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MARICHEV-SAIGO-MAEDA FRACTIONAL INTEGRATION OPERATORS OF THE BESSEL FUNCTIONS

S. D. PUROHIT - D. L. SUTHAR - S. L. KALLA

In this paper, we apply generalized operators of fractional integration involving Appell's function $F_3(.)$ due to Marichev-Saigo-Maeda, to the Bessel function of first kind. The results are expressed in terms of generalized Wright function and hypergeometric functions ${}_pF_q$. Special cases involving this function are mentioned. Results given recently by Kilbas and Sebastian follow as special cases of the theorems establish here.

1. Introduction

The fractional integral operators involving various special functions, have found significant importance and applications in various sub-fields of applicable mathematical analysis. Since last four decades, a number of workers like Love [11], McBride [13], Kalla [4, 5], Kalla and Saxena [6, 7], Saigo [17, 18, 19], Saigo and Maeda [20], Kiryakova [9], etc. have studied in depth, the properties, applications and different extensions of various hypergeometric operators of fractional integration. A detailed account of such operators along with their properties and applications can be found in the research monographs by Miller and Ross [14], Kiryakova [9]-[10] and Debnath and Bhatta [1] etc.

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A useful generalization of the hypergeometric fractional integrals, including the Saigo operators [17]-[19], has been introduced by Marichev [12] (see details in Samko *et al.* [21, p. 194, (10.47) and whole section 10.3]) and later extended and studied by Saigo and Maeda [20, p.393, eqn (4.12) and (4.13)] in term of any complex order with Appell function $F_3(.)$ in the kernel, as follows:

Let $\alpha,\alpha',\beta,\beta',\gamma\in\mathbb{C}$ and $x>0$, then the generalized fractional calculus operators (the Marichev-Saigo-Maeda operators) involving the Appell function, or Horn's F_3-function are defined by the following equations:

$$\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma}f\right)(x) = \frac{x^{-\alpha}}{\Gamma(\gamma)}\int_0^x (x-t)^{\gamma-1}t^{-\alpha'}F_3\left(\alpha,\alpha',\beta,\beta';\gamma;1-\frac{t}{x},1-\frac{x}{t}\right)f(t)dt \quad (\Re(\gamma)>0), \tag{1}$$

and

$$\left(I_{0,-}^{\alpha,\alpha',\beta,\beta',\gamma}f\right)(x) = \frac{x^{-\alpha'}}{\Gamma(\gamma)}\int_x^\infty (t-x)^{\gamma-1}t^{-\alpha}F_3\left(\alpha,\alpha',\beta,\beta';\gamma;1-\frac{x}{t},1-\frac{t}{x}\right)f(t)dt \quad (\Re(\gamma)>0). \tag{2}$$

For the definition of the Appell function $F_3(.)$ the interested reader may refer to the monograph by Srivastava and Karlson [23] (see also Erdélyi *et al.* [2] and Prudnikov *et al.* [16]). Following Saigo *et al.* [20], [22], the left-hand sided and right-hand sided generalized integration of the type (1) and (2) for a power function are given by:

$$\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma}x^{\rho-1}\right)(x) = \Gamma\left[\begin{matrix}\rho,\rho+\gamma-\alpha-\alpha'-\beta,\rho+\beta'-\alpha'\\ \rho+\beta',\rho+\gamma-\alpha-\alpha',\rho+\gamma-\alpha'-\beta\end{matrix}\right]x^{\rho-\alpha-\alpha'+\gamma-1}, \tag{3}$$

where $\Re(\gamma)>0$, $\Re(\rho)>max\{0,\Re(\alpha+\alpha'+\beta-\gamma),\Re(\alpha'-\beta')\}$ and

$$\left(I_{0,-}^{\alpha,\alpha',\beta,\beta',\gamma}x^{\rho-1}\right)(x) = \Gamma\left[\begin{matrix}1-\rho-\gamma+\alpha+\alpha',1-\rho+\alpha+\beta'-\gamma,1-\rho-\beta\\ 1-\rho,1-\rho+\alpha+\alpha'+\beta'-\gamma,1-\rho+\alpha-\beta\end{matrix}\right]x^{\rho-\alpha-\alpha'+\gamma-1}, \tag{4}$$

where $\Re(\gamma)>0, \Re(\rho)<1+min\{\Re(-\beta),\Re(\alpha+\alpha'-\gamma),\Re(\alpha+\beta'-\gamma)\}$. The symbol occurring in (3) and (4) is given by

$$\Gamma\left[\begin{matrix}a,\,b,\,c\\ d,\,e,\,f\end{matrix}\right] = \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(d)\Gamma(e)\Gamma(f)}.$$

