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NEW OSTROWSKI TYPE INEQUALITIES FOR CO-ORDINATED *S*-CONVEX FUNCTIONS IN THE SECOND SENSE

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In this paper some new Ostrowski type inequalities for co-ordinated s-convex functions in the second sense are obtained.

1. Introduction

In 1938, A. Ostrowski proved the following interesting inequality [21]:

Theorem 1.1. *Let* $f:[a,b]\to\mathbb{R}$ *be a differentiable mapping on* (a,b) *whose derivative* $f':(a,b)\to\mathbb{R}$ *is bounded on* (a,b)*, i.e.,* $\|f'\|_\infty := \sup\limits_{t\in(a,b)} |f'(t)| < \infty$*. Then we have the inequality*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \left[\frac{1}{4} + \frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a)\left\|f'\right\|_\infty, \tag{1}$$

for all $x\in[a,b]$*. The constant* $\frac{1}{4}$ *is the best possible.*

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The inequality (1) can be rewritten in equivalent form as:

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \left[\frac{(x-a)^2+(b-x)^2}{2(b-a)}\right]\left\|f'\right\|_\infty.$$

Since 1938 when A. Ostrowski proved his famous inequality, many mathematicians have been working about and around it, in many different directions and with a lot of applications in Numerical Analysis and Probability, etc. Several generalizations of the Ostrowski integral inequality for mappings of bounded variation, Lipschitzian, monotonic, absolutely continuous, convex mappings, s-convex mappings and n-times differentiable mappings with error estimates for some special means and for some numerical quadrature rules are considered by many authors. For recent results and generalizations concerning Ostrowski's inequality see [4, 5, 7, 8, 11, 12, 20, 23–26] and the references therein.

Let us consider now a bidimensional interval $\Delta =: [a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$ and $c<d$, a mapping $f:\Delta\to\mathbb{R}$ is said to be convex on Δ if the inequality

$$f(\lambda x+(1-\lambda)z,\lambda y+(1-\lambda)w)\le \lambda f(x,y)+(1-\lambda)f(z,w),$$

holds for all $(x,y),(z,w)\in\Delta$ and $\lambda\in[0,1]$. The mapping f is said to be concave on the co-ordinates on Δ if the above inequality holds in reversed direction, for all $(x,y),(z,w)\in\Delta$ and $\lambda\in[0,1]$.

A modification for convex (concave) functions on Δ, which is also known as co-ordinated convex (concave) functions, was introduced by S. S. Dragomir [9, 13] as follows:

A function $f:\Delta\to\mathbb{R}$ is said to be convex (concave) on the co-ordinates on Δ if the partial mappings $f_y:[a,b]\to\mathbb{R}, f_y(u)=f(u,y)$ and $f_x:[c,d]\to\mathbb{R}, f_x(v)=f(x,v)$ are convex (concave) where defined for all $x\in[a,b], y\in[c,d]$.

A formal definition for co-ordinated convex (concave) functions may be stated in:

Definition 1.2. [18] A mapping $f:\Delta\to\mathbb{R}$ is said to be convex on the co-ordinates on Δ if the inequality

$$\begin{aligned}&f(tx+(1-t)y,ru+(1-r)w)\\&\le trf(x,u)+t(1-r)f(x,w)+r(1-t)f(y,u)+(1-t)(1-r)f(y,w),\end{aligned}\tag{2}$$

holds for all $t,r\in[0,1]$ and $(x,u),(y,w)\in\Delta$. The mapping f is concave on the co-ordinates on Δ if the inequality (2) holds in reversed direction for all $t,r\in[0,1]$ and $(x,y),(u,w)\in\Delta$.

Clearly, every convex (concave) mapping $f:\Delta\to\mathbb{R}$ is convex (concave) on the co-ordinates. Furthermore, there exists co-ordinated convex (concave) function which is not convex (concave), (see for instance [9, 13]).

