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LACUNARY SEQUENCE SPACES DEFINED BY A MUSIELAK-ORLICZ FUNCTION

KULDIP RAJ - SUNIL K. SHARMA

In this paper we introduce lacunary sequence spaces defined by a Musielak-Orlicz function $\mathcal{M} = (M_k)$ and a sequence of modulus functions $F = (f_k)$. We also make an effort to study some topological properties and inclusion relations between these spaces.

1. Introduction and Preliminaries

Let l_∞ and c denote the Banach spaces of bounded and convergent sequences $x = (x_k)$ normed by $\|x\| = \sup_k |x_k|$, respectively. Let σ be a one-to-one mapping of the set of positive integers into itself such that $\sigma^m(n) = \sigma(\sigma^{m-1}(n)), m = 1, 2, 3, \cdots$. A continuous linear functional φ on l_∞ is said to be an invariant mean or a $\sigma-$mean if and only if

1. $\varphi(x) \geq 0$ when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n,
2. $\varphi(e) = 1$, where $e = (1, 1, 1, \cdots)$ and
3. $\varphi(\{x_{\sigma(n)}\}) = \varphi(\{x_n\})$ for all $x \in l_\infty$.

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For certain kinds of mappings σ every invariant mean φ extends the limit functional on the space c, in the sense that $\varphi(x) = \lim x$ for all $x \in c$. The set of all $\sigma-$convergent sequences will be denoted by V_σ. If $x = (x_n)$, set $Tx = (Tx_n) = (x_{\sigma(n)})$. It can be shown in [20] that

$$V_\sigma = \Big\{x \in l_\infty : \lim_m t_{mn}(x) = le \text{ uniformly in } n,\ l = \sigma - \lim x\Big\}, \qquad (1)$$

where $t_{mn}(x) = (x_n + Tx_n + ... + T^m x_n)/(m+1)$. The special case of (1) in which $\sigma(n) = n+1$ was given by Lorentz [7]. Several authors including Schaefer [20], Mursaleen [12], Savaş [19] and many others have studied invariant convergent sequences.

A bounded sequence $x = (x_k)$ is said to be strongly $\sigma-$convergent to a number l if and only if $(|x_k - l|) \in V_\sigma$ with σ - limit zero (see[13]). By $[V_\sigma]$, we denote the set of all strongly $\sigma-$convergent sequences. It is known that $c \subset [V_\sigma] \subset V_\sigma \subset l_\infty$. By a lacunary sequence $\theta = (i_r)$, $r = 0, 1, 2, \cdots$, where $i_0 = 0$, we shall mean an increasing sequence of non-negative integers $h_r = (i_r - i_{r-1}) \to \infty$ as $r \to \infty$. The intervals determined by θ are denoted by $I_r = (i_{r-1}, i_r]$ and the ratio i_r/i_{r-1} will be denoted by q_r. The space of lacunary strongly convergent sequences N_θ was defined by Freedman [4] as follows:

$$N_\theta = \Big\{x = (x_k) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} |x_k - L| = 0 \text{ for some } L\Big\}.$$

In [6], Kızmaz defined the sequence spaces

$$Z(\Delta) = \Big\{x = (x_k) : (\Delta x_k) \in Z\Big\} \text{ for } Z = \ell_\infty, c \text{ and } c_0,$$

where $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$. Et and Çolak [3] generalized the difference sequence spaces to the sequence spaces

$$Z(\Delta^n) = \Big\{x = (x_k) : (\Delta^n x_k) \in Z\Big\} \text{ for } Z = \ell_\infty, c \text{ and } c_0,$$

where $n \in \mathbb{N}$, $\Delta^0 x = (x_k)$, $\Delta x = (x_k - x_{k+1})$,

$$\Delta^n x = (\Delta^n x_k) = (\Delta^{n-1} x_k - \Delta^{n-1} x_{k+1}).$$

The generalized difference sequence has the following binomial representation

$$\Delta^n(x_k) = \sum_{v=0}^{n} (-1)^v \binom{n}{v} x_{k+v}.$$

An Orlicz function $M : [0,\infty) \to [0,\infty)$ is convex and continuous such that $M(0) = 0$, $M(x) > 0$ for $x > 0$. Let w be the space of all real or complex sequences $x = (x_k)$. Lindenstrauss and Tzafriri [8] used the idea of Orlicz function to define the following sequence space,

$$\ell_M = \Big\{x \in w : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty\Big\}$$

which is called as an Orlicz sequence space. The space ℓ_M is a Banach space with the norm

$$||x|| = \inf\Big\{\rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \le 1\Big\}.$$

It is shown in [8] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p (p \ge 1)$. An Orlicz function M satisfies Δ_2-condition if and only if for any constant $L > 1$ there exists a constant $K(L)$ such that $M(Lu) \le K(L)M(u)$ for all values of $u \ge 0$.
A sequence $\mathcal{M} = (M_k)$ of Orlicz functions is called a Musielak-Orlicz function (see [11], [14]). A sequence $\mathcal{N} = (N_k)$ defined by

$$N_k(v) = \sup\{|v|u - M_k(u) : u \ge 0\},\ k = 1,2,\cdots$$

is called the complementary function of a Musielak-Orlicz function $\mathcal{M}$. For a given Musielak-Orlicz function $\mathcal{M}$, the Musielak-Orlicz sequence space $t_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ are defined as follows:

$$t_{\mathcal{M}} = \Big\{x \in w : I_{\mathcal{M}}(cx) < \infty \text{ for some } c > 0\Big\},$$

$$h_{\mathcal{M}} = \Big\{x \in w : I_{\mathcal{M}}(cx) < \infty \text{ for all } c > 0\Big\},$$

where $I_{\mathcal{M}}$ is a convex modular defined by

$$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty} M_k(x_k),\ x = (x_k) \in t_{\mathcal{M}}.$$

We consider $t_{\mathcal{M}}$ equipped with the Luxemburg norm

$$||x|| = \inf\Big\{k > 0 : I_{\mathcal{M}}\Big(\frac{x}{k}\Big) \le 1\Big\}$$

or equipped with the Orlicz norm

$$||x||^0 = \inf\Big\{\frac{1}{k}\Big(1 + I_{\mathcal{M}}(kx)\Big) : k > 0\Big\}.$$

A Musielak-Orlicz function $\mathcal{M} = (M_k)$ is said to satisfy Δ_2-condition if there exist constants $a, K > 0$ and a sequence $c = (c_k)_{k=1}^{\infty} \in \ell_+^1$ (the positive cone of ℓ^1) such that the inequality

$$M_k(2u) \leq KM_k(u) + c_k$$

holds for all $k \in \mathbb{N}$ and $u \in R_+$, whenever $M_k(u) \leq a$.
Let X be a linear metric space. A function $p : X \to \mathbb{R}$ is called paranorm, if