The Bessel function of the first kind $J_\upsilon(z)$ defined for complex $z \in \mathbb{C}, (z \neq 0)$ and $\upsilon \in \mathbb{C}, (\Re(\upsilon) > -1)$ by [3, 7.2(2)] (see also [15])

$$J_\upsilon(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{\upsilon+2k}}{\Gamma(\upsilon+k+1)k!}. \tag{5}$$

The generalized Wright hypergeometric function ${}_p\psi_q(z)$, for $z \in \mathbb{C}$, complex $a_i, b_j \in \mathbb{C}$, and real $\alpha_i, \beta_j \in \mathbb{R} = (-\infty, \infty)(\alpha_i, \beta_j \neq 0; i = 1,2,...,p; j = 1,2,...,q)$ is defined by:

$${}_p\psi_q(z) = {}_p\psi_q \left[\begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \mid z \right] = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(a_i + \alpha_i k) z^k}{\prod_{j=1}^{q} \Gamma(b_j + \beta_j k) k!}. \tag{6}$$

Wright [24] introduced the generalized Wright function (6) and proved several theorems on the asymptotic expansion of ${}_p\psi_q(z)$ (for instance, see [24, 25, 26]) for all values of the argument z, under the condition:

$$\sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i > -1.$$

The generalized hypergeometric function for complex $a_i, b_j \in \mathbb{C}$ and $b_j \neq 0, -1, \cdots$ $(i = 1,2,\cdots,p; j = 1,2,\cdots,q)$ is given by the power series [2, Section 4.1(1)]:

$${}_pF_q(a_1, \cdots, a_p; b_1, \cdots, b_q; z) = \sum_{r=0}^{\infty} \frac{(a_1)_r \cdots (a_p)_r z^r}{(b_1)_r \cdots (b_q)_r r!}, \tag{7}$$

where for convergence, we have $|z| < 1$ if $p = q+1$ and for any z if $p \leq q$. The function (7) is a special case of the generalized Wright function (6) for $\alpha_1 = \cdots = \alpha_p = \beta_1 = \cdots = \beta_q = 1$:

$${}_pF_q(a_1, \cdots, a_p; b_1, \cdots, b_q; z) = \frac{\prod_{j=1}^{q} \Gamma(b_j)}{\prod_{i=1}^{p} \Gamma(a_i)} \, {}_p\psi_q \left[\begin{array}{c} (a_i, 1)_{1,p} \\ (b_j, 1)_{1,q} \end{array} \mid z \right]. \tag{8}$$

In this paper, we apply the integral operators (1) and (2) to the Bessel function of first kind $J_\upsilon(x)$ and express the image in terms of generalized Wright and hypergeometric functions.

2. Main Results

In this section, we establish image formulas for the Bessel function of first kind involving Saigo-Meada fractional integral operators (1) and (2), in term of the generalized Wright function. These formulas are given by the following theorems:

Theorem 2.1. *Let $\alpha, \alpha', \beta, \beta', \gamma, \upsilon, \rho \in \mathbb{C}$ and $x > 0$ be such that*

$$\Re(\gamma) > 0, \Re(\upsilon) > -1, \quad \Re(\rho+\upsilon) > max\left[0, \Re(\alpha+\alpha'+\beta-\gamma), \Re(\alpha'-\beta')\right], \tag{9}$$

then there hold the formula

$$\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} J_\upsilon(t)\right)(x) = \frac{x^{\rho+\upsilon-\alpha-\alpha'+\gamma-1}}{2^\upsilon} \times {}_3\psi_4\left[\begin{matrix} (\rho+\upsilon,2),(\rho+\upsilon+\gamma-\alpha-\alpha'-\beta,2),(\rho+\upsilon+\beta'-\alpha',2) \\ (\rho+\upsilon+\beta',2),(\rho+\upsilon+\gamma-\alpha-\alpha',2),(\rho+\upsilon+\gamma-\alpha'-\beta,2),(\upsilon+1,1) \end{matrix} \Bigg| -\frac{x^2}{4}\right]. \tag{10}$$

Proof. Using (1) and (5), and then changing the order of integration and summation, which is justified under the conditions stated with Theorem 2.1, we get

$$\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} J_\upsilon(t)\right)(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (1/2)^{\upsilon+2k}}{\Gamma(\upsilon+k+1)k!} \left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho+\upsilon+2k-1}\right)(x).$$

Following the conditions (9), for any $k = 0, 1, 2, \cdots$,

$$\Re(\rho+\upsilon+2k) \geq \Re(\rho+\upsilon) > max\left[0, \Re(\alpha+\alpha'+\beta-\gamma), \Re(\alpha'-\beta')\right].$$

