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ALGEBRAIC PROPERTIES OF BIER SPHERES

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We give a classification of flag Bier spheres, as well as descriptions of the first and second Betti numbers of general Bier spheres. Additionally, we compute the Betti numbers for a specific class of Bier spheres, constructed from skeletons of a full simplex.

1. Introduction

The following construction of Bier spheres was introduced by Thomas Bier in unpublished notes. Let Δ be a simplicial complex on the vertex set $[n] = \{1,\dots,n\}$. We do not assume every vertex to be a face of Δ. The *Alexander dual* of Δ is defined as the simplicial complex

$$\Delta^* = \left\{ \sigma' \subset [n]' \,\middle|\, \overline{\sigma} \notin \Delta \right\},$$

where $\overline{\sigma}$ denotes the complement and $\sigma' = \{ i' \in \{1',\dots,n'\} \mid i \in \sigma \}$ a primed analogue of a subset σ of $[n]$. The *deleted join* of simplicial complexes Δ and Γ on disjoint vertex sets $[n]$ and $[n]'$, respectively, is

$$\Delta \tilde{*} \Gamma = \left\{ \sigma \cup \tau' \,\middle|\, \sigma \in \Delta, \tau' \in \Gamma, \sigma \cap \tau = \emptyset \right\}.$$

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Joining a simplicial complex Δ not equal to $2^{[n]}$ in this way with its Alexander dual yields a simplicial sphere $Bier(\Delta) = \Delta \tilde{*} \Delta^*$, which is called the *Bier sphere* of Δ.

We refer to [1] for many results about this construction, in particular for the fact that $Bier(\Delta)$ is a sphere of dimension $n-2$.

2. Flag Bier spheres

A simplicial complex is called *flag* if all of its minimal non-faces are of cardinality not greater than 2. Since a Bier sphere $Bier(\Delta)$ is flag if and only if both Δ and Δ^* are flag, the property of $Bier(\Delta)$ of being flag imposes a severe restriction on the underlying simplicial complex Δ.

If Δ has a *cone vertex*, i.e. a vertex contained in every facet of Δ, then the corresponding vertex of Δ^* is also a cone vertex. Forwarded in the Bier sphere of Δ these two vertices form a minimal non-face and every facet contains exactly one of them. Thus, $Bier(\Delta)$ is a suspension of the Bier sphere of Δ with the cone vertex removed. It follows that $Bier(\Delta)$ is flag if and only if every Bier sphere of a cone over Δ is flag.

Conversely, a given Bier sphere which is a suspension results from a simplicial complex containing a cone vertex. Thus, every flag Bier sphere is obtained by repeated suspensions of a flag Bier sphere $Bier(\Delta)$, where Δ is not a cone.

Only a few simplicial complexes are not a cone and additionally the underlying complex of a flag Bier sphere as the following proposition shows.

Proposition 2.1. *There are exactly eight simplicial complexes Δ that are not cones, such that $Bier(\Delta)$ is flag. They define up to isomorphism four flag Bier spheres.*

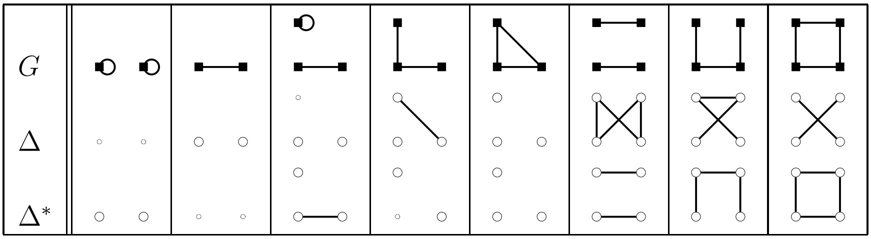
Proof. Let Δ be a flag simplicial complex on the vertex set $[n]$ without cone vertices such that Δ^* is flag. Define a graph G on the vertices $[n]$ with edges the minimal non-faces of cardinality 2 and loops the minimal non-faces of cardinality 1 of Δ. Note that by this definition, a vertex in G with a loop cannot be adjacent to another vertex. This assignment from the set of flag simplicial complexes without cone vertices to the set of graphs is injective.