1. $p(x) \geq 0$, for all $x \in X$,
2. $p(-x) = p(x)$, for all $x \in X$,
3. $p(x+y) \leq p(x) + p(y)$, for all $x, y \in X$,
4. if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [21], Theorem 10.4.2, P-183). For more detail about sequence spaces (see [2], [15], [16]) and references therein.
A sequence space E is said to be solid or normal if $(\alpha_k x_k) \in E$ whenever $(x_k) \in E$ and for all sequences of scalars(α_k) with $|\alpha_k| \leq 1$ (see [14]).
The following inequality will be used throughout the paper. If $0 \leq p_k \leq \sup p_k = G$, $K = \max(1, 2^{G-1})$ then

$$|a_k + b_k|^{p_k} \leq K\{|a_k|^{p_k} + |b_k|^{p_k}\} \tag{2}$$

for all k and $a_k, b_k \in \mathbb{C}$. Also $|a|^{p_k} \leq \max(1, |a|^G)$ for all $a \in \mathbb{C}$.
Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function and X be a locally convex Hausdorff topological linear space whose topology is determined by a set Q of seminorms q. Let $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of strictly positive real numbers. By $w(X)$ we denotes the space of all X-valued sequences. In this paper we define the following sequence spaces:

$$\Big[V_\sigma, \Delta^n, \theta, \mathcal{M}, u, p, q\Big]_1 = \Big\{x \in w(X) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k \in I_r} \Big[M_k\Big(q\Big(\frac{u_k \Delta^n x_{\sigma^k(m)} - l}{\rho}\Big)\Big)\Big]^{p_k} = 0, \text{ uniformly in } m, \text{ for some } \rho > 0\Big\},$$

$$\Big[V_\sigma,\Delta^n,\theta,\mathcal{M},u,p,q\Big]_0=\Big\{x\in w(X):\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(q\Big(\frac{u_k\Delta^n x_{\sigma^k(m)}}{\rho}\Big)\Big)\Big]^{p_k}=0,\quad\text{uniformly in } m,\text{ for some }\ \rho>0\Big\}$$

and

$$\Big[V_\sigma,\Delta^n,\theta,\mathcal{M},u,p,q\Big]_\infty=\Big\{x\in w(X):\sup_{r,m}\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(q\Big(\frac{u_k\Delta^n x_{\sigma^k(m)}}{\rho}\Big)\Big)\Big]^{p_k}<\infty,\quad\text{for some }\ \rho>0\Big\}.$$

If we take $\mathcal{M}(x)=x$, we get

$$\Big[V_\sigma,\Delta^n,\theta,u,p,q\Big]_1=\Big\{x\in w(X):\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\Big[q\Big(\frac{u_k\Delta^n x_{\sigma^k(m)}-l}{\rho}\Big)\Big]^{p_k}=0,\quad\text{uniformly in } m,\text{ for some }\ \rho>0\Big\},$$

$$\Big[V_\sigma,\Delta^n,\theta,u,p,q\Big]_0=\Big\{x\in w(X):\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\Big[q\Big(\frac{u_k\Delta^n x_{\sigma^k(m)}}{\rho}\Big)\Big]^{p_k}=0,\quad\text{uniformly in } m,\text{ for some }\ \rho>0\Big\}$$

and

$$\Big[V_\sigma,\Delta^n,\theta,u,p,q\Big]_\infty=\Big\{x\in w(X):\sup_{r,m}\frac{1}{h_r}\sum_{k\in I_r}\Big[q\Big(\frac{u_k\Delta^n x_{\sigma^k(m)}}{\rho}\Big)\Big]^{p_k}<\infty,\quad\text{for some }\ \rho>0\Big\}.$$

If we take $p=(p_k)=1$ for all k, we get

$$\Big[V_\sigma,\Delta^n,\theta,\mathcal{M},u,q\Big]_1=\Big\{x\in w(X):\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(q\Big(\frac{u_k\Delta^n x_{\sigma^k(m)}-l}{\rho}\Big)\Big)\Big]=0,\quad\text{uniformly in } m,\text{ for some }\ \rho>0\Big\},$$

$$\Big[V_\sigma,\Delta^n,\theta,\mathcal{M},u,q\Big]_0=\Big\{x\in w(X):\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(q\Big(\frac{u_k\Delta^n x_{\sigma^k(m)}}{\rho}\Big)\Big)\Big]=0,\quad\text{uniformly in } m,\text{ for some }\ \rho>0\Big\}$$

and

$$\Big[V_\sigma,\Delta^n,\theta,\mathcal{M},u,q\Big]_\infty=\Big\{x\in w(X):\sup_{r,m}\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(q\Big(\frac{u_k\Delta^n x_{\sigma^k(m)}}{\rho}\Big)\Big)\Big]<\infty,\ \text{for some }\ \rho>0\Big\}.$$

The main purpose of this paper is to introduce and study some lacunary sequence spaces defined by a Musielak-Orlicz function. We examine some topological properties and inclusion relations between the spaces $\Big[V_\sigma,\Delta^n,\theta,\mathcal{M},u,p,q\Big]_Z$ in the second section. Third section devoted to the study of lacunary sequence spaces defined by a sequence of modulus functions. We also examine some topological properties and inclusion relation between the spaces $\Big[V_\sigma,\Delta^n,\theta,F,p,q\Big]_Z$. Throughout the paper Z will denote any one of the notation 0, 1 or ∞.

2. Lacunary sequence spaces defined by a Musielak-Orlicz function

In this section of the paper we study very interesting properties like linearity, paranorm and some attractive inclusion relations between the spaces $\Big[V_\sigma,\Delta^n,\theta,\mathcal{M},u,p,q\Big]_Z$.

