LE MATEMATICHE
Vol. LXVII (2012) – Fasc. I, pp. 75–80
doi: 10.4418/2012.67.1.7

ON THE RELATION BETWEEN BETTI NUMBERS OF AN ARF SEMIGROUP AND ITS BLOWUP

MICHAŁ LASOŃ

In this note we prove the relation between Betti numbers of an Arf semigroup S and its blowup S' in the case when they have the same multiplicity n. The relation is then $\beta_{i,s}(S') = \beta_{i,s+(i+1)n}(S)$.

1. Introduction

Definition 1.1. A numerical semigroup is a subset of $\mathbb{N}$ that is closed under addition, contains zero and with finite complement in $\mathbb{N}$.

In fact it is even a monoid, but in this setting it is usually called as above. A subset T of S generates it if every element of S is a linear combination of elements of T with integer coefficients. Every generating set T contains the minimum set of generators, denoted by $G(S)$, which are exactly nonzero elements that can not be represented as a sum of two nonzero elements of S.

Definition 1.2. The smallest generator in $G(S)$ is called the multiplicity of S and the number of elements in $G(S)$ the embedding dimension of S.

Remark 1.3. The multiplicity is greater or equal to embedding dimension.

Entrato in redazione: 8 agosto 2011

AMS 2010 Subject Classification: 13D99, 20M50, 20M14, 20M25.
Keywords: Betti numbers, Arf semigroup, Blowup of a semigroup.
Author is supported by the grant of Polish MNiSzW N N201 413139.

Definition 1.4. An Arf semigroup is a numerical semigroup S such that for each $n \in S$ the set $S(n)$ is also a semigroup, where $S(n) = \{s - n : s \in S, s \geq n\}$.

Arf property is equivalent to: if $s,t,u \in S$ and $s,t \geq u$ then $s+t-u \in S$.

Proposition 1.5. *For Arf semigroups multiplicity and embedding dimension are equal.*

Proof. Let n_1 be the multiplicity of the Arf semigroup S, we will show that embedding dimension is greater or equal to n_1, which is sufficient due to the above Remark. Let r be any residue class modulo n_1 and let $S_r = \{s \in S : s \equiv r \mod n_1\}$. The set S_r is non empty so there exists a minimal element s in it. If s was not a generator then $s = t + u$ for $t,u \in S \setminus \{0\}$, so thanks to the Arf condition $s - n_1 = t + u - n_1 \in S_r$, which is a contradiction. Hence for every residue class modulo n_1 there is a generator belonging to it. □

Definition 1.6. Let S be a semigroup with $G(S) = \{n_1, \ldots, n_k\}$. Here and in the sequel we assume that $n_1 < \cdots < n_k$. Then the blowup of S is a semigroup S' generated by $n_1, n_2 - n_1, \ldots, n_k - n_1$.

Proposition 1.7. *The blowup of an Arf semigroup S with multiplicity n_1 is a semigroup $S'' = S(n_1) = \{s - n_1 : s \in S, s \geq n_1\}$.*

Proof. Let $G(S) = \{n_1, \ldots, n_k\}$, S' the blow up of S is contained in S'' because all its generators are and S'' is a semigroup since S is Arf. Elements of S'' are clearly contained in S'. □

Proposition 1.8. *The blowup of an Arf semigroup is an Arf semigroup.*

Proof. Let $s', t' \geq u'$ and $s', t', u' \in S'$ the blowup of an Arf semigroup S with multiplicity n_1. Then $s' = s - n_1, t' = t - n_1, u' = u - n_1$ and clearly $s,t \geq u$. Hence $s' + t' - u' = (s + t - u) - n_1 \in S'$. □

Remark 1.9. If $G(S) = \{n_1, \ldots, n_k\}$ for an Arf semigroup S which has the same multiplicity as its blowup S' then $G(S') = \{n_1, n_2 - n_1, \ldots, n_k - n_1\}$. It is because both have the same multiplicity and because they are Arf they have also the same number of generators. In general multiplicity of a blowup can decrease, then some of $n_i - n_1$ are not generators any more. This remark and Propositions 1.7, 1.10 are the only places where the assumption that both semigroups have the same multiplicity is used.

Proposition 1.10. *For an Arf semigroup S with $G(S) = \{n_1, \ldots, n_k\}$ and its blowup S' of the same multiplicity we have that*

- $s \in S \Leftrightarrow s - n_1 \in S'$ *unless* $s = 0$

- $s \in S \Leftrightarrow s - 2n_1 \in S'$ *unless* $s = 0$ *or* $s = n_l$
- $s \in S \Leftrightarrow s \in S'$ *unless* $s = n_l - n_1$ *for* $l \neq 1$.

