+\infty}b_2(s)ds+C_{q-1}\Big\{\Big[2^{q-1}\Phi^{-1}\left(\int_{-\infty}^{+\infty}b(s)ds\right)+\frac{\gamma}{\sigma}\int_{-\infty}^{+\infty}b_1(s)ds$$

$$+\frac{\alpha}{\sigma}\int_{-\infty}^{+\infty}b_2(s)ds\Big]\left(\delta+\int_{-\infty}^{+\infty}c_1(s)ds+\gamma\int_{-\infty}^{+\infty}\frac{1}{\Phi^{-1}(\rho(s))}ds\right)$$

$$+\Big[2^{q-1}\Phi^{-1}\left(\int_{-\infty}^{+\infty}c(s)ds\right)+\frac{\gamma}{\sigma}\int_{-\infty}^{+\infty}c_1(s)ds+\frac{\alpha}{\sigma}\int_{-\infty}^{+\infty}c_2(s)ds\Big]\times$$

$$\left(\gamma-\int_{-\infty}^{+\infty}b_2(s)ds\right)\Big\}.$$

*Proof.* Similar to the proof of Theorem 3.1, we see that

$$|x(t)| = \left|\lim_{t\to+\infty} x(t) - \int_t^{+\infty} x'(s)ds\right|$$

$$\leq \left|\int_t^{+\infty} \frac{1}{\Phi^{-1}(\rho(s))}\Phi^{-1}(\rho(s))x'(s)ds\right|$$

$$+\left|\frac{\lambda\int_{-\infty}^{+\infty} h(s,x(s),x'(s))ds - \delta\lim_{t\to-\infty}\Phi^{-1}(\rho(t))x'(t)}{\gamma}\right|$$

$$\leq \int_{-\infty}^{+\infty} \frac{1}{\Phi^{-1}(\rho(s))}ds \sup_{t\in R}\Phi^{-1}(\rho(t))|x'(t)|$$

$$+\frac{\int_{-\infty}^{+\infty}|h(s,x(s),x'(s))|ds + \delta\sup_{t\in R}\Phi^{-1}(\rho(t))|x'(t)|}{\gamma}$$

$$\leq \left(\frac{\delta}{\gamma} + \int_{-\infty}^{+\infty}\frac{1}{\Phi^{-1}(\rho(s))}ds\right)\sup_{t\in R}\Phi^{-1}(\rho(t))|x'(t)|$$

$$+\frac{\int_{-\infty}^{+\infty}[a_2(s)+b_2(s)|x(s)|+c_2(s)\Phi^{-1}(\rho(s))|x'(s)|]ds}{\gamma}$$

$$\leq \left(\frac{\delta}{\gamma} + \int_{-\infty}^{+\infty}\frac{1}{\Phi^{-1}(\rho(s))}ds\right)\sup_{t\in R}\Phi^{-1}(\rho(t))|x'(t)|$$

$$+\frac{\int_{-\infty}^{+\infty}a_2(s)ds + \int_{-\infty}^{+\infty}b_2(s)ds\sup_{t\in R}|x(t)| + \int_{-\infty}^{+\infty}c_2(s)ds\sup_{t\in R}\Phi^{-1}(\rho(t))|x'(t)|}{\gamma}.$$

It follows that

$$\sup_{t\in R}|x(t)| \leq \left(\frac{\delta}{\gamma} + \int_{-\infty}^{+\infty}\frac{1}{\Phi^{-1}(\rho(s))}ds\right)\sup_{t\in R}\Phi^{-1}(\rho(t))|x'(t)|$$

$$+\frac{\int_{-\infty}^{+\infty}a_2(s)ds + \int_{-\infty}^{+\infty}b_2(s)ds\sup_{t\in R}|x(t)| + \int_{-\infty}^{+\infty}c_2(s)ds\sup_{t\in R}\Phi^{-1}(\rho(t))|x'(t)|}{\gamma}.$$

