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ON THE SOLUTIONS OF FRACTIONAL REACTION-DIFFUSION EQUATIONS

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In this paper, we obtain the solution of a fractional reaction-diffusion equation associated with the generalized Riemann-Liouville fractional derivative as the time derivative and Riesz-Feller fractional derivative as the space-derivative. The results are derived by the application of the Laplace and Fourier transforms in compact and elegant form in terms of Mittag-Leffler function and H-function. The results obtained here are of general nature and include the results investigated earlier by many authors.

1. Introduction

Fractional calculus is a field of applied mathematics that deals with derivatives and integrals of arbitrary orders. In recent years, it has turned out that many phenomena in engineering, physics, chemistry and other sciences can be described very successfully by models using mathematical tools from fractional calculus. For example, the nonlinear oscillation of earthquake can be modeled with fractional derivatives and the fluid-dynamic traffic model with fractional derivatives can eliminate the deficiency arising from the assumption of continuum traffic flow. Fractional derivatives are also used in modeling of many

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chemical processes, mathematical biology and many other problems in physics and engineering. These findings invoked the growing interest of studies of the fractional calculus in various fields such as physics, chemistry and engineering. Fractional reaction-diffusion models are studied due to their usefulness and importance in many areas of science and engineering. The reaction-diffusion equations arise naturally as description models of many evolution problems in the real world, as in chemistry [25,27], biology [18], etc. As is well known, complex behavior is peculiarity of systems modeled by reaction-diffusion equations and the Belousov-Zhabotinskii reaction [17,28] provides a classic example. The reaction-diffusion equations describes a population of diploid individuals (i.e., the ones that carry two genes) distributed in a two-dimensional habitat. The self-organization phenomena are mainly modeled on the base of nonlinear reaction-diffusion systems. At the same time, a separate non-uniform linear equation of reaction-diffusion (considered in the manuscript) also represents a scientific interest and has important applications in different area of science. General models for reaction-diffusion systems are investigated in [5,6,8,14,15] and others.

In the present article, we investigate the solution of a unified fractional reaction-diffusion equation associated with the generalized Riemann-Liouville fractional derivative and the Riesz-Feller derivative. This new model provides the extensions of the models discussed by Mainardi et al. [12], Haubold et al. [9,10] and Saxena et al. [22-24].

The right-sided Riemann-Liouville fractional integral of order α is defined by Miller and Ross [16, p.45], Samko et al. [20]:

$$ {}_{a}^{RL}D_{t}^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{a}^{t} (t-\tau)^{\alpha-1} f(\tau)\, d\tau, \; (t > a) \tag{1} $$

where $\mathrm{Re}(\alpha) > 0$.

The right-sided Riemann-Liouville fractional derivative of order α is defined as

$$ {}_{a}^{RL}D_{t}^{\alpha} f(t) = \left(\frac{d}{dt}\right)^{n} (I_{a}^{n-\alpha} f(t)) \quad (\mathrm{Re}(\alpha) > 0, n = [\mathrm{Re}(\alpha)] + 1), \tag{2} $$

where $[\alpha]$ represents the integer part of the number α.

By denoting the Laplace transform of a sufficiently well-behaved (generalized) function $f(t)$, $\bar{f}(s) = L\{f(t);s\} = \int_0^\infty e^{-st} f(t)dt$, $\mathrm{Re}(s) > 0$, the Caputo time-fractional derivative of order $\alpha\,(m-1 < \alpha \leq m, m \in N)$ turns out to be defined through

$$ L\,\{{}_{0}^{C}D_{t}^{\alpha} f(t);\, s\} = s^{\alpha} \bar{f}(s) - \sum_{r=0}^{m-1} s^{\alpha-1-r} f^{(r)}(0^{+}), \quad m-1 < \alpha \leq m. \tag{3} $$

This leads to define, see e.g. [7,19],

$$ {}_0^C D_t^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(\tau)\, d\tau}{(t-\tau)^{\alpha+1-m}}, & m-1<\alpha<m \\ \frac{d^m}{dt^m} f(t), & \alpha = m. \end{cases} \tag{4} $$

The generalization of the Riemann-Liouville fractional derivative operator (2) and Caputo fractional derivative operator (4) is made by Hilfer [11], by introducing a right-sided fractional derivative operator of two parameters of order α $(0<\alpha<1)$ and type β $(0 \le \beta \le 1)$ in the form

$$ {}_0D_{a+}^{\alpha,\beta} f(t) = \left(I_{a+}^{\beta(1-\alpha)} \frac{d}{dt} (I_{a+}^{(1-\beta)(1-\alpha)} f(t)) \right), \tag{5} $$