Proof. The first equivalence follows from Proposition 1.7, to prove the second one it is enough to see that if $s = w + w'$ for $w, w' \in S \setminus \{0\}$ then $s - 2n_1 = (w - n_1) + (w' - n_1) \in S'$. And also $n_1, n_l - n_1 = n_1 + (n_l - 2n_1)$ are generators of S' so $n_l - 2n_1$ does not belong to S'. The third one follows from the fact that if $s = w + w'$ for $w, w' \in S' \setminus \{0\}$ then due to Arf property of S' $w + w' - n_1 \in S'$ and so $s = (w + w' - n_1) + n_1 \in S$, otherwise $s = n_j - n_1 \notin S$. □

With a numerical semigroup S with $G(S) = \{n_1, \ldots, n_k\}$ and a field k we associate the semigroup ring $R = k[t^{n_1}, \ldots, t^{n_k}]$. Let T be the graded polynomial ring $k[X_1, \ldots, X_k]$ with $deg(X_i) = n_i$. Graded ring R is isomorphic to T/I where I is the ideal describing relations between generators of S (namely I is generated by binomials $X^\alpha - X^\beta$ for which $\Sigma n_i \alpha_i = \Sigma n_i \beta_i$). Then $Tor_i^T(R, k)$ gets a grading induced by the grading of T. We denote the part of degree s by $Tor_i^T(R, k)_s$.

Definition 1.11. The Betti numbers are $\beta_{i,s} = dim_k Tor_i^T(R, k)_s$.

For $s \in S$ let Δ_s be the simplicial complex on the set of vertices $\{1, \ldots, k\}$ consisting of faces $\{n_{i_1}, \ldots, n_{i_j}\}$ such that $s - (n_{i_1} + \cdots + n_{i_j}) \in S$.

Lemma 1.12. $\beta_{i,s} = dim_k \tilde{H}_{i-1}(\Delta_s)$ □

In the above lemma $\tilde{H}$ denotes the reduced homology and $n_1, \ldots, n_k$ has to be elements of $G(S)$, not only a generating set. The above lemma is proven in [2] page 175 Theorem 9.2, see also [1].

2. Main result

Theorem 2.1. *Let S be an Arf semigroup, such that its blowup S' has the same multiplicity n_1, then $\beta_{i,s}(S') = \beta_{i,s+(i+1)n_1}(S)$.*

Proof. Let $G(S) = \{n_1, \ldots, n_k\}$, then by Remark 1.9 $G(S') = \{n'_1 = n_1, n'_2 = n_2 - n_1, \ldots, n'_k = n_k - n_1\}$. Due to Lemma 1.12 we have to show that $H_{i-1}(\Delta_s(S')) = H_{i-1}(\Delta_{s+(i+1)n_1}(S))$.

Let us fix i, denote $t = s + (i+1)n_1$ and compare the simplicial complexes $\Delta'_s := \Delta_s(S')$ and $\Delta_t := \Delta_t(S)$. Since we are looking on the $(i-1)$ homologies we can remove from both of them all simplexes of dimension greater then i and look only on faces of dimension $(i-1)$ - possible cycles or of dimension i - which give possible boundaries. For $m = i, i+1$ let us define the partial matching

$$\Delta'_s \ni \{j_1, \ldots, j_m\} \leftrightarrow \{j_1, \ldots, j_m\} \in \Delta_t$$

and let us classify unmatched simplexes.

For $m = i$ there are two cases.
First case $j_1 \neq 1$. By Proposition 1.10 we have

$$\{j_1, \dots, j_i\} \in \Delta_t \Leftrightarrow t - n_{j_1} - \dots - n_{j_i} = u \in S \Leftrightarrow$$

$$\Leftrightarrow s - (n_{j_1} - n_1) - \dots - (n_{j_i} - n_1) = u - n_1 \in S' \Leftrightarrow \{j_1, \dots, j_i\} \in \Delta'_s$$

unless $u = 0$.
Second case $j_1 = 1$. Again by Proposition 1.10 we have that

$$t - n_1 - \dots - n_{j_i} = u \in S \Leftrightarrow s - n_1 - \dots - (n_{j_i} - n_1) = u - 2n_1 \in S'$$

unless $u = 0$ or $u = n_l$.

For $m = i + 1$ there are also two cases.
First case $j_1 \neq 1$.

$$t - n_{j_1} - \dots - n_{j_i} = u \in S \Leftrightarrow s - (n_{j_1} - n_1) - \dots - (n_{j_i} - n_1) = u \in S'$$

unless $u = n_l - n_1$.
Second case $j_1 = 1$.

$$t - n_1 - \dots - n_{j_i} = u \in S \Leftrightarrow s - n_1 - \dots - (n_{j_i} - n_1) = u - n_1 \in S'$$

unless $u = 0$.

To conclude there are four types of unmatched faces:

1. Δ_t has extra $(i-1)$-dimensional face $\{j_1, \dots, j_i\}$ for $t - n_{j_1} - \dots - n_{j_i} = 0$
2. Δ_t has extra $(i-1)$-dimensional face $\{1, \dots, j_i\}$ for $t - n_1 - \dots - n_{j_i} = n_l$
3. Δ_t has extra i-dimensional face $\{1, \dots, j_{i+1}\}$ for $t - n_1 - \dots - n_{j_{i+1}} = 0$
4. Δ'_s has extra i-dimensional face $\{j_1, \dots, j_{i+1}\}$ for $t - n_1 - \dots - n_{j_{i+1}} = n_l - n_1$ and $j_r \neq 1$

Let us observe that removing extra faces from the simplicial complexes Δ_t and Δ'_s does not change $(i-1)$ homology.