Hence

$$\sup_{t\in R}|x(t)| \leq \frac{\gamma\left(\frac{\delta+\int_{-\infty}^{+\infty}c_2(s)ds}{\gamma} + \int_{-\infty}^{+\infty}\frac{1}{\Phi^{-1}(\rho(s))}ds\right)\sup_{t\in R}\Phi^{-1}(\rho(t))|x'(t)| + \int_{-\infty}^{+\infty}a_2(s)ds}{\gamma - \int_{-\infty}^{+\infty}b_2(s)ds}. \tag{25}$$

The remainder of the proof is similar to that of the proof of Theorem 3.1 and is omitted. The proof is complete. □

**Corollary 3.3.** *Suppose that* $\alpha > 0$, $f, g, h$ *are Carathéodory functions and there exists* $r_1 > 0$ *such that*

$$\int_{-\infty}^{+\infty} g\left(t, x, \frac{1}{\Phi^{-1}(\rho(t))}y\right)dt \leq \frac{r_1}{3}\left[\frac{1}{\alpha} + D_{q-1}\frac{\gamma}{\sigma}\left(\frac{\beta}{\alpha} + \int_{-\infty}^{+\infty}\frac{1}{\Phi^{-1}(\rho(s))}ds\right)\right]^{-1},$$

$$\int_{-\infty}^{+\infty} h\left(t, x, \frac{1}{\Phi^{-1}(\rho(t))}y\right)dt \leq \frac{r_1}{3D_{q-1}}\frac{\sigma}{\alpha}\left(\frac{\beta}{\alpha} + \int_{-\infty}^{+\infty}\frac{1}{\Phi^{-1}(\rho(s))}ds\right)^{-1},$$

$$\int_{-\infty}^{+\infty} f\left(s,x,\frac{1}{\Phi^{-1}(\rho(s))}y\right)dt \le \Phi\left(\frac{r_1}{3\times 2^{q-1}D_{q-1}}\left(\frac{\beta}{\alpha}+\int_{-\infty}^{+\infty}\frac{1}{\Phi^{-1}(\rho(s))}ds\right)^{-1}\right),$$

*where $x,y\in[-r_1,r_1]$. Then BVP(6) has at least one solution if*

$$\frac{\beta}{\alpha}+\int_{-\infty}^{+\infty}\frac{1}{\Phi^{-1}(\rho(s))}ds \ge 1. \tag{26}$$

*Proof.* From Lemma 2.4, $T: X\to X$ is a completely continuous operator. Now we define $U=\{x\in PX:\|x\|<r_1\}$. For any $x\in\partial U$, $\|x\|=r_1$. So

$$\sup_{t\in R}|x(t)|\le r_1,\ \ \sup_{t\in R}\Phi^{-1}(\rho(t))|x'(t)|\le r_1.$$