In a flag simplicial complex, a vertex v with the property that $\{v,w\}$ is a face for all faces $\{w\}$, is a cone vertex. Hence, the graph G contains no isolated vertices (without loop).

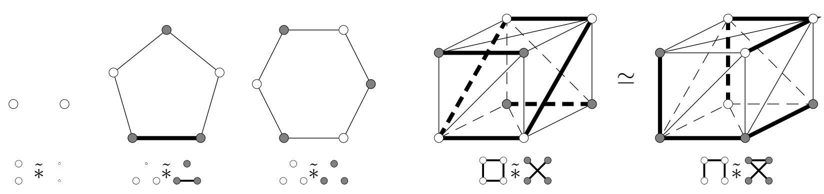
The complements of the facets of Δ correspond to the minimal non-faces of Δ^*. Therefore, the condition that Δ^* is flag implies that the facets of Δ are of cardinality greater or equal to $n-2$. Since a face of Δ induces an independent set in G and vice versa, every independent set in G can be extended to a maximal independent set containing at least $n-2$ elements.

As a result, a vertex v without a loop in G is adjacent to at most two other vertices, otherwise the independent set $\{v\}$ would not be extensible to an independent set with $n-2$ elements. For this reason, components in G are paths or cycles. In a path or a cycle including at least five vertices it is possible to choose an independent set with two elements, which is adjacent to more than two vertices and thus not expandable to an independent set of cardinality $n-2$.

In summary, the types of possible components of G are single vertices with a loop, cycles of length 3 or 4 and paths of length less or equal to 3. Considering the facts that every component provides an independent set of vertices with a number of neighbours and the total number of neighbours of an independent set may not rise above two, there remain eight possibilities for the shape of G. The following list combines them with the corresponding simplicial complexes and their Alexander duals. The small dots symbolize vertices that are not faces.



Since the deleted join is commutative, the graphs on two vertices and the graphs on three vertices, except the cycle, yield isomorphic Bier spheres. The next figure shows all Bier spheres occurring by the above listed pairings of Δ and Δ^*, i.e up to isomorphism all flag Bier spheres which are not suspensions. In particular, it reveals that the graphs on four vertices define isomorphic Bier spheres.



□

For each of these flag Bier spheres exists a labeled cell complex (see chapter 4 of [4]) admitting a cellular resolution of the Stanley-Reisner rings of this Bier sphere.

3. Betti numbers

For the notion of Stanley-Reisner rings and $\mathbb{N}^n$-graded Betti numbers we refer to [2] and [4].

Let $S = K[x_1,\dots,x_n,y_1,\dots,y_n]$ be the polynomial ring in $2n$ variables over a field K. The Stanley-Reisner ring $K[Bier(\Delta)]$ of $Bier(\Delta)$ is the quotient of S by the sum of $\langle x_j y_j \mid j \in [n]\rangle$ and the Stanley-Reisner ideals $I_\Delta \subseteq K[x_1,\dots,x_n]$, $I_{\Delta^*} \subseteq K[y_1,\dots,y_n]$, that is, a minimal non-face of $Bier(\Delta)$ is either of the form $\{j, j'\}$ or a minimal non-face of Δ or Δ^*. In this notation the variables $x_1,\dots,x_n$ correspond to the vertices of Δ and $y_1,\dots,y_n$ to those of Δ^*.