Applying the known result (3) with ρ replaced by $\rho+\upsilon+2k$, we obtain

$$\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} J_\upsilon(t)\right)(x) = \frac{x^{\rho+\upsilon-\alpha-\alpha'+\gamma-1}}{2^\upsilon} \times \sum_{k=0}^{\infty} \frac{\Gamma(\rho+\upsilon+2k)\Gamma(\rho+\upsilon+\gamma-\alpha-\alpha'-\beta+2k)\Gamma(\rho+\upsilon+\beta'-\alpha'+2k)(-x^2)^k}{\Gamma(\rho+\upsilon+\beta'+2k)\Gamma(\rho+\upsilon+\gamma-\alpha-\alpha'+2k)\Gamma(\rho+\upsilon+\gamma-\alpha'-\beta+2k)4^k k!}. \tag{11}$$

Interpreting the right-hand side of the above equation, in view of the definition (6), we arrive at the result (10). □

Theorem 2.2. *Let $\alpha, \alpha', \beta, \beta', \gamma, \upsilon, \rho \in \mathbb{C}$ and $x > 0$ be such that*

$$\Re(\gamma) > 0, \Re(\upsilon) > -1, \quad \Re(\rho-\upsilon) < 1 + min\left[\Re(-\beta),\ \Re(\alpha+\alpha'-\gamma), \Re(\alpha+\beta'-\gamma)\right], \tag{12}$$

then the following formula holds true:

$$\left(I_{0,-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} J_\upsilon(1/t)\right)(x) = \frac{x^{\rho-\upsilon-\alpha-\alpha'+\gamma-1}}{2^\upsilon} \times {}_3\psi_4\left[\begin{matrix} (1-\rho-\gamma+\alpha+\alpha'+\upsilon,2),(1-\rho+\alpha+\beta'-\gamma+\upsilon,2),(1-\rho-\beta+\upsilon,2) \\ (1-\rho+\upsilon,2),(1-\rho-\gamma+\alpha+\alpha'+\beta'+\upsilon,2),(1-\rho+\alpha-\beta+\upsilon,2),(\upsilon+1,1) \end{matrix} \Bigg| -\frac{1}{4x^2}\right]. \tag{13}$$

Proof. On making use of the definitions (2) and (5), and changing the order of integration and summation, which is justified under the conditions stated with Theorem 2.2, we have

$$\left(I_{0,-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} J_\upsilon(1/t)\right)(x) \;=\; \sum_{k=0}^{\infty} \tfrac{(-1)^k (1/2)^{\upsilon+2k}}{\Gamma(\upsilon+k+1)k!} \left(I_{0,-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-\upsilon-2k-1}\right)(x)$$

Following the conditions given in (12), for any $k = 0,1,2,\cdots$, we have

$$\begin{aligned}\Re(\rho-\upsilon-2k-1) &\le 1+\Re(\rho-1-\upsilon)\\ &< 1+min\left[\Re(-\beta),\Re(\alpha+\alpha'-\gamma),\Re(\alpha+\beta'-\gamma)\right].\end{aligned}$$

Now, on applying the formula (4) with ρ replaced by $\rho-\upsilon-2k$, we obtain

$$\begin{aligned}\left(I_{0,-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} J_\upsilon(1/t)\right)(x) &= \frac{x^{\rho-\upsilon-\alpha-\alpha'+\gamma-1}}{2^\upsilon} \sum_{k=0}^{\infty} \tfrac{\Gamma(1-\rho-\gamma+\alpha+\alpha'+\upsilon+2k)}{\Gamma(1-\rho+\upsilon+2k)}\\ &\times \tfrac{\Gamma(1-\rho+\alpha+\beta'-\gamma+\upsilon+2k)\Gamma(1-\rho-\beta+\upsilon+2k)(-1)^k}{\Gamma(1-\rho-\gamma+\alpha+\alpha'+\beta'+\upsilon+2k)\Gamma(1-\rho+\alpha-\beta+\upsilon+2k)(4x^2)^k k!}. \qquad (14)\end{aligned}$$

In view of the definition of the generalized Wright function given by (6), the above equation leads to the result (13). □

Now, we consider other variations of the above theorems, that is, we establish image formulas for the Bessel function $J_\upsilon(x)$ under the operators (1) and (2), in term of the generalized hypergeometric function ${}_pF_q$. By the Legendre duplication formula [2, 1.2(15)], namely

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) \quad (z\in\mathbb{C}), \qquad (15)$$

and the Pochhammer symbol $(z)_k$, defined for $z\in\mathbb{C}$ and $k\in\mathbb{N}$ by

$$(z)_0 = 1, \quad (z)_k = z(z+1)\cdots(z+k+1) \quad (k\in\mathbb{N}_0, \mathbb{N}_0 = \mathbb{N}\cup\{0\}), \qquad (16)$$

there holds the formula

$$(z)_{2k} = 2^{2k}\left(\frac{z}{2}\right)_k\left(\frac{z+1}{2}\right)_k \quad (z\in\mathbb{C}, k\in\mathbb{N}). \qquad (17)$$

First we consider the formula (10) of Theorem 2.1.