By the assumptions, using Lemma 2.3 and (20), we have

$$\begin{aligned}
&\sup_{t\in R}|(Tx)(t)|\\
&=\sup_{t\in R}\left|B_x+\int_{-\infty}^{t}\frac{1}{\Phi^{-1}(\rho(s))}\Phi^{-1}\left(A_x+\int_s^{+\infty}f(r,x(r),x'(r))dr\right)ds\right|\\
&\le\frac{\int_{-\infty}^{+\infty}|g(r,x(r),x'(t))|dr+\beta\left|\Phi^{-1}\left(A_x+\int_{-\infty}^{+\infty}f(r,x(r),x'(r))dr\right)\right|}{\alpha}\\
&+\left|\int_{-\infty}^{t}\frac{1}{\Phi^{-1}(\rho(s))}\Phi^{-1}\left(A_x+\int_s^{+\infty}f(r,x(r),x'(r))dr\right)ds\right|\\
&\le\frac{\int_{-\infty}^{+\infty}|g(r,x(r),x'(t))|dr}{\alpha}+\left(\frac{\beta}{\alpha}+\int_{-\infty}^{+\infty}\frac{1}{\Phi^{-1}(\rho(s))}ds\right)\times\\
&\Phi^{-1}\left(2\int_{-\infty}^{+\infty}|f(r,x(r),x'(r))|dr\right.\\
&\left.+\Phi\left(\frac{\gamma\int_{-\infty}^{+\infty}|g(r,x(r),x'(r))|dr+\alpha\int_{-\infty}^{+\infty}|h(r,x(r),x'(r))|dr}{\sigma}\right)\right)\\
&\le\frac{\int_{-\infty}^{+\infty}|g(r,x(r),x'(r))|dr}{\alpha}+\left(\frac{\beta}{\alpha}+\int_{-\infty}^{+\infty}\frac{1}{\Phi^{-1}(\rho(s))}ds\right)\times\\
&2^{q-1}D_{q-1}\Phi^{-1}\left(\int_{-\infty}^{+\infty}|f(r,x(r),x'(r))|dr\right)+\left(\frac{\beta}{\alpha}+\int_{-\infty}^{+\infty}\frac{1}{\Phi^{-1}(\rho(s))}ds\right)\\
&\times D_{q-1}\frac{\gamma\int_{-\infty}^{+\infty}|g(r,x(r),x'(r))|dr+\alpha\int_{-\infty}^{+\infty}|h(r,x(r),x'(r))|dr}{\sigma}\\
&=2^{q-1}D_{q-1}\left(\frac{\beta}{\alpha}+\int_{-\infty}^{+\infty}\frac{1}{\Phi^{-1}(\rho(s))}ds\right)\Phi^{-1}\left(\int_{-\infty}^{+\infty}|f(r,x(r),x'(r))|dr\right)\\
&+\left[\frac{1}{\alpha}+D_{q-1}\frac{\gamma}{\sigma}\left(\frac{\beta}{\alpha}+\int_{-\infty}^{+\infty}\frac{1}{\Phi^{-1}(\rho(s))}ds\right)\right]\int_{-\infty}^{+\infty}|g(r,x(r),x'(r))|dr\\
&+D_{q-1}\frac{\alpha}{\sigma}\left(\frac{\beta}{\alpha}+\int_{-\infty}^{+\infty}\frac{1}{\Phi^{-1}(\rho(s))}ds\right)\int_{-\infty}^{+\infty}|h(r,x(r),x'(r))|dr\\
&\le\frac{r_1}{3}+\frac{r_1}{3}+\frac{r_1}{3}=r_1.
\end{aligned}$$

Furthermore, use (26), we get similarly that

$$\sup_{t\in R}\Phi^{-1}(\rho(t))|(Tx)'(t)| = \sup_{t\in R}\left|\Phi^{-1}\left(A_x+\int_t^{+\infty} f(r,x(r),x'(r))dr\right)\right|$$

$$< 2^{q-1}D_{q-1}\Phi^{-1}\left(\int_{-\infty}^{+\infty}|f(r,x(r),x'(r))|dr\right)$$

$$+\left(\frac{\beta}{\alpha}+\int_{-\infty}^{+\infty}\frac{1}{\Phi^{-1}(\rho(s))}ds\right)D_{q-1}\frac{\gamma\int_{-\infty}^{+\infty}|g(r,x(r),x'(r))|dr+\alpha\int_{-\infty}^{+\infty}|h(r,x(r),x'(r))|dr}{\sigma}$$

$$= 2^{q-1}D_{q-1}\left(\frac{\beta}{\alpha}+\int_{-\infty}^{+\infty}\frac{1}{\Phi^{-1}(\rho(s))}ds\right)\Phi^{-1}\left(\int_{-\infty}^{+\infty}|f(r,x(r),x'(r))|dr\right)$$

$$+\left[\frac{1}{\alpha}+D_{q-1}\frac{\gamma}{\sigma}\left(\frac{\beta}{\alpha}+\int_{-\infty}^{+\infty}\frac{1}{\Phi^{-1}(\rho(s))}ds\right)\right]\int_{-\infty}^{+\infty}|g(r,x(r),x'(r))|dr$$

$$+D_{q-1}\frac{\alpha}{\sigma}\left(\frac{\beta}{\alpha}+\int_{-\infty}^{+\infty}\frac{1}{\Phi^{-1}(\rho(s))}ds\right)\int_{-\infty}^{+\infty}|h(r,x(r),x'(r))|dr$$

$$\le \frac{r_1}{3}+\frac{r_1}{3}+\frac{r_1}{3}=r_1.$$

Choose $p^*=0$, $U=\{x\in X: ||x||<r_1+1\}$ and $C=\overline{U}$. It is easy to see that $\|Tx\|<\|x\|$ for all $x\in\partial U$. Similarly as the process in Theorem 3.1, the result follows. The proof is complete. □