For subsets σ and τ of $[n]$ we write $\mathbf{x}^\sigma$ and $\mathbf{y}^\tau$ for the monomials $\Pi_{i\in\sigma} x_i$ and $\Pi_{i\in\tau} y_i$, respectively, and $\beta_{i,\sigma\tau'}(K[Bier(\Delta)])$ for the i-th Betti number of $K[Bier(\Delta)]$ in squarefree degree $(a_1,\dots,a_{2n}) \in \{0,1\}^{2n}$, where $a_i = 1$ if and only if $i \in \sigma$ or $i-n \in \tau$. Analogously the notations $\beta_{i,\sigma}(K[\Delta])$ and $\beta_{i,\tau'}(K[\Delta^*])$ are used for the Betti numbers of $K[\Delta]$ and $K[\Delta^*]$. It suffices to consider Betti numbers in squarefree degrees, since all non-zero Betti numbers of Stanley-Reisner rings lie in these degrees.

The main tool in the following is the *lcm-lattice* L_Δ of Δ (confer [2]). This lattice is the set of unions of minimal non-faces of Δ, ordered by inclusion. For σ in L_Δ the open lower interval $(\emptyset,\sigma)_\Delta$ below σ is $\{\tau \in L_\Delta \mid \tau \subsetneq \sigma\}$. The poset $(\emptyset,\sigma)_\Delta$ will be considered as an abstract simplicial complex whose faces are given by chains in $(\emptyset,\sigma)_\Delta$.

Theorem 3.1 (Theorem 2.1 of [2]). *For $i \geq 1$ and $\sigma \in L_\Delta$ we have*

$$\beta_{i,\sigma}(K[\Delta]) = \dim \tilde{H}_{i-2}((\emptyset,\sigma)_\Delta)$$

If $\sigma \notin L_\Delta$, then the corresponding Betti number of $K[\Delta]$ vanishes.

Here, as in the sequel, $\tilde{H}_i(\Delta)$ denotes the reduced homology of degree i of the simplicial complex Δ over the field K. This theorem justifies the first proposition about the Betti numbers of Bier spheres:

Proposition 3.2. *For all $\sigma \subset [n]$ it holds*

$$\beta_{i,\sigma}(K[Bier(\Delta)]) = \beta_{i,\sigma}(K[\Delta]) \text{ and } \beta_{i,\sigma'}(K[Bier(\Delta)]) = \beta_{i,\sigma'}(K[\Delta^*]).$$

Proof. This follows from Theorem 3.1 by observing $(\emptyset,\sigma)_{Bier(\Delta)} = (\emptyset,\sigma)_\Delta$ and $(\emptyset,\sigma')_{Bier(\Delta)} = (\emptyset,\sigma')_{\Delta^*}$. □

In other words, the Betti numbers of the Stanley-Reisner rings of Δ and Δ^* arise one to one as the Betti numbers of $K[Bier(\Delta)]$ in the degrees of the form $(\mathbf{a},0,\dots,0)$ and $(0,\dots,0,\mathbf{a})$, respectively, with $\mathbf{a}$ in $\{0,1\}^n$. For the Betti numbers of $K[Bier(\Delta)]$ of a different type, the *mixed* Betti numbers, we can give a necessary condition for being non-zero.

Lemma 3.3. *Let $\sigma, \tau \subset [n]$ be non-empty sets such that $\beta_{i,\sigma\tau'}(K[Bier(\Delta)]) \neq 0$ for an i. Then $\sigma \cap \tau \neq \emptyset$. Equivalently, there exists an index $j \in [n]$ such that x_j and y_j both divide $\mathbf{x}^\sigma \mathbf{y}^\tau$.*

Proof. The condition $\beta_{i,\sigma\tau'}(K[Bier(\Delta)]) \neq 0$ implies that the subset $\sigma \cup \tau'$ is in $L_{Bier(\Delta)}$ and therewith an union of minimal non-faces of $Bier(\Delta)$. Under the assumption $\sigma \cap \tau = \emptyset$, this union contains only minimal non-faces of Δ or Δ^* (since a minimal non-face of the type $\{j, j'\}$ of $Bier(\Delta)$ causes a common element in σ and τ). Hence, σ and τ' are non-faces of Δ and Δ^*, respectively. But $\sigma \notin \Delta$ implies that $\overline{\sigma}'$ is a face of Δ^*, so τ' cannot be a subset of it. It follows $\sigma \cap \tau \neq \emptyset$. □