Theorem 2.1. *For any Musielak-Orlicz function $\mathcal{M}=(M_k)$ and for a bounded sequence of positive real numbers $p=(p_k)$, the spaces $\Big[V_\sigma,\Delta^n,\theta,\mathcal{M},u,p,q\Big]_Z$ are linear over the field of complex numbers $\mathbb{C}$.*

Proof. Let $x=(x_k),\ y=(y_k)\in\Big[V_\sigma,\Delta^n,\theta,\mathcal{M},u,p,q]_0$ and let $\alpha,\beta\in\mathbb{C}$. Then there exist positive numbers ρ_1 and ρ_2 such that

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(q\Big(\frac{u_k\Delta^n x_{\sigma^k(m)}}{\rho_1}\Big)\Big)\Big]^{p_k}=0$$

and

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(q\Big(\frac{u_k\Delta^n y_{\sigma^k(m)}}{\rho_2}\Big)\Big)\Big]^{p_k}=0.$$

Define $\rho_3=\max(2|\alpha|\rho_1,2|\beta|\rho_2)$. Since $\mathcal{M}=(M_k)$ is non-decreasing and convex, q is a seminorm and so by using inequality (2), we have

$$\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(q\Big(\frac{\alpha u_k\Delta^n x_{\sigma^k(m)}+\beta u_k\Delta^n y_{\sigma^k(m)}}{\rho_3}\Big)\Big)\Big]^{p_k}$$
$$\leq \frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(q\Big(\frac{\alpha u_k\Delta^n x_{\sigma^k(m)}}{\rho_3}\Big)+q\Big(\frac{\beta u_k\Delta^n y_{\sigma^k(m)}}{\rho_3}\Big)\Big)\Big]^{p_k}$$
$$\leq K\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(q\Big(\frac{u_k\Delta^n x_{\sigma^k(m)}}{\rho_1}\Big)\Big)\Big]^{p_k}+K\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(q\Big(\frac{u_k\Delta^n y_{\sigma^k(m)}}{\rho_2}\Big)\Big)\Big]^{p_k}$$
$$\to 0 \quad \text{as } r\to\infty \quad \text{uniformly in } m.$$

This proves that $\Big[V_\sigma,\Delta^n,\theta,\mathcal{M},u,p,q\Big]_0$ is a linear space. Similarly, we can prove that $\Big[V_\sigma,\Delta^n,\theta,\mathcal{M},u,p,q\Big]_1$ and $\Big[V_\sigma,\Delta^n,\theta,\mathcal{M},u,p,q\Big]_\infty$ are linear spaces. □

Theorem 2.2. *For any Musielak-Orlicz function $\mathcal{M}=(M_k)$ and $p=(p_k)$ be a bounded sequence of positive real numbers, the spaces $\Big[V_\sigma,\Delta^n,\theta,\mathcal{M},u,p,q\Big]_Z$ are paranormed spaces, paranormed defined by*

$$g(x)=\inf\Big\{\rho^{\frac{p_n}{H}}:\Big(\sup_{k\geq 1}\Big[M_k\Big(q\Big(\frac{u_k\Delta^n x_{\sigma^k}(m)}{\rho}\Big)\Big)\Big]^{p_k}\Big)^{\frac{1}{H}}\leq 1,\ \rho>0,$$
$$\textit{uniformly in } m\Big\},$$

where $H=\max(1,\sup_k p_k)$.

Proof. We shall prove the result for the case $\Big[V_\sigma,\Delta^n,\theta,\mathcal{M},u,p,q\Big]_\infty$. Clearly $g(x)=g(-x)$ and $g(\theta)=0$ where θ is the zero sequence of X. Let $x=(x_k),\ y=(y_k)\in\Big[V_\sigma,\Delta^n,\theta,\mathcal{M},u,p,q\Big]_\infty$. Then there exist positive numbers ρ_1 and ρ_2 such that

$$\sup_{k\geq 1}\Big[M_k\Big(q\Big(\frac{u_k\Delta^n x_{\sigma^k}(m)}{\rho_1}\Big)\Big)\Big]^{p_k}\leq 1,\ \text{ uniformly in } m$$

and

$$\sup_{k\geq 1}\Big[M_k\Big(q\Big(\frac{u_k\Delta^n y_{\sigma^k}(m)}{\rho_2}\Big)\Big)\Big]^{p_k}\leq 1,\ \text{ uniformly in } m.$$

Let $\rho = \rho_1 + \rho_2$ and by using Minkowski's inequality, we have

$$\begin{aligned}
&\sup_{k\geq 1}\Big[M_k\Big(q\Big(\frac{u_k\Delta^n x_{\sigma^k}(m) + u_k\Delta^n y_{\sigma^k}(m)}{\rho}\Big)\Big)\Big]^{p_k}\\
&= \sup_{k\geq 1}\Big[M_k\Big(q\Big(\frac{u_k\Delta^n x_{\sigma^k}(m) + u_k\Delta^n y_{\sigma^k}(m)}{\rho_1+\rho_2}\Big)\Big)\Big]^{p_k}\\
&\leq \Big(\frac{\rho_1}{\rho_1+\rho_2}\Big)\sup_{k\geq 1}\Big[M_k\Big(q\Big(\frac{u_k\Delta^n x_{\sigma^k}(m)}{\rho_1}\Big)\Big)\Big]^{p_k}\\
&+ \Big(\frac{\rho_2}{\rho_1+\rho_2}\Big)\sup_{k\geq 1}\Big[M_k\Big(q\Big(\frac{u_k\Delta^n y_{\sigma^k}(m)}{\rho_2}\Big)\Big)\Big]^{p_k}\\
&\leq 1, \text{ uniformly in } m.
\end{aligned}$$

Hence

$$\begin{aligned}
&g(x+y)\\
&= \inf\Big\{(\rho_1+\rho_2)^{\frac{p_n}{H}} : \Big(\sup_{k\geq 1}\Big[M_k\Big(q\Big(\frac{u_k\Delta^n x_{\sigma^k}(m) + u_k\Delta^n y_{\sigma^k}(m)}{\rho}\Big)\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1,\\
&\qquad \rho > 0, \text{ uniformly in } m\Big\}\\
&\leq \inf\Big\{(\rho_1)^{\frac{p_n}{H}} : \Big(\sup_{k\geq 1}\Big[M_k\Big(q\Big(\frac{u_k\Delta^n x_{\sigma^k}(m)}{\rho_1}\Big)\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1,\\
&\qquad \rho_1 > 0, \text{ uniformly in } m\Big\}\\
&+ \inf\Big\{(\rho_2)^{\frac{p_n}{H}} : \Big(\sup_{k\geq 1}\Big[M_k\Big(q\Big(\frac{u_k\Delta^n y_{\sigma^k}(m)}{\rho_2}\Big)\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1,\\
&\qquad \rho_2 > 0, \text{ uniformly in } m\Big\}\\
&= g(x) + g(y).
\end{aligned}$$