The main result proved concerning the co-ordinated convex function from [9, 13] is given in:

Theorem 1.3. *[9] Suppose that $f:\Delta\to\mathbb{R}$ is co-ordinated convex on Δ. Then one has the inequalities:*

$$\begin{aligned}
&f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
&\leq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right]\\
&\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx\\
&\leq \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)\,dx+\frac{1}{b-a}\int_a^b f(x,d)\,dx\right.\\
&\left.+\frac{1}{d-c}\int_c^d f(a,y)\,dy+\frac{1}{d-c}\int_c^d f(b,y)\,dy\right]\\
&\leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.
\end{aligned}\tag{3}$$

The above inequalities are sharp. The inequalities in (3) hold in reverse direction if the mapping f is concave.

The concept of s-convex functions on the co-ordinates in the second sense was introduced by Alomari and Darus in [1] as a generalization of the co-ordinated convexity in:

Definition 1.4. [1] Consider the bidimensional interval $\Delta=:[a,b]\times[c,d]$ in $[0,\infty)^2$ with $a<b$ and $c<d$. The mapping $f:\Delta\to\mathbb{R}$ is s-convex in the second sense on Δ if

$$f(\lambda x+(1-\lambda)z,\lambda y+(1-\lambda)w)\leq\lambda^s f(x,y)+(1-\lambda)^s f(z,w),$$

holds for all $(x,y),(z,w)\in\Delta$, $\lambda\in[0,1]$ with some fixed $s\in(0,1]$.

A function $f:\Delta=:[a,b]\times[c,d]\subseteq[0,\infty)^2\to\mathbb{R}$ is called s-convex in the second sense on the co-ordinates on Δ if the partial mappings $f_y:[a,b]\to\mathbb{R}$, $f_y(u)=f(u,y)$ and $f_x:[c,d]\to\mathbb{R}$, $f_x(v)=f(x,v)$, are s-convex in the second sense for all $y\in[c,d]$, $x\in[a,b]$ and $s\in(0,1]$, i.e., the partial mappings f_y and f_x are s-convex in the second sense with some fixed $s\in(0,1]$.

A formal definition of co-ordinated s-convex function in second sense may be stated as follows:

Definition 1.5. A function $f : \Delta =: [a,b] \times [c,d] \subseteq [0,\infty]^2 \to \mathbb{R}$ is called s-convex in the second sense on the co-ordinates on Δ if

$$\begin{aligned} & f(tx+(1-t)y, ru+(1-r)w) \\ & \leq t^s r^s f(x,u) + t^s(1-r)^s f(x,w) + r^s(1-t)^s f(y,u) + (1-t)^s(1-r)^s f(y,w), \end{aligned} \tag{4}$$

holds for all $t,r \in [0,1]$ and $(x,u),(y,w) \in \Delta$, for some fixed $s \in (0,1]$. The mapping f is s-concave on the co-ordinates on Δ if the inequality (4) holds in reversed direction for all $t,r \in [0,1]$ and $(x,y),(u,w) \in \Delta$ with some fixed $s \in (0,1]$.

In [5], Alomari et al. also proved a variant of inequalities given above by (3) for s-convex functions in the second sense on the co-ordinates on a rectangle from the plane $\mathbb{R}^2$:

Theorem 1.6. [1] *Suppose $f : \Delta = [a,b] \times [c,d] \subseteq [0,\infty)^2 \to [0,\infty)$ is s-convex function in the second sense on the co-ordinates on Δ. Then one has the inequalities:*

$$\begin{aligned} & 4^{s-1} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ & \leq 2^{s-2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right)dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right)dy\right] \\ & \leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx \\ & \leq \frac{1}{2(s+1)}\left[\frac{1}{b-a}\int_a^b [f(x,c)+f(x,d)]\,dx \right. \\ & \left. + \frac{1}{d-c}\int_a^b [f(a,y)+f(b,y)]\,dy\right] \\ & \leq \frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{(s+1)^2}. \end{aligned} \tag{5}$$