It is interesting to observe that for $\beta = 0$, (5) reduces to the classical Riemann-Liouville fractional derivative operator (2). On the other hand, for $\beta = 1$ it yields the Caputo fractional derivative operator defined by (4). The Laplace transform formula for this operator is given by Hilfer [11]

$$ L\, \{{}_0D_{0+}^{\alpha,\beta} f(t)\, ;\, s\} = s^{\alpha}\bar{f}(s) - s^{\beta(\alpha-1)} I_{0+}^{(1-\beta)(1-\alpha)} f(0^+), \;\; (0 < \alpha < 1), \tag{6} $$

where the initial value term $I_{0+}^{(1-\beta)(1-\alpha)} f(0^+)$, involves the Riemann-Liouville fractional integral operator of order $(1-\beta)\,(1-\alpha)$ evaluated in the limit as $t \to 0^+$. This derivative is also called the Hilfer fractional derivative. For more details and properties of this operator see Tomovski et al. [26].

Following Feller [3,4], it is conventional to define the Riesz-Feller space-fractional derivative of order α and skewness θ in terms of its Fourier transform as

$$ F\, \{{}_xD_{\theta}^{\alpha} f(x)\, ;\, k\} = -\, \psi_{\alpha}^{\theta}(k)\, f^*(k), \tag{7} $$

where

$$ \psi_{\alpha}^{\theta}(k) = |\, k\,|^{\alpha} \exp\left[i\, (\mathrm{sign} k)\, \frac{\theta\pi}{2} \right], 0 < \alpha \le 2,\; |\theta| \le \min\{\alpha, 2-\alpha\}. $$

2. Fractional reaction-diffusion equation

In this section, we will investigate the solution of the fractional reaction-diffusion equation (8). This system is a generalized form of the reaction-diffusion equation recently studied by Manne et al. [13]. The result is given in the form of the following Theorem.

Theorem 2.1. *Consider the fractional reaction-diffusion equation associated with the generalized Riemann-Liouville fractional derivative and Riesz-Feller fractional derivative in the form*

$$ {}_0D_t^{\alpha,\beta}N(x,t) \;=\; \eta\, {}_xD_\theta^{\gamma}N(x,t) + \;\phi(x,t), \tag{8} $$

where $\eta, t > 0$, $x \in R$; $\gamma, \theta, \alpha, \beta$ *are real parameters with the conditions* $0 < \gamma \leq 2$, $|\theta| \leq \min(\gamma, 2-\gamma)$, $0 < \alpha \leq 1$, $0 < \beta \leq 1$ *and the initial conditions*

$$ I_{0+}^{(1-\beta)(1-\alpha)}N(x,0^+) \;=\; f(x) \;\; ; \;\; \text{for } x \in R, \lim_{|x| \to \pm\infty} N(x,t) \;=\; 0, \; t > \; 0, \tag{9} $$

η *is diffusion constant,* $\phi(x,t)$ *is a nonlinear function belonging to the area of reaction-diffusion,* ${}_0D_t^{\alpha,\beta}$ *is the generalized Riemann-Liouville fractional derivative operator, defined by (5),* $I_{0+}^{(1-\beta)(1-\alpha)}N(x,0^+)$, *involves the Riemann-Liouville fractional integral operator of order* $(1-\beta)(1-\alpha)$ *evaluated in the limit as* $t \to 0^+$. *Then for the solution of (8), subject to the above constraints, there holds the formula*

$$ N(x,t) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} f^*(k)t^{\alpha-\beta(\alpha-1)-1}E_{\alpha,\alpha-\beta(\alpha-1)}[-\eta\; \psi_\gamma^\theta(k)t^\alpha]\exp(-ikx)dk $$

$$ +\frac{1}{2\pi}\int_0^t u^{\alpha-1}du\int_{-\infty}^{\infty}\phi^*(k,t-u)\,E_{\alpha,\alpha}[-\eta\;\psi_\gamma^\theta(k)u^\alpha]\exp(-ikx)\,dk. \tag{10} $$

In equation (10) and following, $E_{\alpha,\beta}(z)$ *denotes the generalized Mittag-Leffler function [1,2,21].*

Proof. Applying the Laplace transform with respect to the time variable t and using the initial conditions (9), we find that

$$ s^\alpha \bar{N}(x,s) - s^{\beta(\alpha-1)}\, f(x) = \;\eta\, {}_xD_\theta^\gamma \bar{N}(x,s) \;+\; \bar{\phi}(x,s). \tag{11} $$

If we apply the Fourier transform with respect to space variable x and use the formula (7), it yields

$$ s^\alpha \bar{N}^*(k,s) - s^{\beta(\alpha-1)}f^*(k) \;=\; -\,\eta\;\psi_\gamma^\theta(k)\,\bar{N}^*(k,s) \;+\; \bar{\phi}^*(k,s). \tag{12} $$