Corollary 2.3. *Let the condition of Theorem 2.1 be satisfied, and let* $\rho+\upsilon$, $\rho+\upsilon+\gamma-\alpha-\alpha'-\beta$, $\rho+\upsilon+\beta'-\alpha'\neq 0,-1,\cdots$, *then there holds the formula:*

$$\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}J_\upsilon(t)\right)(x)=\frac{x^{\rho+\upsilon-1}}{2^\upsilon}\frac{\Gamma(\rho+\upsilon)\Gamma(\rho+\upsilon+\gamma-\alpha-\alpha'-\beta)}{\Gamma(\rho+\upsilon+\beta')\Gamma(\rho+\upsilon+\gamma-\alpha-\alpha')\Gamma(\upsilon+1)}\times$$
$$\frac{\Gamma(\rho+\upsilon+\beta'-\alpha')}{\Gamma(\rho+\upsilon+\gamma-\alpha'-\beta)}\,{}_6F_7\left[\begin{array}{c}\frac{\rho+\upsilon}{2},\frac{\rho+\upsilon+1}{2},\frac{\rho+\upsilon+\gamma-\alpha-\alpha'-\beta}{2},\frac{\rho+\upsilon+\gamma-\alpha-\alpha'-\beta+1}{2},\\ \upsilon+1,\frac{\rho+\upsilon+\beta'}{2},\frac{\rho+\upsilon+\beta'+1}{2},\frac{\rho+\upsilon+\gamma-\alpha-\alpha'}{2},\frac{\rho+\upsilon+\gamma-\alpha-\alpha'+1}{2}\end{array}\right.,$$
$$\left.\begin{array}{c}\frac{\rho+\upsilon+\beta'-\alpha'}{2},\frac{\rho+\upsilon+\beta'-\alpha'+1}{2}\\ \frac{\rho+\upsilon+\gamma-\alpha'-\beta}{2},\frac{\rho+\upsilon+\gamma-\alpha'-\beta+1}{2}\end{array}\;\middle|\,-\frac{x^2}{4}\right]. \tag{18}$$

Proof. To prove the above result, we make use of the well-known identity

$$\Gamma(z+k)=(z)_k\Gamma(z)\qquad(z\in\mathbb{C},k\in\mathbb{N}), \tag{19}$$

and the formula (17), in equation (11), then we have

$$\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}J_\upsilon(t)\right)(x)=\sum_{k=0}^{\infty}\frac{(-1)^k(1/2)^{\upsilon+k}}{\Gamma(\upsilon+k+1)k!}$$
$$\times\frac{\Gamma(\rho+\upsilon)\Gamma(\rho+\upsilon+\gamma-\alpha-\alpha'-\beta)\Gamma(\rho+\upsilon+\beta'-\alpha')}{\Gamma(\rho+\upsilon+\beta')\Gamma(\rho+\upsilon+\gamma-\alpha-\alpha')\Gamma(\rho+\upsilon+\gamma-\alpha'-\beta)}$$
$$\times\frac{(\rho+\upsilon)_{2k}(\rho+\upsilon+\gamma-\alpha-\alpha'-\beta)_{2k}(\rho+\upsilon+\beta'-\alpha')_{2k}}{(\rho+\upsilon+\beta')_{2k}(\rho+\upsilon+\gamma-\alpha-\alpha')_{2k}\Gamma(\rho+\upsilon+\gamma-\alpha'-\beta)_{2k}}x^{\rho+\upsilon+2k-1}=$$
$$\frac{x^{\rho+\upsilon-1}}{2^\upsilon}\frac{\Gamma(\rho+\upsilon)\Gamma(\rho+\upsilon+\gamma-\alpha-\alpha'-\beta)\Gamma(\rho+\upsilon+\beta'-\alpha')}{\Gamma(\rho+\upsilon+\beta')\Gamma(\rho+\upsilon+\gamma-\alpha-\alpha')\Gamma(\rho+\upsilon+\gamma-\alpha'-\beta)\Gamma(\upsilon+1)}$$
$$\times\sum_{k=0}^{\infty}\frac{(\frac{\rho+\upsilon}{2})_k(\frac{\rho+\upsilon+1}{2})_k(\frac{\rho+\upsilon+\gamma-\alpha-\alpha'-\beta}{2})_k(\frac{\rho+\upsilon+\gamma-\alpha-\alpha'-\beta+1}{2})_k}{(\upsilon+1)_k(\frac{\rho+\upsilon+\beta'}{2})_k(\frac{\rho+\upsilon+\beta'+1}{2})_k(\frac{\rho+\upsilon+\gamma-\alpha-\alpha'}{2})_k(\frac{\rho+\upsilon+\gamma-\alpha-\alpha'+1}{2})_k}$$
$$\times\frac{(\frac{\rho+\upsilon+\beta'-\alpha'}{2})_k(\frac{\rho+\upsilon+\beta'-\alpha'+1}{2})_k}{(\frac{\rho+\upsilon+\gamma-\alpha'-\beta}{2})_k(\frac{\rho+\upsilon+\gamma-\alpha'-\beta+1}{2})_k}\frac{(-x^2/4)^k}{k!}. \tag{20}$$