In recent years, many authors have proved several inequalities for co-ordinated convex functions. These studies include, among others, the works in [1–3, 6, 9, 15, 17–20, 22, 27] (see also the references therein). Alomari et al. [1]-[3], proved several Hermite-Hadamard type inequalities for co-ordinated s-convex functions. Bakula et. al [6], proved Jensen's inequality for convex functions on the co-ordinates from the rectangle from the plan $\mathbb{R}^2$. Dragomir [9], proved the Hermite-Hadamard type inequalities for co-ordinated convex functions. Hwang et. al [15], also proved some Hermite-Hadamard type inequalities for co-ordinated convex function of two variables by considering some mappings directly associated with the Hermite-Hadamard type inequality for co-ordinated convex mappings of two variables. Latif et. al [17]-[20], proved some inequalities of Hermite-Hadamard type for differentiable co-ordinated convex functions, for product of two co-ordinated convex mappings, for co-ordinated h-convex mappings and also proved some Ostrowski type inequalities for co-ordinated convex mappings. Özdemir et. al [22], proved Hadamard's type inequalities for co-ordinated m-convex and (α, m)-convex functions. Sarikaya, et. al [27] proved Hermite-Hadamard type inequalities for differentiable co-ordinated convex function. For further inequalities on co-ordinated convex functions see also the references in the above cited papers.

In the present paper, we establish new Ostrowski type inequalities for co-ordinated s-convex functions in second sense similar to those from [20].

2. Main Results

To establish our main results we need the following identity:

Lemma 2.1. [20] *Let $f : \Delta \to \mathbb{R}$ be a twice partial differentiable mapping on Δ°. If $\frac{\partial^2 f}{\partial r \partial t} \in L(\Delta)$, then the following identity holds:*

$$\begin{aligned} & f(x,y) + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(u,v)\,dv\,du - A \\ & = \frac{(x-a)^2 (y-c)^2}{(b-a)(d-c)} \int_0^1 \int_0^1 rt \frac{\partial^2}{\partial r \partial t} f(tx + (1-t)a, ry + (1-r)c)\,dr\,dt \\ & - \frac{(x-a)^2 (d-y)^2}{(b-a)(d-c)} \int_0^1 \int_0^1 rt \frac{\partial^2}{\partial r \partial t} f(tx + (1-t)a, ry + (1-r)d)\,dr\,dt \\ & - \frac{(b-x)^2 (y-c)^2}{(b-a)(d-c)} \int_0^1 \int_0^1 rt \frac{\partial^2}{\partial r \partial t} f(tx + (1-t)b, ry + (1-r)c)\,dr\,dt \\ & + \frac{(b-x)^2 (d-y)^2}{(b-a)(d-c)} \int_0^1 \int_0^1 rt \frac{\partial^2}{\partial r \partial t} f(tx + (1-t)b, ry + (1-r)d)\,dr\,dt, \end{aligned} \tag{6}$$

for all $(x,y) \in \Delta$, where

$$A = \frac{1}{d-c} \int_c^d f(x,v)\,dv + \frac{1}{b-a} \int_a^b f(u,y)\,du.$$

We begin with the following result:

Theorem 2.2. *Let $\Delta = [a,b] \times [c,d] \subseteq [0,\infty)^2 \to \mathbb{R}$ be a twice partial differentiable mapping on Δ° such that $\frac{\partial^2 f}{\partial r \partial t} \in L(\Delta)$. If $\left| \frac{\partial^2 f}{\partial r \partial t} \right|$ is s-convex in the second* *sense on the co-ordinates on* Δ *with* $s\in(0,1]$ *and* $\left|\frac{\partial^2}{\partial r\partial t}f(x,y)\right|\le M$*,* $(x,y)\in\Delta$*, then the following inequality holds:*