The face F_0 of type 1 $\{j_1, \dots, j_i\}$ for $t - n_{j_1} - \dots - n_{j_i} = 0$ is not a member of any cycle. Namely if $\delta(\Sigma \alpha_r F_r + \alpha_0 F_0) = 0$ we look at the coefficient of $\{j_2, \dots, j_i\}$. If it was a boundary of a face $\{l, j_2, \dots, j_i\} \in \Delta_t$ we would get $t - n_l - n_{j_2} - \dots - n_{j_i} = n_l - n_{j_1} \in S$, which is a contradiction. So this coefficient is α_0 and it is equal to 0 hence removing F_0 does not affect $(i-1)$ homology.

The face F_0 of type 4 $\{j_1, \dots, j_{i+1}\}$ for $t - n_{j_1} - \dots - n_{j_{i+1}} = n_l - n_1$ for $j_r \neq 1$ does not create any new boundary in Δ'_s. We have that for any $r = 1, \dots, i+1$

$$t - n_1 - n_{j_1} - \dots - n_{j_{r-1}} - n_{j_{r+1}} - \dots - n_{j_{i+1}} = (n_l - n_1) + (n_r - n_1) \in S'$$

so $\{1, j_1, \dots, j_{r-1}, j_{r+1}, \dots, j_{i+1}\} \in \Delta'_s$ and is matched.
We have that $\delta(\{1, j_1, \dots, j_{i+1}\}) = \Sigma\alpha_j F_j + \alpha_0 F_0$ is a linear combination of matched faces and F_0 and the boundary of this linear combination is equal to 0, so $\Sigma\alpha_r \delta F_r + \alpha_0 \delta F_0 = 0$, hence δF_0 does not enlarge dimension of subspace of boundaries.

Consider the face F_0 of type 2 $\{1, \dots, j_i\}$ for $t - n_1 - \cdots - n_{j_i} = n_l$. As in the case of faces of type 1 we look for other faces with $\{j_2, \dots, j_i\}$ as a member of its boundary. If there is no such then F_0 is not a member of any cycle and removing it does not change $(i-1)$ homology. If there is one, with an extra vertex $r \neq 1$, then $t - n_r - \cdots - n_{j_i} = n_l + n_1 - n_r = u \in S$. We get $n_l - n_1 = (u - n_1) + (n_r - n_1) \in S' + S'$ so $u = n_1, r = l$, since $n_l - n_1$ is a generator of S'. Hence $l \neq j_2, \dots, j_i$ and all such faces F_0 are members of a boundary of a face of type 3 $\{1, \dots, j_i, l\}$ for $t - n_1 - \cdots - n_{j_i} - n_l = 0$. We have also that two different faces of type 3 have disjoint boundaries, because they have sum equal to 0. So faces of type 2 and 3 can be considered together separately for different faces of type 3.
Let us fix a face of type 3 $\{1, j_2, \dots, j_{i+1}\}$ and i its subfaces of type 2. We want to show that adding them does not change $(i-1)$ homology. By looking at the boundary simplexes $\{j_2, \dots, j_{r-1}, j_{r+1}, \dots, j_{i+1}\}$ for $r = 2, \dots, i+1$ we get i equations that has to be satisfied for a linear combination to be a cycle. So by adding extra faces to Δ_t a cycle can contain $i+1$ new simplexes of dimension $(i-1)$ but has to satisfy i new linearly independent equations. This enlarge the dimension of kernel of δ_{i-1} by at most one. But $\delta_i(\{1, j_2, \dots, j_{i+1}\})$ is a boundary so it is also a cycle, hence the dimension of kernel is greater by exactly one. The dimension of the image of δ_i is also greater by one since by adding a face of type 3 we add a new boundary which contains new faces, hence the $(i-1)$ homology is the same. □

Acknowledgements

We would like to thank very much Ralf Fröberg for introduction to this subject, posing this problem and many interesting talks. We are also grateful to Mats Boij and Alexander Engström for their help.

REFERENCES

[1] W. Bruns - J. Herzog, *Semigroup rings and simplicial complexes*, J. Pure and App. Algebra 122 (1997), 185–208.

[2] E. Miller - B. Sturmfels, *Combinatorial Commutative Algebra*, Graduate Texts in Mathematics, Springer, 2004.

MICHAŁ LASOŃ
Institute of Mathematics of the Polish Academy of Sciences
Śniadeckich 8, 00-956 Warszawa, Poland
Theoretical Computer Science Department
Faculty of Mathematics and Computer Science
Jagiellonian University
S. Łojasiewicza 6, 30-348 Kraków, Poland
e-mail: michalason@gmail.com