**Corollary 3.4.** *Suppose that $\alpha>0$, $f,g,h$ are Carathéodory functions and*

$$\lim_{d\to+\infty}\frac{\max_{x,y\in[-d,d]}\sup_{t\in R}\int_{-\infty}^{+\infty} f\left(s,x,\frac{1}{\Phi^{-1}(\rho(s))}y\right)dt}{d^{p-1}}=0,$$

$$\lim_{d\to+\infty}\frac{\max_{x,y\in[-d,d]}\int_{-\infty}^{+\infty} g\left(t,x,\frac{1}{\Phi^{-1}(\rho(t))}y\right)dt}{d}=0,$$

$$\lim_{d\to+\infty}\frac{\max_{x,y\in[-d,d]}\int_{-\infty}^{+\infty} h\left(t,x,\frac{1}{\Phi^{-1}(\rho(t))}y\right)dt}{d}=0.$$

*Then BVP(6) has at least one solution if (26) holds.*

*Proof.* Let

$$\varepsilon^*=\min\left\{\frac{1}{3}\left[\frac{1}{\alpha}+D_{q-1}\frac{\gamma}{\sigma}\left(\frac{\beta}{\alpha}+\int_{-\infty}^{+\infty}\frac{1}{\Phi^{-1}(\rho(s))}ds\right)\right]^{-1},\right.$$

$$\frac{1}{3D_{q-1}}\frac{\sigma}{\alpha}\left(\frac{\beta}{\alpha}+\int_{-\infty}^{+\infty}\frac{1}{\Phi^{-1}(\rho(s))}ds\right)^{-1},$$

$$\left.\Phi\left(\frac{1}{3\times 2^{q-1}D_{q-1}}\left(\frac{\beta}{\alpha}+\int_{-\infty}^{+\infty}\frac{1}{\Phi^{-1}(\rho(s))}ds\right)^{-1}\right)\right\}.$$

Then, there exists $r > 0$, such that

$$\int_{-\infty}^{+\infty} g\left(t,x,\frac{1}{\Phi^{-1}(\rho(t))}y\right)dt \le \frac{1}{3}\left[\frac{1}{\alpha}+D_{q-1}\frac{\gamma}{\sigma}\left(\frac{\beta}{\alpha}+\int_{-\infty}^{+\infty}\frac{1}{\Phi^{-1}(\rho(s))}ds\right)\right]^{-1},$$

$$\int_{-\infty}^{+\infty} h\left(t,x,\frac{1}{\Phi^{-1}(\rho(t))}y\right)dt \le \frac{1}{3D_{q-1}}\frac{\sigma}{\alpha}\left(\frac{\beta}{\alpha}+\int_{-\infty}^{+\infty}\frac{1}{\Phi^{-1}(\rho(s))}ds\right)^{-1},$$

$$\int_{-\infty}^{+\infty} f\left(t,x,\frac{1}{\Phi^{-1}(\rho(t))}y\right)dt \le \Phi\left(\frac{1}{3\times 2^{q-1}D_{q-1}}\left(\frac{\beta}{\alpha}+\int_{-\infty}^{+\infty}\frac{1}{\Phi^{-1}(\rho(s))}ds\right)^{-1}\right).$$

By Corollary 3.3, BVP(6) has at least one solution. The proof is complete. □

The proofs of the following two corollaries are similar to those of Corollaries 3.3 and 3.4 and the details are omitted.