$$\left|f(x,y)+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(u,v)\,dvdu-A\right| \le \frac{M}{(s+1)^2}\left[\frac{(x-a)^2+(b-x)^2}{b-a}\right]\left[\frac{(y-c)^2+(d-y)^2}{d-c}\right], \tag{7}$$

for all $(x,y)\in\Delta$*, where A is defined in Lemma 2.1.*

Proof. By Lemma 2.1, we have that the following inequality holds:

$$\begin{aligned}&\left|f(x,y)+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(u,v)\,dvdu-A\right|\\ &\le\frac{(x-a)^2(y-c)^2}{(b-a)(d-c)}\int_0^1\int_0^1 rt\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)a,ry+(1-r)c)\right|drdt\\ &+\frac{(x-a)^2(d-y)^2}{(b-a)(d-c)}\int_0^1\int_0^1 rt\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)a,ry+(1-r)d)\right|drdt\\ &+\frac{(b-x)^2(y-c)^2}{(b-a)(d-c)}\int_0^1\int_0^1 rt\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)b,ry+(1-r)c)\right|drdt\\ &+\frac{(b-x)^2(d-y)^2}{(b-a)(d-c)}\int_0^1\int_0^1 rt\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)b,ry+(1-r)d)\right|drdt,\end{aligned} \tag{8}$$

for all $(x,y)\in\Delta$.

Using the co-ordinated s-convexity of $\left|\frac{\partial^2 f}{\partial r\partial t}\right|$, we have that the following inequality holds:

$$\begin{aligned}&\int_0^1\int_0^1 rt\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)a,ry+(1-r)c)\right|drdt\\ &\le\left|\frac{\partial^2}{\partial r\partial t}f(x,y)\right|\int_0^1\int_0^1 t^{s+1}r^{s+1}drdt+\left|\frac{\partial^2}{\partial r\partial t}f(x,c)\right|\int_0^1\int_0^1 t^{s+1}r(1-r)^s drdt\\ &\quad+\left|\frac{\partial^2}{\partial r\partial t}f(a,y)\right|\int_0^1\int_0^1 r^{s+1}t(1-t)^s drdt\\ &\quad+\left|\frac{\partial^2}{\partial r\partial t}f(a,c)\right|\int_0^1\int_0^1 rt(1-t)^s(1-r)^s drdt.\end{aligned} \tag{9}$$

Since

$$\int_0^1\int_0^1 t^{s+1}r^{s+1}drdt=\frac{1}{(s+2)^2},$$

$$\int_0^1\int_0^1 t^{s+1}r(1-r)^s\,drdt=\int_0^1\int_0^1 r^{s+1}t(1-t)^s\,drdt=\frac{1}{(s+1)(s+2)^2},$$

$$\int_0^1\int_0^1 rt(1-t)^s(1-r)^s\,drdt=\frac{1}{(s+1)^2(s+2)^2}$$

and

$$\left|\frac{\partial^2}{\partial r\partial t}f(x,y)\right|\le M,(x,y)\in\Delta,$$

where we have used the Euler Beta function and its to evaluate the above integrals.
Hence from (9), we obtain

$$\begin{aligned}&\int_0^1\int_0^1 rt\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)a,ry+(1-r)c)\right|drdt\\&\qquad\le\frac{M}{(s+2)^2}+\frac{2M}{(s+1)(s+2)^2}+\frac{M}{(s+1)^2(s+2)^2}=\frac{M}{(s+1)^2}\end{aligned}\tag{10}$$

Analogously, we also have

$$\int_0^1\int_0^1 rt\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)a,ry+(1-r)d)\right|drdt\le\frac{M}{(s+1)^2},\tag{11}$$

$$\int_0^1\int_0^1 rt\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)b,ry+(1-r)c)\right|drdt\le\frac{M}{(s+1)^2}\tag{12}$$

and

$$\int_0^1\int_0^1 rt\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)b,ry+(1-r)d)\right|drdt\le\frac{M}{(s+1)^2}.\tag{13}$$