Thus, in accordance with (7), we get the result (18). □

Similarly, from Theorem 2.2, one can easily obtain the following result:

Corollary 2.4. *Let conditions of Theorem 2.2 be satisfied, and let* $\alpha+\alpha'+\upsilon-$

$\rho+1,\ \alpha+\beta'-\gamma+\upsilon-\rho+1,\ -\beta+\upsilon-\rho+1\neq 0,-1,\cdots$. *Then*

$$\left(I_{0,-}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}J_{\upsilon}(1/t)\right)(x)=\frac{x^{\rho-\upsilon-\alpha-\alpha'+\gamma-1}}{2^{\upsilon}}\ \frac{\Gamma(\alpha+\alpha'+\upsilon-\gamma-\rho+1)}{\Gamma(\upsilon-\rho+1)\Gamma(\upsilon+1)}$$
$$\times\frac{\Gamma(\alpha+\beta'-\gamma+\upsilon-\rho+1)\Gamma(-\beta+\upsilon-\rho+1)}{\Gamma(\alpha+\alpha'+\beta'+\upsilon-\gamma-\rho+1)\Gamma(\alpha-\beta+\upsilon-\rho+1)}\ {}_6F_7\left[\begin{array}{c}\frac{\alpha+\alpha'+\upsilon-\gamma-\rho+1}{2},\\ \upsilon+1,\frac{\upsilon-\rho+1}{2},\frac{\upsilon-\rho+2}{2},\end{array}\right.$$
$$\begin{array}{c}\frac{\alpha+\alpha'+\upsilon-\gamma-\rho+2}{2},\frac{\alpha+\beta'-\gamma+\upsilon-\rho+1}{2},\frac{\alpha+\beta'-\gamma+\upsilon-\rho+2}{2},\frac{-\beta+\upsilon-\rho+1}{2},\frac{-\beta+\upsilon-\rho+2}{2}\\ \frac{\alpha+\alpha'+\beta'+\upsilon-\gamma-\rho+1}{2},\frac{\alpha+\alpha'+\beta'+\upsilon-\gamma-\rho+2}{2},\frac{\alpha-\beta+\upsilon-\rho+1}{2},\frac{\alpha-\beta+\upsilon-\rho+2}{2}\end{array};$$
$$\left.\mid-\frac{1}{4x^2}\right].\tag{21}$$

3. Special Cases

In this section, we derive certain image formulas for the cosine and sine functions under the fractional integral operators (1) and (2), in terms of the generalized Wright function.

For $\upsilon=-1/2$, the Bessel function $J_{\upsilon}(z)$ given by (5) coincides with the cosine function, excluding the multiplier $(2/\pi z)^{1/2}$, as under [3]:

$$J_{-(1/2)}(z)=\left(\frac{2}{\pi z}\right)^{1/2}\cos z.\tag{22}$$

From Theorem 2.1 we obtain the following:

Corollary 3.1. *Let* $\alpha,\alpha',\beta,\beta',\gamma,\rho\in\mathbb{C}$ *and* $x>0$ *be such that*

$$\Re(\gamma)>0,\Re(\rho)>max\left[0,\Re(\alpha+\alpha'+\beta-\gamma),\Re(\alpha'-\beta')\right],\tag{23}$$

then

$$\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}\cos t\right)(x)=\pi^{1/2}x^{\rho-\alpha-\alpha'+\gamma-1}$$
$$\times\ {}_3\psi_4\left[\begin{array}{c}(\rho,2),(\rho+\gamma-\alpha-\alpha'-\beta,2),(\rho+\beta'-\alpha',2)\\(\rho+\beta',2),(\rho+\gamma-\alpha-\alpha',2),(\rho+\gamma-\alpha'-\beta,2),(\frac{1}{2},1)\end{array}\mid-\frac{x^2}{4}\right].\tag{24}$$

Proof. On setting $\upsilon=-1/2$ into the result (10), and using (22), we have

$$\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-3/2}\cos t\right)(x)\ =\pi^{1/2}x^{\rho-\alpha-\alpha'+\gamma-3/2}$$
$$\times\ {}_3\psi_4\left[\begin{array}{c}(\rho-\frac{1}{2},2),(\rho+\gamma-\alpha-\alpha'-\beta-\frac{1}{2},2),(\rho+\beta'-\alpha'-\frac{1}{2},2)\\(\rho+\beta'-\frac{1}{2},2),(\rho+\gamma-\alpha-\alpha'-\frac{1}{2},2),(\rho+\gamma-\alpha'-\beta-\frac{1}{2},2),(\frac{1}{2},1)\end{array}\mid-\frac{x^2}{4}\right].\tag{25}$$

Now, if we replace ρ by $\rho+1/2$, then the conditions given by (9) transform to the conditions (23) and equation (25) yields to the result (24). $\square$

The next statement follows from the Theorem 2.2 by setting $\upsilon = -1/2$ in result (13) and taking (5) and (22) into account.