Finally, we prove that the scalar multiplication is continuous. Let λ be any complex number. By definition, we have

$$\begin{aligned}
g(\lambda x) &= \inf\Big\{\rho^{\frac{p_n}{H}} : \Big(\sup_{k\geq 1}\Big[M_k\Big(q\Big(\frac{\lambda u_k\Delta^n x_{\sigma^k}(m)}{\rho}\Big)\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1,\\
&\qquad \rho > 0, \text{ uniformly in } m\Big\}\\
&= \inf\Big\{(|\lambda| t)^{\frac{p_n}{H}} : \Big(\sup_{k\geq 1}\Big[M_k\Big(q\Big(\frac{u_k\Delta^n x_{\sigma^k}(m)}{t}\Big)\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1,\ t > 0,\\
&\qquad \text{uniformly in } m\Big\},
\end{aligned}$$

where $t = \frac{\rho}{|\lambda|}$. This completes the proof of the theorem. □

Theorem 2.3. *Let $\mathcal{M}' = (M'_k)$ and $\mathcal{M}'' = (M''_k)$ be two Musielak-Orlicz functions. Then we have* $\Big[V_\sigma,,\Delta^n\theta,\mathcal{M}',u,p,q\Big]_Z \cap \Big[V_\sigma,\Delta^n,\theta,\mathcal{M}'',u,p,q\Big]_Z \subseteq \Big[V_\sigma,\Delta^n,\theta,\mathcal{M}'+\mathcal{M}'',u,p,q\Big]_Z$.

Proof. The proof is easy so we omit it. □

Theorem 2.4. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers and q_1, q_2 are two seminorms on X. Then*

$$\Big[V_\sigma,\Delta^n,\theta,\mathcal{M},u,p,q_1\Big]_Z \cap \Big[V_\sigma,\Delta^n,\theta,\mathcal{M},u,p,q_2\Big]_Z \neq \varnothing.$$

Proof. The zero elements belongs to $\Big[V_\sigma,\Delta^n,\theta,\mathcal{M},u,p,q_1\Big]_Z$ and $\Big[V_\sigma,\Delta^n,\theta,\mathcal{M},u,p,q_2\Big]_Z$, thus the intersection is non empty. □

Theorem 2.5. *For any Musielak-Orlicz function $\mathcal{M} = (M_k)$, let q_1, q_2 be two seminorms on X. Then the following results holds:*
(i) If q_1 is stronger than q_2, then

$$\Big[V_\sigma,\Delta^n,\theta,\mathcal{M}',u,p,q_1\Big]_Z \subset \Big[V_\sigma,\theta,\mathcal{M}',u,p,q_2\Big]_Z,$$

(ii)

$$\Big[V_\sigma,\Delta^n,\theta,\mathcal{M},u,p,q_1\Big]_Z \cap \Big[V_\sigma,\Delta^n,\theta,\mathcal{M},u,p,q_2\Big]_Z \subset \Big[V_\sigma,\theta,\mathcal{M},u,p,q_1+q_2\Big]_Z.$$

Proof. The proof is easy so we omit it. □

Theorem 2.6. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function. Then*

$$\Big[V_\sigma,\Delta^n,\theta,\mathcal{M},u,p,q\Big]_0 \subset \Big[V_\sigma,\Delta^n,\theta,\mathcal{M},u,p,q\Big]_1 \subset \Big[V_\sigma,\Delta^n,\theta,\mathcal{M},u,p,q\Big]_\infty.$$

Proof. The inclusion $\Big[V_\sigma,\Delta^n,\theta,\mathcal{M},u,p,q\Big]_0 \subset \Big[V_\sigma,\Delta^n,\theta,\mathcal{M},u,p,q\Big]_1$ is obvious. Let $x = (x_k) \in \Big[V_\sigma,\Delta^n,\theta,\mathcal{M},u,p,q\Big]_1$. Then we have

$$\begin{aligned}
&\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(q\Big(\frac{u_k\Delta^n x_{\sigma^k(m)}}{\rho}\Big)\Big)\Big]^{p_k} \\
&\leq \frac{K}{h_r}\sum_{k\in I_r}\Big[M_k\Big(q\Big(\frac{u_k\Delta^n x_{\sigma^k(m)-l}}{\rho}\Big)\Big)\Big]^{p_k} + \frac{K}{h_r}\sum_{k\in I_r}\Big[M_k\Big(q\Big(\frac{l}{\rho}\Big)\Big)\Big]^{p_k} \\
&\leq \frac{K}{h_r}\sum_{k\in I_r}\Big[M_k\Big(q\Big(\frac{u_k\Delta^n x_{\sigma^k(m)-l}}{\rho}\Big)\Big)\Big]^{p_k} + K\max\Big\{1,\Big[M_k\Big(q\Big(\frac{l}{\rho}\Big)\Big)\Big]^G\Big\}.
\end{aligned}$$

Thus $x=(x_k)\in\Big[V_\sigma,\Delta^n,\theta,\mathcal{M},u,p,q\Big]_\infty$. This completes the proof of the theorem. □

Theorem 2.7. *Let $\mathcal{M}=(M_k)$ be a Musielak-Orlicz function and $\theta=(i_r)$ be a lacunary sequence. Then the following result holds:*
(i) If $\liminf_r q_r>1$, then $\Big[V_\sigma,\Delta^n,\mathcal{M},u,p,q\Big]_Z\subset\Big[V_\sigma,\Delta^n,\theta,\mathcal{M},u,p,q\Big]_Z$,
(ii) If $\limsup_r q_r<\infty$, then $\Big[V_\sigma,\Delta^n,\theta,\mathcal{M}',u,p,q_1\Big]_Z\subset\Big[V_\sigma,\Delta^n,\mathcal{M}',u,p,q_2\Big]_Z$,
(iii) If $1<\liminf_r q_r\le\limsup_r q_r<\infty$, then

$$\Big[V_\sigma,\Delta^n,\mathcal{M},u,p,q\Big]_Z=\Big[V_\sigma,\Delta^n,\theta,\mathcal{M},u,p,q\Big]_Z.$$

Proof. The proof is easy so we omit it. □

Theorem 2.8. *Let $0<p_k\le t_k$ and $\left(\frac{t_k}{p_k}\right)$ be bounded. Then*

$$\Big[V_\sigma,\Delta^n,\theta,\mathcal{M},u,t,q\Big]_Z\subset\Big[V_\sigma,\Delta^n,\theta,\mathcal{M},u,p,q\Big]_Z.$$