The first Betti numbers can be read off the definition of the Bier sphere: They correspond to the minimal non-faces of Δ, of Δ^* and to the deleted edges $x_j y_j$, where $j \in [n]$ is an index such that $\{x_j\} \in \Delta$ and $\{y_j\} \in \Delta^*$ (if one of these vertices is a non-face, then it replaces $x_j y_j$ in a set of minimal generators of the Stanley-Reisner ideal of $Bier(\Delta)$). The second mixed Betti numbers are described by the following result.

Proposition 3.4. *Let $\sigma, \tau \subset [n]$ be non-empty sets. Then the following three conditions are equivalent:*

(1) $\beta_{2,\sigma\tau'}(K[Bier(\Delta)]) \neq 0$.

(2) $\beta_{2,\sigma\tau'}(K[Bier(\Delta)]) = 1$.

(3) There are exactly two minimal non-faces ρ_1, ρ_2 of $Bier(\Delta)$ such that $\rho_i \subset \sigma \cup \tau'$ for $i = 1, 2$. These two satisfy $\rho_1 \cup \rho_2 = \sigma \cup \tau'$.

If these conditions hold and $Bier(\Delta)$ has no minimal non-faces of cardinality 1, then at least one of those two minimal non-faces is of the form $\{j, j'\}$.

Proof. If $\beta_{2,\sigma\tau'}(K[Bier(\Delta)])$ is non-zero, the subset $\sigma \cup \tau'$ of $[n] \cup [n]'$ is the union of minimal non-faces of $Bier(\Delta)$ and there is a $j \in [n]$ with $\{j, j'\} \subseteq \sigma \cup \tau'$, by means of Lemma 3.3. According to Theorem 3.1, the computation of $\beta_{2,\sigma\tau'}(K[Bier(\Delta)])$ via $\dim \tilde{H}_0((\emptyset, \sigma \cup \tau')_{Bier(\Delta)})$ implies that $(\emptyset, \sigma \cup \tau')_{Bier(\Delta)}$ contains at least two connected components. Thus, there exist two minimal non-faces $\rho_1, \rho_2 \subseteq \sigma \cup \tau'$ of $Bier(\Delta)$ lying in different connected components of $(\emptyset, \sigma \cup \tau')_{Bier(\Delta)}$. Note that this is equivalent to $\rho_1 \cup \rho_2 = \sigma \cup \tau'$ (If the union $\rho_1 \cup \rho_2$ was a proper subset of $\sigma \cup \tau'$, it would connect the non-faces ρ_1 and ρ_2 in $(\emptyset, \sigma \cup \tau')_{Bier(\Delta)}$). Since $\{j, j'\}$ is a non-face of $Bier(\Delta)$, we can claim without loss of generality $j \in \rho_1$ and $j \notin \rho_2$. We will prove by contradiction that there is no third minimal non-face $\rho_3 \subseteq \sigma \cup \tau'$ of $Bier(\Delta)$.

Assume the contrary. As a minimal non-face of $Bier(\Delta)$, ρ_3 is of the form $\{k,k'\}$ for a $k \in [n]$ or a minimal non-face of Δ or Δ^*.