**Corollary 3.5.** *Suppose that $\gamma > 0$, $f,g,h$ are Carathéodory functions and there exists $r_1 > 0$ such that*

$$\int_{-\infty}^{+\infty} g\left(t,x,\frac{1}{\Phi^{-1}(\rho(t))}y\right)dt \le \frac{r_1}{3}\left[\frac{1}{\gamma}+D_{q-1}\frac{\alpha}{\sigma}\left(\frac{\delta}{\gamma}+\int_{-\infty}^{+\infty}\frac{1}{\Phi^{-1}(\rho(s))}ds\right)\right]^{-1},$$

$$\int_{-\infty}^{+\infty} h\left(t,x,\frac{1}{\Phi^{-1}(\rho(t))}y\right)dt \le \frac{r_1}{3D_{q-1}}\frac{\sigma}{\gamma}\left(\frac{\beta}{\alpha}+\int_{-\infty}^{+\infty}\frac{1}{\Phi^{-1}(\rho(s))}ds\right)^{-1},$$

$$\int_{-\infty}^{+\infty} f\left(s,x,\frac{1}{\Phi^{-1}(\rho(s))}y\right)dt \le \Phi\left(\frac{r_1}{3\times 2^{q-1}D_{q-1}}\left(\frac{\delta}{\gamma}+\int_{-\infty}^{+\infty}\frac{1}{\Phi^{-1}(\rho(s))}ds\right)^{-1}\right),$$

*where $x,y \in [-r_1,r_1]$. Then BVP(6) has at least one solution if*

$$\frac{\delta}{\gamma}+\int_{-\infty}^{+\infty}\frac{1}{\Phi^{-1}(\rho(s))}ds \ge 1. \tag{27}$$

*Proof.* By the assumptions, use Lemma 2.3 and (20), we have

$$\sup_{t\in R}|(Tx)(t)| =$$

$$\sup_{t\in R}\left|B_x+\int_{-\infty}^{t}\frac{1}{\Phi^{-1}(\rho(s))}\Phi^{-1}\left(A_x+\int_s^{+\infty}f(r,x(r),x'(r))dr\right)ds\right|$$

$$\leq \frac{\int_{-\infty}^{+\infty}|h(r,x(r),x'(t))|dr+\delta\left|\Phi^{-1}(A_x)\right|}{\gamma}$$

$$+\left|\int_{-\infty}^{t}\frac{1}{\Phi^{-1}(\rho(s))}\Phi^{-1}\left(A_x+\int_s^{+\infty}f(r,x(r),x'(r))dr\right)ds\right|$$

$$\leq 2^{q-1}D_{q-1}\left(\frac{\delta}{\gamma}+\int_{-\infty}^{+\infty}\frac{1}{\Phi^{-1}(\rho(s))}ds\right)\Phi^{-1}\left(\int_{-\infty}^{+\infty}|f(r,x(r),x'(r))|dr\right)$$

$$+\left[\frac{1}{\gamma}+D_{q-1}\frac{\alpha}{\sigma}\left(\frac{\delta}{\gamma}+\int_{-\infty}^{+\infty}\frac{1}{\Phi^{-1}(\rho(s))}ds\right)\right]\int_{-\infty}^{+\infty}|h(r,x(r),x'(r))|dr$$

$$+D_{q-1}\frac{\gamma}{\sigma}\left(\frac{\beta}{\alpha}+\int_{-\infty}^{+\infty}\frac{1}{\Phi^{-1}(\rho(s))}ds\right)\int_{-\infty}^{+\infty}|g(r,x(r),x'(r))|dr$$

$$\leq \frac{r_1}{3}+\frac{r_1}{3}+\frac{r_1}{3}=r_1.$$

The remainder of the proof is omitted. □

**Corollary 3.6.** *Suppose that $\gamma>0$, $f,h,h$ are Carathéodory functions and*

$$\lim_{d\to+\infty}\frac{\max_{x,y\in[-d,d]}\sup_{t\in R}\int_{-\infty}^{+\infty}f\left(s,x,\frac{1}{\Phi^{-1}(\rho(s))}y\right)dt}{d^{p-1}}=0,$$

$$\lim_{d\to+\infty}\frac{\max_{x,y\in[-d,d]}\int_{-\infty}^{+\infty}g\left(t,x,\frac{1}{\Phi^{-1}(\rho(t))}y\right)dt}{d}=0,$$

$$\lim_{d\to+\infty}\frac{\max_{x,y\in[-d,d]}\int_{-\infty}^{+\infty}h\left(t,x,\frac{1}{\Phi^{-1}(\rho(t))}y\right)dt}{d}=0.$$

*Then BVP(6) has at least one solution if (27) holds.*

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