Proof. We shall prove it for the case $Z=1$. Let $x=(x_k)\in\Big[V_\sigma,\Delta^n,\theta,\mathcal{M},u,t,q\Big]_1$. We write

$$S_k=\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(q\Big(\frac{u_k\Delta^n x_{\sigma^k(m)}-l}{\rho}\Big)\Big)\Big]^{p_k}$$

and $(\mu_k)=\left(\frac{p_k}{t_k}\right)$ for all $k\in\mathbb{N}$. Then $0<\mu_k\le 1$ for all $k\in\mathbb{N}$. Take $0<\mu<\mu_k$ for all $k\in\mathbb{N}$. Define the sequence u_k and v_k as follows:
For $S_k\ge 1$, let $u_k=S_k$ and $v_k=0$ and for $S_k<1$, let $u_k=0$ and $v_k=S_k$. Then clearly for all $k\in\mathbb{N}$, we have $S_k=u_k+v_k, S_k^{\mu_k}=u_k^{\mu_k}+v_k^{\mu_k}$. Now it follows that $u_k^{\mu_k}\le u_k\le S_k$ and $v_k^{\mu_k}\le v_k^{\mu}$. Therefore,

$$\frac{1}{h_r}\sum_{k\in I_r}S_k^{\mu_k}=\frac{1}{h_r}\sum_{k\in I_r}(u_k^{\mu_k}+v_k^{\mu_k})\le\frac{1}{h_r}\sum_{k\in I_r}S_k+\frac{1}{h_r}\sum_{k\in I_r}v_k^{\mu}.$$

Now for each k,

$$\begin{aligned}\frac{1}{h_r}\sum_{k\in I_r}v_k^{\mu}&=\sum_{k\in I_r}\Big(\frac{1}{h_r}v_k\Big)^{\mu}\Big(\frac{1}{h_r}\Big)^{1-\mu}\\&\le\Big(\sum_{k\in I_r}\Big[\Big(\frac{1}{h_r}v_k\Big)^{\mu}\Big]^{\frac{1}{\mu}}\Big)^{\mu}\Big(\sum_{k\in I_r}\Big[\Big(\frac{1}{h_r}\Big)^{1-\mu}\Big]^{\frac{1}{1-\mu}}\Big)^{1-\mu}\\&=\Big(\frac{1}{h_r}\sum_{k\in I_r}v_k\Big)^{\mu}\end{aligned}$$

and so

$$\frac{1}{h_r}\sum_{k\in I_r} S_k^{\mu_k} \le \frac{1}{h_r}\sum_{k\in I_r} S_k + \Big(\frac{1}{h_r}\sum_{k\in I_r} v_k\Big)^{\mu}.$$

Hence $x=(x_k)\in \Big[V_\sigma,\Delta^n,\theta,\mathcal{M},u,p,q\Big]_1$. Similarly we can prove other cases. □

Theorem 2.9. *The sequence space* $\Big[V_\sigma,\Delta^n,\theta,\mathcal{M},u,p,q\Big]_\infty$ *is solid.*

Proof. Let $x=(x_k)\in \Big[V_\sigma,\Delta^n,\theta,\mathcal{M},u,p,q\Big]_\infty$, that is

$$\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(q\Big(\frac{u_k\Delta^n x_{\sigma^k(m)}}{\rho}\Big)\Big)\Big]^{p_k} < \infty.$$

Let (α_k) be a sequence of scalars such that $|\alpha_k|\le 1$ for all $k\in\mathbb{N}$. Thus we have

$$\begin{aligned}\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(q\Big(\frac{\alpha_k u_k\Delta^n x_{\sigma^k(m)}}{\rho}\Big)\Big)\Big]^{p_k} &\le \frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(q\Big(\frac{u_k\Delta^n x_{\sigma^k(m)}}{\rho}\Big)\Big)\Big]^{p_k}\\ &< \infty.\end{aligned}$$

This shows that $(\alpha_k x_k)\in \Big[V_\sigma,\Delta^n,\theta,\mathcal{M},u,p,q\Big]_\infty$ for all sequences of scalars (α_k) with $|\alpha_k|\le 1$ for all $k\in\mathbb{N}$, whenever $(x_k)\in \Big[V_\sigma,\Delta^n,\theta,\mathcal{M},u,p,q\Big]_\infty$. Hence the space $\Big[V_\sigma,\Delta^n,\theta,\mathcal{M},u,p,q\Big]_\infty$ is a solid sequence space. This completes the proof of the theorem. □

Corollary 2.10. *The sequence space* $\Big[V_\sigma,\Delta^n,\theta,\mathcal{M},u,p,q\Big]_\infty$ *is monotone.*

Proof. It is easy to prove so we omit the details. □

3. Lacunary sequence spaces defined by a sequence of modulus functions

A modulus function is a function $f:[0,\infty)\to[0,\infty)$ such that

1. $f(x)=0$ if and only if $x=0$,
2. $f(x+y)\le f(x)+f(y)$ for all $x\ge 0$, $y\ge 0$,
3. f is increasing
4. f is continuous from right at 0.

It follows that f must be continuous everywhere on $[0,\infty)$. The modulus function may be bounded or unbounded. For example, if we take $f(x)=\frac{x}{x+1}$, then $f(x)$ is bounded. If $f(x)=x^p$, $0<p<1$, then the modulus $f(x)$ is unbounded. Subsequentially, modulus function has been discussed in ([1], [9], [10], [17], [18]) and references therein.
Let $F=(f_k)$ be a sequence of modulus function, X be a locally convex Hausdorff topological linear space whose topology is determined by a set Q of seminorms q. Let $p=(p_k)$ be a bounded sequence of positive real numbers and $u=(u_k)$ be a sequence of strictly positive real numbers. By $w(X)$ be denote the space of all X-valued sequences. In this section we define the following sequence spaces:

$$\Big[V_\sigma,\Delta^n,\theta,F,p,q\Big]_1=\Big\{x=(x_k)\in w(X):\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\Big[f_k\Big(q\Big(\frac{\Delta^n x_{\sigma^k(m)}-l}{\rho}\Big)\Big)\Big]^{p_k}=0,\quad \text{uniformly in } m, \text{ for some } \rho>0\Big\},$$

$$\Big[V_\sigma,\Delta^n,\theta,F,p,q\Big]_0=\Big\{x=(x_k)\in w(X):\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\Big[f_k\Big(q\Big(\frac{\Delta^n x_{\sigma^k(m)}}{\rho}\Big)\Big)\Big]^{p_k}=0,\quad \text{uniformly in } m, \text{ for some } \rho>0\Big\}$$

and

$$\Big[V_\sigma,\Delta^n,\theta,F,p,q\Big]_\infty=\Big\{x=(x_k)\in w(X):\sup_{r,m}\frac{1}{h_r}\sum_{k\in I_r}\Big[f_k\Big(q\Big(\frac{\Delta^n x_{\sigma^k(m)}}{\rho}\Big)\Big)\Big]^{p_k}<\infty,\quad \text{for some } \rho>0\Big\}.$$