Corollary 3.2. *Let $\alpha, \alpha', \beta, \beta', \gamma, \rho \in \mathbb{C}$ and $x > 0$ be such that*

$$\Re(\gamma) > 0, \Re(\rho) < min\left[\Re(-\beta), \Re(\alpha+\alpha'-\gamma), \Re(\alpha+\beta'-\gamma)\right], \qquad (26)$$

then

$$\left(I_{0,-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho} \cos(1/t)\right)(x) = \pi^{1/2} x^{\rho-\alpha-\alpha'+\gamma}$$
$$\times {}_3\psi_4\left[\begin{matrix}(\rho-\gamma+\alpha+\alpha',2),(-\rho+\alpha+\beta'-\gamma,2),(-\rho-\beta,2)\\(-\rho,2),(-\rho-\gamma+\alpha+\alpha'+\beta',2),(-\rho+\alpha-\beta,2),(\frac{1}{2},1)\end{matrix} \mid -\frac{1}{4x^2}\right]. \qquad (27)$$

The next statements give image formulas for the cosine function under the Saigo-Maeda fractional integral operators, in terms of the generalized hypergeometric series (7), follow from Corollaries 2.3 and 2.4 with $\upsilon = -1/2$.

Corollary 3.3. *Let the conditions of Corollary 3.1 be satisfied, and let ρ, $\rho+\gamma-\alpha-\alpha'-\beta$, $\rho+\beta'-\alpha' \neq 0, -1, \cdots$, then the following formula holds true*

$$\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} \cos t\right)(x) = \frac{\Gamma(\rho)\Gamma(\rho+\gamma-\alpha-\alpha'-\beta)\Gamma(\rho+\beta'-\alpha')\ x^{\rho-\alpha-\alpha'+\gamma-1}}{\Gamma(\rho+\beta')\Gamma(\rho+\gamma-\alpha-\alpha')\Gamma(\rho+\gamma-\alpha'-\beta)}$$
$$\times\ {}_6F_7\left[\begin{matrix}\frac{\rho}{2}, \frac{\rho+1}{2}, \frac{\rho+\gamma-\alpha-\alpha'-\beta}{2}, \frac{\rho+\gamma-\alpha-\alpha'-\beta+1}{2}, \frac{\rho+\beta'-\alpha'}{2}, \frac{\rho+\beta'-\alpha'+1}{2}\\ \frac{1}{2}, \frac{\rho+\beta'}{2}, \frac{\rho+\beta'+1}{2}, \frac{\rho+\gamma-\alpha-\alpha'}{2}, \frac{\rho+\gamma-\alpha-\alpha'+1}{2}, \frac{\rho+\gamma-\alpha'-\beta}{2}, \frac{\rho+\gamma-\alpha'-\beta+1}{2}\end{matrix} \mid -\frac{x^2}{4}\right]. \qquad (28)$$

Corollary 3.4. *Let conditions of Corollary 3.2 be satisfied, and let $\alpha+\alpha'-\gamma-\rho$, $\alpha+\beta'-\gamma-\rho$, $-\beta-\rho \neq 0, -1, \cdots$. Then*

$$\left(I_{0,-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho} cos(1/t)\right)(x) = \frac{\Gamma(\alpha+\alpha'-\gamma-\rho)\Gamma(\alpha+\beta'-\gamma-\rho)x^{\rho-\alpha-\alpha'+\gamma}}{\Gamma(-\rho)\Gamma(\alpha+\alpha'+\beta'-\gamma-\rho)} \frac{\Gamma(-\beta-\rho)}{\Gamma(\alpha-\beta-\rho)}$$
$$\times\ {}_6F_7\left[\begin{matrix}\frac{\alpha+\alpha'-\gamma-\rho}{2}, \frac{\alpha+\alpha'-\gamma-\rho+1}{2}, \frac{\alpha+\beta'-\gamma-\rho}{2}, \frac{\alpha+\beta'-\gamma-\rho+1}{2}, \frac{-\beta-\rho}{2}, \frac{-\beta-\rho+1}{2}\\ \frac{1}{2}, \frac{-\rho}{2}, \frac{-\rho+1}{2}, \frac{\alpha+\alpha'+\beta'-\gamma-\rho}{2}, \frac{\alpha+\alpha'+\beta'-\gamma-\rho+1}{2}, \frac{\alpha-\beta-\rho}{2}, \frac{\alpha-\beta-\rho+1}{2}\end{matrix} \mid -\frac{1}{4x^2}\right]. \qquad (29)$$