Solving for $\bar{N}^*(k,s)$, it gives

$$ \bar{N}^*(k,s) \;=\; \frac{f^*(k)\,s^{\beta(\alpha-1)}}{s^\alpha + \eta\;\psi_\gamma^\theta(k)} + \frac{\bar{\phi}^*(k,s)}{s^\alpha + \eta\;\psi_\gamma^\theta(k)}. \tag{13} $$

If we take the inverse Laplace transform of (13) and apply the formula

$$ L^{-1}\left[\frac{s^{\beta-1}}{s^\alpha + a};t\right] = t^{\alpha-\beta}\,E_{\alpha,\alpha-\beta+1}(-a\,t^\alpha), \tag{14} $$

where Re $(s) > 0$, Re $(\alpha - \beta + 1) > 0$, it is seen that

$$N^*(k,t) = f^*(k)\, t^{\alpha-\beta(\alpha-1)-1} E_{\alpha,\alpha-\beta(\alpha-1)}[-\eta\, \psi_\gamma^\theta(k)\, t^\alpha]$$

$$+ \int_0^t \phi^*(k,t-u)\, u^{\alpha-1} E_{\alpha,\alpha}[-\eta\, \psi_\gamma^\theta(k) u^\alpha] du. \qquad (15)$$

Finally, the required solution (10) is obtained by taking inverse Fourier transform of (15). □

3. Special Cases

If we set $\beta = 0$, then the Hilfer fractional derivative (5) reduces to a Riemann-Liouville fractional derivative (2) and the theorem yields the following result derived by Haubold et al. [10].

Corollary 3.1. *Consider the fractional reaction-diffusion model*

$$ {}_0^{RL}D_t^\alpha N(x,t) = \eta\, {}_xD_\theta^\gamma N(x,t) + \phi(x,t), \qquad (16)$$

where $\eta, t > 0,\ x \in R;\ \gamma, \theta, \alpha$ *are real parameters with the constraints* $0 < \gamma \leq 2,\quad |\theta| \leq \min\ (\gamma, 2-\gamma),\ \ 0 < \alpha \leq 1,$ *and the initial conditions*

$$ {}_0^{RL}D_t^{\alpha-1} N\,(x,0) = f(x)\ ;\ \text{for } x \in R, \lim_{|x| \to \pm\infty} N\,(x,t) = 0,\ t > 0, \qquad (17)$$

where ${}_0^{RL}D_t^\alpha$ *is the Riemann-Liouville fractional derivative operator of order* α *defined by (2),* $[{}_0^{RL}D_t^{\alpha-1} u(x,0)]$ *means the Riemann-Liouville fractional partial derivative of u(x,t) with respect to t of order* $\alpha-1$ *evaluated at t = 0,* η *is a diffusion constant and* $\phi(x,t)$ *is a nonlinear function belonging to the area of reaction-diffusion. Then the solution of (16), subject to the above initial conditions, is given by*

$$N\,(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f^*(k) t^{\alpha-1} E_{\alpha,\alpha}[-\eta\ \psi_\gamma^\theta(k) t^\alpha] \exp(-ikx) dk$$

$$+\frac{1}{2\pi} \int_0^t u^{\alpha-1} du \int_{-\infty}^{\infty} \phi^*(k,t-u)\, E_{\alpha,\alpha}[-\eta\ \psi_\gamma^\theta(k) u^\alpha] \exp\,(-ikx)\, dk. \qquad (18)$$

If we take $\beta = 1$, then the Hilfer fractional derivative (5) reduces to a Caputo fractional derivative operator (4) and it yields the following result which is the similar result as derived by Haubold et al. [9].

Corollary 3.2. *Consider the fractional reaction-diffusion model*

$$ {}_0^C D_t^{\alpha} N(x,t) = \eta \, {}_x D_{\theta}^{\gamma} N(x,t) + \phi(x,t), \tag{19} $$

where $\eta, t > 0$, $x \in R$; γ, θ, α *are real parameters with the constraints* $0 < \gamma \leq 2$, $|\theta| \leq \min(\gamma, 2-\gamma)$, $0 < \alpha \leq 1$, *and the initial conditions*

$$ N(x,0) = f(x) \; ; \; \text{for } x \in R, \lim_{|x| \to \pm\infty} N(x,t) = 0, \; t > 0, \tag{20} $$

where ${}_0^C D_t^{\alpha}$ *is the Caputo fractional derivative operator of order* α, η *is a diffusion constant and* $\phi(x,t)$ *is a nonlinear function belonging to the area of reaction-diffusion. Then the solution of (19), subject to the above initial conditions, is given by*

$$ N(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f^*(k) E_{\alpha,1}[-\eta \, \psi_{\gamma}^{\theta}(k) t^{\alpha}] \exp(-ikx) dk $$

$$ + \frac{1}{2\pi} \int_0^t u^{\alpha-1} du \int_{-\infty}^{\infty} \phi^*(k, t-u) \, E_{\alpha,\alpha}[-\eta \, \psi_{\gamma}^{\theta}(k) u^{\alpha}] \exp(-ikx) \, dk. \tag{21} $$

Next, if we set $\beta = 1$, $f(x) = \delta(x)$ and $\phi(x,t) = 0$, where $\delta(x)$ is the Dirac-delta function, then we arrive at the following interesting result obtained by Mainardi et al. [12].