Consider the first case. Since $j \in \rho_1$ and ρ_1 is a minimal non-face of $Bier(\Delta)$, it is either equal to $\{j,j'\}$ or a subset of $[n]$. Consequently, if k' was an element of ρ_1, it would follow $k = j$ and therewith $\rho_3 \subseteq \rho_1$, which is not possible for two unequal minimal non-faces. Therefore, the inclusion $\rho_3 \subseteq \sigma \cup \tau' = \rho_1 \cup \rho_2$ implies $k' \in \rho_2$. As ρ_2 and ρ_3 are not subsets of one another, $\{k'\} \subsetneq \rho_2$ and k is not in ρ_2. Hence, $\rho_2 \subseteq [n]'$ and the element k is in ρ_1. Since $k' \notin \rho_1$ and ρ_1 is not a subset of ρ_3, it holds $\rho_1 \subseteq [n]$ and $\{k\} \subsetneq \rho_1$. This yields

$$\rho_1 \cup \rho_3 = \rho_1 \cup \{k'\} \subsetneq \rho_1 \cup \rho_2 \text{ and } \rho_3 \cup \rho_2 = \{k\} \cup \rho_2 \subsetneq \rho_1 \cup \rho_2,$$

which is equivalent to the fact that ρ_1 and ρ_3 as well as ρ_2 and ρ_3 (hence, also ρ_1 and ρ_2) lie in the same connected component, in contradiction to the assumptions.

If ρ_3 is a minimal non-face of Δ, i.e. $\rho_3 \subseteq [n]$, the minimal non-face ρ_1 cannot be a subset of $[n]$. Otherwise $j' \in \rho_2$. Since $j \notin \rho_2$, this implies $\rho_2 \subseteq [n]'$ and the inclusion $\rho_1 \cup \rho_3 \subseteq \sigma \cup \tau' = \rho_1 \cup \rho_2$ yields $\rho_3 \subseteq \rho_1$. Thus, $\rho_1 = \{j,j'\}$. As ρ_3 is not a subset of ρ_1, there exist $k \in [n] \setminus \{j\}$ with $k \in \rho_3$. The minimal non-face ρ_2 contains k, therewith it is of the form $\{k,k'\}$ or a subset of $[n]$. Both cases imply $j' \notin \rho_2$. Consequently, the inclusion $\rho_2 \cup \rho_3 \subsetneq \sigma \cup \tau' = \rho_1 \cup \rho_2$ is strict. I.e. ρ_2 and ρ_3 are in the same connected component, thus, ρ_1 and ρ_3 are not, which is expressed by $\rho_1 \cup \rho_3 = \sigma \cup \tau' = \rho_1 \cup \rho_2$. Since ρ_1 and ρ_2 are disjoint, the minimal non-face ρ_2 is a subset of ρ_3. This is a contradiction.

Finally consider the case that ρ_3 is a minimal non-face of Δ^*, i.e. $\rho_3 \subseteq [n]'$. As $j \notin \rho_2$, it holds $\rho_2 \cup \rho_3 \subsetneq \sigma \cup \tau' = \rho_1 \cup \rho_2$. Thus, ρ_2 and ρ_3 lie in the same connected component, which prevents ρ_3 from being in the same connected component as ρ_1. Therefore, $\rho_1 \cup \rho_3 = \sigma \cup \tau' = \rho_1 \cup \rho_2$. If ρ_1 was a subset of $[n]$, this would imply $\rho_3 \subseteq \rho_2$. Hence, $\rho_1 = \{j,j'\}$ and the minimal non-faces ρ_2 and ρ_3 may only differ by the element j', this means one of them is contradictory included in the other.

The remaining implications of the proposition follow easily by considering $(\emptyset, \sigma \cup \tau')_{Bier(\Delta)}$. If there are exactly two minimal non-faces ρ_1, ρ_2 of $Bier(\Delta)$ such that $\rho_1 \cup \rho_2 = \sigma \cup \tau'$, the number of connected components in $(\emptyset, \sigma \cup \tau')_{Bier(\Delta)}$ is two and by Theorem 3.1 this gives rise to $\beta_{2,m}(K[Bier(\Delta)]) = 1 \neq 0$. □