Again, for $\upsilon = 1/2$, the Bessel function $J_\upsilon(z)$ coincides with the sine function, excluding the multiplier $(2/\pi z)^{1/2}$, that is

$$J_{(1/2)}(z) = \left(\frac{2}{\pi z}\right)^{1/2} \sin z. \qquad (30)$$

Then, from Theorem 2.1 we obtain the following result:

Corollary 3.5. *Let $\alpha,\alpha',\beta,\beta',\gamma,\rho\in\mathbb{C}$ and $x>0$ be such that condition (23) is satisfied. Then*

$$\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-2}\sin t\right)(x)=\frac{\pi^{1/2}}{2}x^{\rho-\alpha-\alpha'+\gamma-1}$$
$$\times\ {}_3\psi_4\left[\begin{matrix}(\rho,2),(\rho+\gamma-\alpha-\alpha'-\beta,2),(\rho+\beta'-\alpha',2)\\(\rho+\beta',2),(\rho+\gamma-\alpha-\alpha',2),(\rho+\gamma-\alpha'-\beta,2),(\frac{3}{2},1)\end{matrix}\right.$$
$$\left.\mid-\frac{x^2}{4}\right].\quad(31)$$

Proof. Setting $\upsilon=1/2$ in equation (10), and using the relation (30), we have

$$\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-3/2}\sin t\right)(x)\ =\frac{\pi^{1/2}}{2}x^{\rho-\alpha-\alpha'+\gamma-1/2}\times$$
$${}_3\psi_4\left[\begin{matrix}(\rho+\frac{1}{2},2),(\rho+\gamma-\alpha-\alpha'-\beta+\frac{1}{2},2),(\rho+\beta'-\alpha'+\frac{1}{2},2)\\(\rho+\beta'+\frac{1}{2},2),(\rho+\gamma-\alpha-\alpha'+\frac{1}{2},2),(\rho+\gamma-\alpha'-\beta+\frac{1}{2},2),(\frac{3}{2},1)\end{matrix}\right.$$
$$\left.\mid-\frac{x^2}{4}\right].\quad(32)$$

If we replace ρ by $\rho-1/2$, then condition (9) transform to condition (23) and equation (32) yields the desired result of Corollary 3.5. □

The next statements give rise to the image formulas of the sine function under fractional integral operators, follows from Theorem 2.2, Corollary 2.3 and Corollary 2.4 with $\upsilon=1/2$.

Corollary 3.6. *Let $\alpha,\alpha',\beta,\beta',\gamma,\rho\in\mathbb{C}$ and $x>0$ be such that*

$$\Re(\gamma)>0,\Re(\rho)<1+min\left[\Re(-\beta),\Re(\alpha+\alpha'-\gamma),\Re(\alpha+\beta'-\gamma)\right],\quad(33)$$

then

$$\left(I_{0,-}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho}\sin(1/t)\right)(x)=\frac{\pi^{1/2}}{2}x^{\rho-\alpha-\alpha'+\gamma-1}$$
$$\times\ {}_3\psi_4\left[\begin{matrix}(\rho-\gamma+\alpha+\alpha'+1,2),(1-\rho+\alpha+\beta'-\gamma,2),(1-\rho-\beta,2)\\(1-\rho,2),(1-\rho-\gamma+\alpha+\alpha'+\beta',2),(1-\rho+\alpha-\beta,2),(\frac{3}{2},1)\end{matrix}\mid-\frac{1}{4x^2}\right].\quad(34)$$

Corollary 3.7. *Let the conditions of Corollary 3.5 be satisfied, and let ρ, $\rho+\gamma-\alpha-\alpha'-\beta$, $\rho+\beta'-\alpha'\neq,0,-1,\cdots$, then there holds*

$$\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-2}\sin t\right)(x)=x^{\rho-\alpha-\alpha'+\gamma-1}\frac{\Gamma(\rho)\Gamma(\rho+\gamma-\alpha-\alpha'-\beta)\Gamma(\rho+\beta'-\alpha')}{\Gamma(\rho+\beta')\Gamma(\rho+\gamma-\alpha-\alpha')\Gamma(\rho+\gamma-\alpha'-\beta)}$$
$$\times\ {}_6F_7\left[\begin{matrix}\frac{\rho}{2},\frac{\rho+1}{2},\frac{\rho+\gamma-\alpha-\alpha'-\beta}{2},\frac{\rho+\gamma-\alpha-\alpha'-\beta+1}{2},\frac{\rho+\beta'-\alpha'}{2},\frac{\rho+\beta'-\alpha'+1}{2}\\\frac{3}{2},\frac{\rho+\beta'}{2},\frac{\rho+\beta'+1}{2},\frac{\rho+\gamma-\alpha-\alpha'}{2},\frac{\rho+\gamma-\alpha-\alpha'+1}{2},\frac{\rho+\gamma-\alpha'-\beta}{2},\frac{\rho+\gamma-\alpha'-\beta+1}{2}\end{matrix}\right.$$
$$\left.\mid-\frac{x^2}{4}\right].\quad(35)$$