Now by making use of the inequalities (10)-(13) and the fact that

$$\begin{aligned}&(x-a)^2(y-c)^2+(x-a)^2(d-y)^2+(b-x)^2(y-c)^2+(b-x)^2(d-y)^2\\&=\left[(x-a)^2+(b-x)^2\right]\left[(y-c)^2+(d-y)^2\right],\end{aligned}$$

we get the inequality (7). This completes the proof. □

The corresponding version for powers of the absolute value of the partial derivative is incorporated in the following result:

Theorem 2.3. $\Delta=[a,b]\times[c,d]\subseteq[0,\infty)^2\to\mathbb{R}$ *be a twice partial differentiable mapping on* Δ° *such that* $\frac{\partial^2 f}{\partial r\partial t}\in L(\Delta)$. *If* $\left|\frac{\partial^2 f}{\partial r\partial t}\right|^q$ *is s-convex in the second sense* *on the co-ordinates on* Δ, p, $q>1$, $\frac{1}{p}+\frac{1}{q}=1$ *and* $\left|\frac{\partial^2}{\partial r\partial t}f(x,y)\right|\leq M$, $(x,y)\in\Delta$, *then the following inequality holds:*

$$\left|f(x,y)+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(u,v)\,dvdu-A\right| \leq \frac{M}{(1+p)^{\frac{2}{p}}}\left(\frac{2}{s+1}\right)^{\frac{2}{q}}\left[\frac{(x-a)^2+(b-x)^2}{b-a}\right]\left[\frac{(y-c)^2+(d-y)^2}{d-c}\right], \quad (14)$$

for all $(x,y)\in\Delta$, *where* A *is defined in Lemma 2.1.*

Proof. By Lemma 2.1 and using the Hölder inequality for double integrals, we have that inequality holds:

$$\left|f(x,y)+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(u,v)\,dvdu-A\right|\leq\left(\int_0^1\int_0^1 r^pt^p\,drdt\right)^{\frac{1}{p}}$$
$$\times\left[\frac{(x-a)^2(y-c)^2}{(b-a)(d-c)}\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)a,ry+(1-r)c)\right|^q drdt\right)^{\frac{1}{q}}\right.$$
$$+\frac{(x-a)^2(d-y)^2}{(b-a)(d-c)}\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)a,ry+(1-r)d)\right|^q drdt\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^2(y-c)^2}{(b-a)(d-c)}\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)b,sy+(1-s)c)\right|^q drdt\right)^{\frac{1}{q}}$$
$$\left.+\frac{(b-x)^2(d-y)^2}{(b-a)(d-c)}\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)b,ry+(1-r)d)\right|^q drdt\right)^{\frac{1}{q}}\right], \quad (15)$$

for all $(x,y)\in\Delta$.

Since $\left|\frac{\partial^2 f}{\partial r\partial t}\right|^q$ is s-convex in the second sense on the co-ordinates on Δ and $\left|\frac{\partial^2}{\partial r\partial t}f(x,y)\right|\leq M$, $(x,y)\in\Delta$, we have

$$\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)a,ry+(1-r)c)\right|^q drdt$$
$$\leq\left|\frac{\partial^2}{\partial r\partial t}f(x,y)\right|^q\int_0^1\int_0^1 t^sr^s\,drdt+\left|\frac{\partial^2}{\partial r\partial t}f(a,c)\right|^q\int_0^1\int_0^1(1-t)^s(1-r)^s\,drdt$$
$$+\left|\frac{\partial^2}{\partial r\partial t}f(a,y)\right|^q\int_0^1\int_0^1 r^s(1-t)^s\,drdt+\left|\frac{\partial^2}{\partial r\partial t}f(x,c)\right|^q\int_0^1\int_0^1 t^s(1-r)^s\,drdt$$
$$=\frac{4M^q}{(s+1)^2}.$$