If we take $F(x)=x$, we get

$$\Big[V_\sigma,\Delta^n,\theta,p,q\Big]_1=\Big\{x=(x_k)\in w(X):\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\Big[q\Big(\frac{\Delta^n x_{\sigma^k(m)}-l}{\rho}\Big)\Big]^{p_k}=0\quad \text{uniformly in } m, \text{ for some } \rho>0\Big\},$$

$$\Big[V_\sigma,\Delta^n,\theta,p,q\Big]_0=\Big\{x=(x_k)\in w(X):\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\Big[q\Big(\frac{\Delta^n x_{\sigma^k(m)}}{\rho}\Big)\Big]^{p_k}=0,\quad \text{uniformly in } m, \text{ for some } \rho>0\Big\}$$

and

$$\Big[V_\sigma,\Delta^n,\theta,p,q\Big]_\infty=\Big\{x=(x_k)\in w(X):\sup_{r,m}\frac{1}{h_r}\sum_{k\in I_r}\Big[q\Big(\frac{\Delta^n x_{\sigma^k(m)}}{\rho}\Big)\Big]^{p_k}<\infty,$$
$$\text{for some }\ \rho>0\Big\}.$$

If we take $p=(p_k)=1$ for all k, we get

$$\Big[V_\sigma,\Delta^n,\theta,F,q\Big]_1=\Big\{x=(x_k)\in w(X):\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\Big[f_k\Big(q\Big(\frac{\Delta^n x_{\sigma^k(m)}-l}{\rho}\Big)\Big)\Big]$$
$$=0,\quad\text{uniformly in } m,\text{ for some }\ \rho>0\Big\},$$

$$\Big[V_\sigma,\Delta^n,\theta,F,q\Big]_0=\Big\{x=(x_k)\in w(X):\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\Big[f_k\Big(q\Big(\frac{\Delta^n x_{\sigma^k(m)}}{\rho}\Big)\Big)\Big]=0,$$
$$\text{uniformly in } m,\text{ for some }\ \rho>0\Big\}$$

and

$$\Big[V_\sigma,\Delta^n,\theta,F,q\Big]_\infty=\Big\{x=(x_k)\in w(X):\sup_{r,m}\frac{1}{h_r}\sum_{k\in I_r}\Big[f_k\Big(q\Big(\frac{\Delta^n x_{\sigma^k(m)}}{\rho}\Big)\Big)\Big]<\infty,$$
$$\text{for some }\ \rho>0\Big\}.$$

The main purpose of this section is to study some topological properties and some inclusion relations between of the spaces $\Big[V_\sigma,\Delta^n,\theta,F,p,q\Big]_Z$.

Theorem 3.1. *Let $F=(f_k)$ be a sequence of modulus functions and $p=(p_k)$ be a bounded sequence of positive real numbers. Then the spaces $\Big[V_\sigma,\Delta^n,\theta,F,$ $p,q\Big]_Z$, $Z=0,1,\infty$ are linear over the field of complex numbers $\mathbb{C}$.*

Proof. We shall prove the result for the case $\Big[V_\sigma,\Delta^n,\theta,F,p,q\Big]_0$. Let $x=(x_k)$, $y=(y_k)\in\Big[V_\sigma,\Delta^n,\theta,F,p,q\Big]_0$ and let $\alpha,\beta\in\mathbb{C}$. Then there exist positive numbers ρ_1 and ρ_2 such that

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\Big[f_k\Big(q\Big(\frac{\Delta^n x_{\sigma^k(m)}}{\rho_1}\Big)\Big)\Big]^{p_k}=0,\ \text{ uniformly in } m$$

and

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\Big[f_k\Big(q\Big(\frac{\Delta^n y_{\sigma^k(m)}}{\rho_2}\Big)\Big)\Big]^{p_k}=0,\ \text{uniformly in } m.$$

Define $\rho_3=\max(2|\alpha|\rho_1,2|\beta|\rho_2)$. Since $F=(f_k)$ is non-decreasing, q is a semi-norm and so by using inequality (2), we have

$$\begin{aligned}&\frac{1}{h_r}\sum_{k\in I_r}\Big[f_k\Big(q\Big(\frac{\alpha\Delta^n x_{\sigma^k(m)}+\beta\Delta^n y_{\sigma^k(m)}}{\rho_3}\Big)\Big)\Big]^{p_k}\\ &\qquad\le\frac{1}{h_r}\sum_{k\in I_r}\Big[f_k\Big(q\Big(\frac{\alpha\Delta^n x_{\sigma^k(m)}}{\rho_3}\Big)+q\Big(\frac{\beta\Delta^n y_{\sigma^k(m)}}{\rho_3}\Big)\Big)\Big]^{p_k}\\ &\le K\frac{1}{h_r}\sum_{k\in I_r}\Big[f_k\Big(q\Big(\frac{\Delta^n x_{\sigma^k(m)}}{\rho_1}\Big)\Big)\Big]^{p_k}+K\frac{1}{h_r}\sum_{k\in I_r}\Big[f_k\Big(q\Big(\frac{\Delta^n y_{\sigma^k(m)}}{\rho_2}\Big)\Big)\Big]^{p_k}\to 0\\ &\qquad\qquad\text{as } r\to\infty \text{ uniformly in } m.\end{aligned}$$

This proves that $\Big[V_\sigma,\Delta^n,\theta,F,p,q\Big]_0$ is a linear space. Similarly, we can prove that $\Big[V_\sigma,\Delta^n,\theta,F,p,q\Big]_1$ and $\Big[V_\sigma,\Delta^n,\theta,F,p,q\Big]_\infty$ are linear spaces. □

Theorem 3.2. *Let $F=(f_k)$ be a sequence of modulus functions and $p=(p_k)$ be a bounded sequence of positive real numbers. Then the spaces $\Big[V_\sigma,\Delta^n,\theta,F,p,q\Big]_Z$ are paranormed spaces, paranormed defined by*

$$g^*(x)=\inf\Big\{\rho^{\frac{p_n}{H}}:\Big[\sup_{k\ge1}f_k\Big(q\Big(\frac{\Delta^n x_{\sigma^k}(m)}{\rho}\Big)\Big)^{p_k}\Big]^{\frac{1}{H}}\le 1,\rho>0\ \textit{uniformly in } m\Big\},$$

where $H=\max(1,\sup_k p_k)$.