Corollary 3.8. *Let conditions of Corollary 3.6 be satisfied, and let* $\alpha+\alpha'-\rho$, $\alpha+\beta'-\gamma-\rho$, $-\beta-\rho\neq 0,-1,\cdots$. *Then*

$$\left(I_{0,-}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho}\sin(1/t)\right)(x)=x^{\rho-\alpha-\alpha'+\gamma-1}$$
$$\times\frac{\Gamma(\alpha+\alpha'-\gamma-\rho+1)\Gamma(\alpha+\beta'-\gamma-\rho+1)\Gamma(-\beta-\rho+1)}{\Gamma(1-\rho)\Gamma(\alpha+\alpha'+\beta'-\gamma-\rho+1)\Gamma(\alpha-\beta-\rho+1)}\times$$
$${}_6F_7\left[\begin{matrix}\frac{\alpha+\alpha'-\gamma-\rho+1}{2},\frac{\alpha+\alpha'-\gamma-\rho+2}{2},\frac{\alpha+\beta'-\gamma-\rho+1}{2},\frac{\alpha+\beta'-\gamma-\rho+2}{2},\frac{-\beta-\rho+1}{2},\frac{-\beta-\rho+2}{2}\\ \frac{3}{2},\frac{-\rho+1}{2},\frac{-\rho+2}{2},\frac{\alpha+\alpha'+\beta'-\gamma-\rho+1}{2},\frac{\alpha+\alpha'+\beta'-\gamma-\rho+2}{2},\frac{\alpha-\beta-\rho+1}{2},\frac{\alpha-\beta-\rho+2}{2}\end{matrix}\right.$$
$$\left.\Big|-\frac{1}{4x^2}\right]. \quad (36)$$

4. Concluding observations

In this section, we consider some consequences of the main results derived in the preceding sections. If we set $\alpha'=0$ in the operators (1) and (2), then by the known identities due to Saxena and Saigo [22, p.93, eqn. (2.15) and (2.16)], we have:

$$\left(I_{0,+}^{\alpha,\,0,\,\beta,\,\beta',\,\gamma}f\right)(x)=\left(I_{0,x}^{\gamma,\,\alpha-\gamma,\,-\beta}f\right)(x), \quad (37)$$

and

$$\left(J_{0,-}^{\alpha,\,0,\,\beta,\,\beta',\,\gamma}f\right)(x)=\left(I_{x,\infty}^{\gamma,\,\alpha-\gamma,\,-\beta}f\right)(x), \quad (38)$$

where the hypergeometric operators that appear in the right-hand side are due to Saigo [17], defined as:

$$I_{0,x}^{\alpha,\beta,\eta}f(x)=\frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}\,{}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-t/x\right)f(t)dt, \quad (39)$$

and

$$I_{x,\infty}^{\alpha,\beta,\eta}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^\infty(t-x)^{\alpha-1}\,t^{-\alpha-\beta}{}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-x/t\right)f(t)dt$$
$$(\alpha,\beta,\gamma\in C;\Re(\alpha)>0). \quad (40)$$

Now, we observe that the results given by Kilbas and Sebastian [8, pp.873-882, Theorems 1 to 12] follow from the results derived in this paper, if we set at $\alpha'=0$ and take the identities (37) and (38) into account.

Further, it can be easily seen that the Riemann-Liouville, Weyl and Eedélyi-Kober fractional calculus operators are special cases of Saigo's operators (39) and (40). Therefore, the results obtained in this article are useful in deriving certain composition formulas involving Riemann-Liouville, Weyl and Erdélyi-Kober fractional calculus operators and Bessel function of first kind.

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SUNIL D. PUROHIT
Department of Basic-Sciences (Mathematics)
College of Technology and Engineering
M.P. University of Agriculture and Technology
Udaipur-313001, India.
e-mail: sunil_a_purohit@yahoo.com

DAYA L. SUTHAR
Department of Mathematics
Alwar Institute of Engineering & Technology
North Ext., MIA, Alwar, INDIA.
e-mail: dd_suthar@yahoo.co.in

SHYAM L. KALLA
Department of Computer Engineering
Vyas Institute of Higher Education
Jodhpur-342008, INDIA.
e-mail: shyamkalla@gmail.com