Similarly, we also have the following inequalities:

$$\int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial r \partial t} f(tx + (1-t)a, ry + (1-r)d) \right|^q drdt \le \frac{4M^q}{(s+1)^2},$$

$$\int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial r \partial t} f(tx + (1-t)b, ry + (1-r)c) \right|^q drdt \le \frac{4M^q}{(s+1)^2}$$

and

$$\int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial r \partial t} f(tx + (1-t)b, ry + (1-r)d) \right|^q drdt \le \frac{4M^q}{(s+1)^2}.$$

Using the fact

$$\int_0^1 \int_0^1 r^p t^p drdt = \frac{1}{(1+p)^2}$$

and the above inequalities in (15), we get (14). This completes the proof of the theorem. □

A different approach leads us to the following result:

Theorem 2.4. *Let* $\Delta = [a,b] \times [c,d] \subseteq [0,\infty)^2 \to \mathbb{R}$ *be a twice partial differentiable mapping on* Δ° *such that* $\frac{\partial^2 f}{\partial r \partial t} \in L(\Delta)$. *If* $\left| \frac{\partial^2 f}{\partial r \partial t} \right|^q$ *is s-convex on the co-ordinates on* Δ, $q \ge 1$ *and* $\left| \frac{\partial^2}{\partial r \partial t} f(x,y) \right| \le M$, $(x,y) \in \Delta$, *then the following inequality holds:*

$$\left| f(x,y) + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(u,v)\, dvdu - A \right| \le \frac{M}{4} \left(\frac{2}{s+1} \right)^{\frac{2}{q}} \left[\frac{(x-a)^2 + (b-x)^2}{b-a} \right] \left[\frac{(y-c)^2 + (d-y)^2}{d-c} \right], \quad (16)$$

for all $(x,y) \in \Delta$, *where* A *is defined in Lemma 2.1.*

Proof. Suppose $q \ge 1$. From Lemma 2.1 and using the power mean inequality for double integrals, we have

$$\left|f(x,y)+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(u,v)\,dvdu-A\right|\le\left(\int_0^1\int_0^1 rtdrdt\right)^{1-\frac{1}{q}}$$
$$\times\left[\frac{(x-a)^2(y-c)^2}{(b-a)(d-c)}\left(\int_0^1\int_0^1 tr\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)a,ry+(1-r)c)\right|^q drdt\right)^{\frac{1}{q}}\right.$$
$$+\frac{(x-a)^2(y-d)^2}{(b-a)(d-c)}\left(\int_0^1\int_0^1 tr\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)a,ry+(1-r)d)\right|^q drdt\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^2(y-c)^2}{(b-a)(d-c)}\left(\int_0^1\int_0^1 tr\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)b,ry+(1-r)c)\right|^q drdt\right)^{\frac{1}{q}}$$
$$\left.+\frac{(b-x)^2(d-y)^2}{(b-a)(d-c)}\left(\int_0^1\int_0^1 tr\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)b,ry+(1-r)d)\right|^q drdt\right)^{\frac{1}{q}}\right] \tag{17}$$

for all $(x,y)\in\Delta$.
By similar argument as in Theorem 2.3 that $\left|\frac{\partial^2 f}{\partial r\partial t}\right|^q$ is s-convex on the co-ordinates on Δ in the second sense and $\left|\frac{\partial^2}{\partial r\partial t}f(x,y)\right|\le M$, $(x,y)\in\Delta$, we have

$$\int_0^1\int_0^1 tr\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)a,ry+(1-r)c)\right|^q drdt$$
$$\le\left|\frac{\partial^2}{\partial r\partial t}f(x,y)\right|^q\int_0^1\int_0^1 t^{s+1}r^{s+1}drdt$$
$$+\left|\frac{\partial^2}{\partial r\partial t}f(x,c)\right|^q\int_0^1\int_0^1 t^{s+1}r(1-r)^s drdt$$
$$+\left|\frac{\partial^2}{\partial r\partial t}f(a,y)\right|^q\int_0^1\int_0^1 t(1-t)^s r^{s+1}drdt$$
$$+\left|\frac{\partial^2}{\partial r\partial t}f(a,c)\right|^q\int_0^1\int_0^1 t(1-t)^s r(1+r)^s drdt$$
$$=\frac{M^q}{(s+2)^2}+\frac{M^q}{(s+1)(s+2)^2}+\frac{M^q}{(s+1)(s+2)^2}+\frac{M^q}{(s+1)^2(s+2)^2}$$
$$=\frac{M^q}{(s+1)^2}.$$