Corollary 3.3. *The solution of the fractional diffusion equation*

$$ \frac{\partial^{\alpha} N(x,t)}{\partial t^{\alpha}} = \eta \, {}_x D_{\theta}^{\gamma} N(x,t), \; \eta > 0, \; x \in R, \; 0 < \alpha \leq 1, 0 < \gamma \leq 2, \tag{22} $$

with the initial conditions

$$ N(x,0) = \delta(x), \; \lim_{x \to \pm\infty} N(x,t) = 0, \tag{23} $$

where η *is a diffusion constant and* $\delta(x)$ *is the Dirac-delta function, is given by*

$$ N(x,t) = \frac{1}{\gamma |x|} H_{3,3}^{2,1} \left[\frac{|x|}{(\eta t^{\alpha})^{1/\gamma}} \middle|_{(1,1/\gamma),(1,1),(1,\rho)}^{(1,1/\gamma),(1,\alpha/\gamma),(1,\rho)} \right], \gamma > \alpha, \tag{24} $$

where $\rho = \frac{\gamma - \theta}{2\gamma}$.

If we set $\alpha = 1$, $0 < \gamma < 2$; $\theta \leq \min\{\gamma, 2-\gamma\}$, then equation (22) reduces to a space-fractional diffusion equation

$$ \frac{\partial N(x,t)}{\partial t} = \eta \, {}_x D_{\theta}^{\gamma} N(x,t), \; \eta > 0, \; x \in R, \tag{25} $$

with the initial conditions $N(x,0) = \delta(x)$, $\lim_{x \to \pm\infty} N(x,t) = 0$, where η is a diffusion constant and $\delta(x)$ is the Dirac-delta function. Then the solution of equation (25) is given by

$$L_{\gamma}^{\theta}(x) = \frac{1}{\gamma(\eta t)^{1/\gamma}} H_{2,2}^{1,1}\left[\frac{(\eta t)^{1/\gamma}}{|x|}\Big|_{(1/\gamma,1/\gamma),(\rho,\rho)}^{(1,1),(\rho,\rho)}\right], 0 < \gamma < 1, |\theta| < \gamma, \quad (26)$$

where $\rho = \frac{\gamma-\theta}{2\gamma}$.
The density represented by the above expression is known as γ-stable Lévy density.
Finally, if we take $\alpha = 1/2$, $\beta = 1$, in equation (8), then we get the following result which is the same result as derived by Haubold et al. [9].

Corollary 3.4. *Consider the following fractional reaction-diffusion model*

$$D_t^{1/2} N(x,t) = \eta\, {}_xD_{\theta}^{\gamma} N(x,t) + \phi(x,t), \quad (27)$$

where η, $t > 0$, $x \in R$; γ, θ are real parameters with the constraints $0 < \gamma \leq 2$, $|\theta| \leq \min(\gamma, 2-\gamma)$, and the initial conditions

$$N(x,0) = f(x), \text{ for } x \in R, \lim_{x \to \pm\infty} N(x,t) = 0. \quad (28)$$

Here η is diffusion constant and $\phi(x,t)$ is a nonlinear function belonging to the area of reaction-diffusion. Then for the solution of (27), subject to the above initial conditions, there holds the formula

$$N(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f^*(k)\, E_{1/2,1}(-\eta\, t^{\alpha}\, \psi_{\gamma}^{\theta}(k)) \exp(-ikx)\, dk$$

$$+\frac{1}{2\pi}\int_0^t u^{-1/2} du \int_{-\infty}^{\infty} \phi^*(k,t-u)\, E_{\frac{1}{2},\frac{1}{2}}(-\eta\, u^{1/2}\, \psi_{\gamma}^{\theta}(k)) \exp(-ikx)\, dk. \quad (29)$$

If we take $\theta = 0$ in (29), then it reduces to the result obtained by Saxena et al. [22] for the fractional reaction-diffusion equation.

4. Conclusions

In this paper, we have presented a solution of a fractional reaction-diffusion equation. The solution has been developed in terms of the generalized Mittag-Leffler and H-functions in a compact and elegant form with the help of Laplace and Fourier transforms and their inverses. Most of the results obtained are in a form suitable for numerical computation. The importance of the derived results lies in the fact that numerous results on fractional reaction, fractional diffusion, anomalous diffusion problems, and fractional telegraph equations scattered in the literature can be derived, as special cases, of the results investigated in this article.

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