Proof. We shall prove the theorem for the case $\Big[V_\sigma,\Delta^n,\theta,F,p,q\Big]_\infty$. Clearly, $g^*(x)=g(-x)$ and $g^*(\theta)=0$ where θ is the zero sequence of X. Let $x=(x_k),y=(y_k)\in\Big[V_\sigma,\Delta^n,\theta,F,p,q\Big]_\infty$. Then there exist positive numbers ρ_1 and ρ_2 such that

$$\sup_{k\ge1}f_k\Big(q\Big(\frac{\Delta^n x_{\sigma^k}(m)}{\rho_1}\Big)\Big)^{p_k}\le 1,\ \text{uniformly in } m$$

and

$$\sup_{k\ge1}f_k\Big(q\Big(\frac{\Delta^n y_{\sigma^k}(m)}{\rho_2}\Big)\Big)^{p_k}\le 1,\ \text{uniformly in } m.$$

Let $\rho = \rho_1 + \rho_2$ and by using Minkowski's inequality, we have

$$\sup_{k\geq 1} f_k\Big(q\Big(\frac{\Delta^n x_{\sigma^k}(m) + \Delta^n y_{\sigma^k}(m)}{\rho}\Big)\Big)^{p_k} = \sup_{k\geq 1} f_k\Big(q\Big(\frac{\Delta^n x_{\sigma^k}(m) + \Delta^n y_{\sigma^k}(m)}{\rho_1+\rho_2}\Big)\Big)\Big]^{p_k}$$
$$\leq \Big(\frac{\rho_1}{\rho_1+\rho_2}\Big)\sup_{k\geq 1} f_k\Big[q\Big(\frac{\Delta^n x_{\sigma^k}(m)}{\rho_1}\Big)\Big]^{p_k} + \Big(\frac{\rho_2}{\rho_1+\rho_2}\Big)\sup_{k\geq 1} f_k\Big[q\Big(\frac{\Delta^n y_{\sigma^k}(m)}{\rho_2}\Big)\Big]^{p_k}$$
$$\leq 1, \text{uniformly in } m.$$

Hence

$$\begin{aligned} g^*(x+y) = \inf\Big\{(\rho_1+\rho_2)^{\frac{p_n}{H}} : \Big(\sup_{k\geq 1} f_k\Big(q\Big(\frac{\Delta^n x_{\sigma^k}(m) + \Delta^n y_{\sigma^k}(m)}{\rho}\Big)\Big)^{p_k}\Big)^{\frac{1}{H}} \\ \leq 1, \rho > 0, \text{ uniformly in } m\Big\} \\ \leq \inf\Big\{(\rho_1)^{\frac{p_n}{H}} : \Big(\sup_{k\geq 1} f_k\Big(q\Big(\frac{\Delta^n x_{\sigma^k}(m)}{\rho_1}\Big)\Big)^{p_k}\Big)^{\frac{1}{H}} \leq 1, \rho_1 > 0, \\ \text{uniformly in } m\Big\} \\ + \inf\Big\{(\rho_2)^{\frac{p_n}{H}} : \Big(\sup_{k\geq 1} f_k\Big(q\Big(\frac{\Delta^n y_{\sigma^k}(m)}{\rho_2}\Big)\Big)^{p_k}\Big)^{\frac{1}{H}} \leq 1, \rho_2 > 0, \\ \text{uniformly in } m\Big\} \\ = g^*(x) + g^*(y). \end{aligned}$$

Finally, we prove that the scalar multiplication is continuous. Let λ be any complex number. By definition, we have

$$\begin{aligned} g^*(\lambda x) = \inf\Big\{\rho^{\frac{p_n}{H}} : \Big(\sup_{k\geq 1} f_k\Big(q\Big(\frac{\lambda\Delta^n x_{\sigma^k}(m)}{\rho}\Big)\Big)^{p_k}\Big)^{\frac{1}{H}} \leq 1, \rho > 0, \\ \text{uniformly in } m\Big\} \\ = \inf\Big\{(|\lambda| t)^{\frac{p_n}{H}} : \Big(\sup_{k\geq 1} f_k\Big(q\Big(\frac{\Delta^n x_{\sigma^k}(m)}{t}\Big)\Big)^{p_k}\Big)^{\frac{1}{H}} \leq 1, t > 0, \\ \text{uniformly in } m\Big\}, \end{aligned}$$

where $t = \frac{\rho}{|\lambda|}$. This completes the proof of the theorem. □

Theorem 3.3. *Let $F' = (f_k')$ and $F'' = (f_k'')$ be two sequences of modulus functions. Then we have*

$$\Big[V_\sigma, \Delta^n, \theta, F', p, q\Big]_Z \cap \Big[V_\sigma, \Delta^n, \theta, F'', u, p, q\Big]_Z \subseteq \Big[V_\sigma, \Delta^n, \theta, F' + F'', u, p, q\Big]_Z.$$

Proof. The proof is easy so we omit it. □

Theorem 3.4. *Let $F = (F_k)$ be a sequence of modulus functions and $p = (p_k)$ be a bounded sequence of positive real numbers. Then for any two seminorms q_1 and q_2 on X, we have $\left[V_\sigma, \Delta^n, \theta, F, p, q_1\right]_Z \cap \left[V_\sigma, \Delta^n, \theta, F, p, q_2\right]_Z \neq \phi$.*

Proof. Since the zero element belongs to $\left[V_\sigma, \Delta^n, \theta, F, p, q_1\right]_Z$ and $\left[V_\sigma, \Delta^n, \theta, F, p, q_2\right]_Z$ and thus the intersection is non empty. □

Theorem 3.5. *Let $F = (F_k)$ be a sequence of modulus functions. Then*

$$\left[V_\sigma, \Delta^n, \theta, F, p, q\right]_0 \subset [V_\sigma, \Delta^n, \theta, F, p, q\Big]_1 \subset [V_\sigma, \Delta^n, \theta, F, p, q\Big]_\infty.$$

Proof. The inclusion $\left[V_\sigma, \Delta^n, \theta, F, p, q\right]_0 \subset \left[V_\sigma, \Delta^n, \theta, F, p, q\right]_1$ is obvious. Let $x = (x_k) \in \left[V_\sigma, \Delta^n, \theta, F, p, q\right]_1$. Then we have

$$\frac{1}{h_r} \sum_{k \in I_r} \left[f_k\Big(q\Big(\frac{\Delta^n x_{\sigma^k(m)}}{\rho}\Big)\Big)\right]^{p_k}$$

$$\leq K \frac{1}{h_r} \sum_{k \in I_r} \left[f_k\Big(q\Big(\frac{\Delta^n x_{\sigma^k(m)-l}}{\rho}\Big)\Big)\right]^{p_k} + K \frac{1}{h_r} \sum_{k \in I_r} \left[f_k\Big(q\Big(\frac{l}{\rho}\Big)\Big)\right]^{p_k}$$

$$\leq \frac{K}{h_r} \sum_{k \in I_r} \left[f_k\Big(q\Big(\frac{\Delta^n x_{\sigma^k(m)-l}}{\rho}\Big)\Big)\right]^{p_k} + K \max\left\{1, \left[f_k\Big(q\Big(\frac{l}{\rho}\Big)\Big)\right]^G\right\}.$$