In a similar way, we also have that the following inequalities:

$$\int_0^1\int_0^1 tr\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)a,ry+(1-r)d)\right|^q drdt\le\frac{M^q}{(s+1)^2}$$

$$\int_0^1\int_0^1 tr\left|\frac{\partial^2}{\partial r\partial t}f\left(tx+(1-t)b,ry+(1-r)c\right)\right|^q drdt \le \frac{M^q}{(s+1)^2}$$

and

$$\int_0^1\int_0^1 tr\left|\frac{\partial^2}{\partial r\partial t}f\left(tx+(1-t)b,ry+(1-r)d\right)\right|^q drdt \le \frac{M^q}{(s+1)^2}.$$

Now using the above inequalities and

$$\int_0^1\int_0^1 rtdrdt = \frac{1}{4}$$

in (17), we get the desired inequality (16). This completes the proof. □

Remark 2.5. Since $(1+p)^{\frac{1}{p}} < 2$, $p > 1$ and accordingly, we have

$$\frac{1}{2} < \frac{1}{(1+p)^{\frac{1}{p}}},\ p > 1$$

which gives

$$\frac{1}{4} < \frac{1}{(1+p)^{\frac{2}{p}}},\ p > 1.$$

This reveals that the the inequality (16) gives tighter estimate than that of the inequality (14).

Remark 2.6. From the inequalities proved above in Theorem 2.2-Theorem 2.4, one can get several midpoint type inequalities by setting $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$. However the details are left to the interested reader.

Now we drive some results with co-ordinated s-concavity property instead of co-ordinated s-convexity.

Theorem 2.7. $\Delta = [a,b]\times[c,d] \subseteq [0,\infty)^2 \to \mathbb{R}$ *be a twice partial differentiable mapping on* Δ° *such that* $\frac{\partial^2 f}{\partial r\partial t} \in L(\Delta)$. *If* $\left|\frac{\partial^2 f}{\partial r\partial t}\right|^q$ *is s-concave on the co-ordinates* *on* Δ *and* $p, q > 1$, $\frac{1}{p}+\frac{1}{q}=1$, *then the inequality*

$$\begin{aligned}
&\left|f(x,y)+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(u,v)\,dvdu-A\right| \\
&\leq \frac{4^{\frac{s-1}{q}}}{(1+p)^{\frac{2}{p}}(b-a)(d-c)}\left[(x-a)^2(y-c)^2\left|\frac{\partial^2}{\partial r\partial t}f\left(\frac{x+a}{2},\frac{c+y}{2}\right)\right|\right. \\
&\quad +(x-a)^2(d-y)^2\left|\frac{\partial^2}{\partial r\partial t}f\left(\frac{x+a}{2},\frac{d+y}{2}\right)\right| \\
&\quad +(b-x)^2(y-c)^2\left|\frac{\partial^2}{\partial r\partial t}f\left(\frac{b+a}{2},\frac{y+c}{2}\right)\right| \\
&\quad \left.+(b-x)^2(d-y)^2\left|\frac{\partial^2}{\partial r\partial t}f\left(\frac{b+x}{2},\frac{d+y}{2}\right)\right|\right], \qquad (18)
\end{aligned}$$

hods for all $(x,y)\in\Delta$*, where* A *is defined in Lemma 2.1.*

Proof. From Lemma 2.1 and using the Hölder inequality for double integrals, we have that inequality holds:

$$\begin{aligned}
&\left|f(x,y)+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(u,v)\,dvdu-A\right| \\
&\leq\left(\int_0^1\int_0^1 r^p t^p\,drdt\right)^{\frac{1}{p}} \\
&\times\left[\frac{(x-a)^2(y-c)^2}{(b-a)(d-c)}\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)a,ry+(1-r)c)\right|^q drdt\right)^{\frac{1}{q}}\right. \\
&+\frac{(x-a)^2(d-y)^2}{(b-a)(d-c)}\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)a,ry+(1-r)d)\right|^q drdt\right)^{\frac{1}{q}} \\
&+\frac{(b-x)^2(y-c)^2}{(b-a)(d-c)}\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)b,ry+(1-r)c)\right|^q drdt\right)^{\frac{1}{q}} \\
&\left.+\frac{(b-x)^2(d-y)^2}{(b-a)(d-c)}\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f(tx+(1-t)b,ry+(1-r)d)\right|^q drdt\right)^{\frac{1}{q}}\right], \qquad (19)
\end{aligned}$$

for all $(x,y)\in\Delta$.
Since $\left|\frac{\partial^2 f}{\partial r\partial t}\right|^q$ is s-concave on the co-ordinates on Δ, so an application of (5) with

inequalities in reversed direction, gives us the following inequalities:

$$\begin{aligned}\int_0^1\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f\left(tx+(1-t)a,ry+(1-r)c\right)\right|^q drdt\\ \leq 2^{s-2}\left[\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f\left(tx+(1-t)a,\frac{y+c}{2}\right)\right|^q dt\right.\\ \left.+\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f\left(\frac{x+a}{2},ry+(1-r)c\right)\right|^q dr\right]\\ \leq 4^{s-1}\left|\frac{\partial^2}{\partial r\partial t}f\left(\frac{x+a}{2},\frac{y+c}{2}\right)\right|^q,\end{aligned}\tag{20}$$

$$\begin{aligned}\int_0^1\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f\left(tx+(1-t)a,ry+(1-r)d\right)\right|^q dsdt\\ \leq 2^{s-2}\left[\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f\left(tx+(1-t)a,\frac{d+y}{2}\right)\right|^q dt\right.\\ \left.+\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f\left(\frac{x+a}{2},ry+(1-r)c\right)\right|^q dr\right]\\ \leq 4^{s-1}\left|\frac{\partial^2}{\partial r\partial t}f\left(\frac{x+a}{2},\frac{d+y}{2}\right)\right|^q,\end{aligned}\tag{21}$$

$$\begin{aligned}\int_0^1\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f\left(tx+(1-t)b,ry+(1-r)c\right)\right|^q drdt\\ \leq 2^{s-2}\left[\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f\left(tx+(1-t)a,\frac{y+c}{2}\right)\right|^q dt\right.\\ \left.+\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f\left(\frac{b+x}{2},sy+(1-s)c\right)\right|^q dr\right]\\ \leq 4^{s-1}\left|\frac{\partial^2}{\partial r\partial t}f\left(\frac{b+a}{2},\frac{y+c}{2}\right)\right|^q\end{aligned}\tag{22}$$

and

$$\int_0^1\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f\left(tx+(1-t)b,ry+(1-r)d\right)\right|^q drdt$$
$$\leq 2^{s-2}\left[\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f\left(tx+(1-t)b,\frac{d+y}{2}\right)\right|^q dt\right.$$
$$\left.+\int_0^1\left|\frac{\partial^2}{\partial r\partial t}f\left(\frac{b+x}{2},ry+(1-r)d\right)\right|^q dr\right]$$
$$\leq 4^{s-1}\left|\frac{\partial^2}{\partial r\partial t}f\left(\frac{b+x}{2},\frac{d+y}{2}\right)\right|^q. \quad (23)$$

By making use of (20)-(23) in (19), we obtain (18). Thus the proof of the theorem is complete. □

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