Thus $x = (x_k) \in \left[V_\sigma, \Delta^n, \theta, F, p, q\right]_\infty$. This completes the proof of the theorem. □

Theorem 3.6. *Let $F = (f_k)$ be a sequence of modulus functions and let $\theta = (i_r)$ be a lacunary sequence. Then the following result holds:*
(i) If $\liminf_r q_r > 1$. Then $\left[V_\sigma, \Delta^n, F, p, q\right]_Z \subset \left[V_\sigma, \Delta^n, \theta, F, p, q\right]_Z$,
(ii) if $\limsup_r q_r < \infty$. Then $\left[V_\sigma, \Delta^n, \theta, F, p, q\right]_Z \subset \left[V_\sigma, \Delta^n, F, p, q\right]_Z$,
(iii) if $1 < \liminf_r q_r \leq \limsup_r q_r < \infty$. Then

$$\left[V_\sigma, \Delta^n, F, p, q\right]_Z = \left[V_\sigma, \Delta^n, \theta, F, p, q\right]_Z.$$

Proof. It is easy to prove so we omit it. □

Theorem 3.7. *Let* $0 < p_k \leq t_k$ *and* $(\frac{t_k}{p_k})$ *be bounded. Then*

$$\Big[V_\sigma, \Delta^n, \theta, F, t, q\Big]_Z \subset \Big[V_\sigma, \Delta^n, \theta, F, p, q\Big]_Z.$$

Proof. We will prove it for the case $Z = 1$. Let $x = (x_k) \in \Big[V_\sigma, \Delta^n, \theta, F, t, q\Big]_1$. We write

$$S_k = \frac{1}{h_r} \sum_{k \in I_r} \Big[f_k\Big(q\Big(\frac{\Delta^n x_{\sigma^k(m)}}{\rho}\Big)\Big)\Big]^{p_k}$$

and $(\mu_k) = \big(\frac{p_k}{t_k}\big)$ for all $k \in \mathbb{N}$. Then $0 < \mu_k \leq 1$ for all $k \in \mathbb{N}$. Take $0 < \mu < \mu_k$ for all $k \in \mathbb{N}$. Define the sequence u_k and v_k as follows:
For $S_k \geq 1$, let $u_k = S_k$ and $v_k = 0$ and for $S_k < 1$, let $u_k = 0$ and $v_k = S_k$. Then clearly for all $k \in \mathbb{N}$, we have $S_k = u_k + v_k, S_k^{\mu_k} = u_k^{\mu_k} + v_k^{\mu_k}$. Now it follows that $u_k^{\mu_k} \leq u_k \leq S_k$ and $v_k^{\mu_k} \leq v_k^{\mu}$. Therefore

$$\frac{1}{h_r} \sum_{k \in I_r} S_k^{\mu_k} = \frac{1}{h_r} \sum_{k \in I_r} (u_k^{\mu_k} + v_k^{\mu_k}) \leq \frac{1}{h_r} \sum_{k \in I_r} S_k + \frac{1}{h_r} \sum_{k \in I_r} v_k^{\mu}.$$

Now for each k,

$$\begin{aligned}
\frac{1}{h_r} \sum_{k \in I_r} v_k^{\mu} &= \sum_{k \in I_r} \Big(\frac{1}{h_r} v_k\Big)^{\mu} \Big(\frac{1}{h_r}\Big)^{1-\mu} \\
&\leq \Big(\sum_{k \in I_r} \Big[\Big(\frac{1}{h_r} v_k\Big)^{\mu}\Big]^{\frac{1}{\mu}}\Big)^{\mu} \Big(\sum_{k \in I_r} \Big[\Big(\frac{1}{h_r}\Big)^{1-\mu}\Big]^{\frac{1}{1-\mu}}\Big)^{1-\mu} \\
&= \Big(\frac{1}{h_r} \sum_{k \in I_r} v_k\Big)^{\mu}
\end{aligned}$$

and so

$$\frac{1}{h_r} \sum_{k \in I_r} S_k^{\mu_k} \leq \frac{1}{h_r} \sum_{k \in I_r} S_k + \Big(\frac{1}{h_r} \sum_{k \in I_r} v_k\Big)^{\mu}.$$

Hence $x = (x_k) \in \Big[V_\sigma, \Delta^n, \theta, F, p, q\Big]_1$. Similarly we can prove for other cases. □

Theorem 3.8. *The sequence space* $\Big[V_\sigma, \Delta^n, \theta, F, p, q\Big]_\infty$ *is solid.*

Proof. Let $x = (x_k) \in \Big[V_\sigma, \Delta^n, \theta, F, p, q\Big]_\infty$. Then

$$\frac{1}{h_r} \sum_{k \in I_r} \Big[f_k\Big(q\Big(\frac{\Delta^n x_{\sigma^k(m)}}{\rho}\Big)\Big)\Big]^{p_k} < \infty.$$

Let (α_k) be a sequence of scalars such that $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$. Thus we have

$$\begin{aligned} \frac{1}{h_r}\sum_{k\in I_r}\Big[f_k\Big(q\Big(\frac{\alpha_k\Delta^n x_{\sigma^k(m)}}{\rho}\Big)\Big)\Big]^{p_k} &\leq \frac{1}{h_r}\sum_{k\in I_r}\Big[f_k\Big(q\Big(\frac{\Delta^n x_{\sigma^k(m)}}{\rho}\Big)\Big)\Big]^{p_k} \\ &< \infty. \end{aligned}$$

This shows that $(\alpha_k x_k) \in \Big[V_\sigma, \Delta^n, \theta, F, p, q\Big]_\infty$ for all sequences of scalars (α_k) with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$, whenever $(x_k) \in \Big[V_\sigma, \Delta^n, \theta, F, p, q\Big]_\infty$. Hence the space $\Big[V_\sigma, \Delta^n, \theta, F, p, q\Big]_\infty$ is a solid sequence space. This completes the proof of the theorem. □

Corollary 3.9. *The sequence space* $\Big[V_\sigma, \Delta^n, \theta, F, p, q\Big]_\infty$ *is monotone.*

Proof. It is easy to prove so we omit the details. □

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KULDIP RAJ
School of Mathematics
Shri Mata Vaishno Devi University,
Katra-182320, J & K, India.
e-mail: kuldipraj68@gmail.com

SUNIL K. SHARMA
School of Mathematics
Shri Mata Vaishno Devi University,
Katra-182320, J & K, India.
e-mail: sunilksharma42@yahoo.co.in