Keeping in mind the possible types of minimal non-faces in $Bier(\Delta)$, one can use the third condition of the proposition above for to classify all occurring degrees of mixed second Betti numbers. It holds $\beta_{2,\sigma\tau'}(K[Bier(\Delta)]) = 1$ for $\sigma, \tau \neq \emptyset$, if and only if one of the following conditions is fulfilled for two minimal non-faces ρ_1, ρ_2 of $Bier(\Delta)$ with $\rho_1 \cup \rho_2 = \sigma \cup \tau'$:

- $\rho_1 = \{j\}$ or $\{j'\}$
- $\rho_1 = \{j, j'\}, \rho_2 = \{k, k'\}$ for $j \neq k$ and $\{j,k\} \in \Delta, \{j',k'\} \in \Delta^*$
- $\rho_1 = \{j, j'\}$ and $j \in \rho_2 \subseteq [n]$ (or $j' \in \rho_2 \subseteq [n]'$)
- $\rho_1 = \{j, j'\}$, $j \notin \rho_2 \subseteq [n]$ and Δ has no minimal non-face $\{j\} \cup \pi$ with $\pi \subseteq \rho_2$ (or $j' \notin \rho_2 \subseteq [n]'$ and Δ^* has no minimal non-face $\{j'\} \cup \pi$ with $\pi \subseteq \rho_2$).

It might be tempting to conjecture that all mixed Betti numbers take only the values 0 and 1, but this is wrong: Consider a simplicial complex Δ with multigraded Betti number greater than 1, say $\beta_{i,\sigma}(K[\Delta]) = l > 1$. Since $Bier(\Delta)$ is a sphere, it holds by Gorenstein duality: $\beta_{j,\overline{\sigma}[n]'}(K[Bier(\Delta)]) = l > 1$, where $j = 2n - (n-1) - i$ is the index corresponding to i in this duality.

4. An extended Example: Skeletons of a full simplex

For natural numbers $n, k \geq 0$ let $\Delta_n = \{\, \sigma \subseteq [n] \,\}$ denote the full simplex on n vertices and $\Delta_n^k = \{\, \sigma \subseteq [n] \mid \#\sigma \leq k \,\}$ its k-skeleton. Note that the dimension of the k-skeleton is $k-1$. The Alexander dual is $(\Delta_n^k)^* = \Delta_n^{n-k-1}$. In this section the Betti numbers of the Bier sphere $Bier(\Delta_n^k)$ will be computed. A helpful tool is Hochster's Formula, see [4, Corollary 5.12]:

Theorem 4.1. *Let Δ be a simplicial complex and σ be a subset of its set of vertices. Then*

$$\beta_{i,\sigma}(K[\Delta]) = \dim \tilde{H}_{\#\sigma - i - 1}(\Delta|_\sigma). \tag{1}$$

Here, $\Delta|_\sigma$ denotes the restricted complex $\Delta|_\sigma = \{\, \tau \in \Delta \mid \tau \subset \sigma \,\}$. In order to compute the right-hand side of (1), the Mayer-Vietoris sequence will be used, see [3, p. 149]:

Theorem 4.2. *Let Δ be a simplicial complex and A and B subcomplexes such that $\Delta = A \cup B$. Then there is an exact sequence:*

$$\begin{aligned} \ldots \to \tilde{H}_i(A \cap B) \to \tilde{H}_i(A) \oplus \tilde{H}_i(B) \to \tilde{H}_i(\Delta) \to \tilde{H}_{i-1}(A \cap B) \to \ldots \\ \ldots \to \tilde{H}_{-1}(\Delta) \to 0 \quad . \end{aligned}$$

For a vertex v of Δ, we define the *deletion* of v in Δ as the set $del_\Delta(v) = \{\tau \in \Delta \mid v \notin \tau\}$ and the *link* of v in Δ as $lk_\Delta(v) = \{\tau \in \Delta \mid \tau \cup \{\, v \,\} \in \Delta \text{ and } v \notin \tau\}$.

Corollary 4.3. *Let Δ be a simplicial complex and v a vertex of Δ. Then there is an exact sequence:*

$$\begin{aligned} \ldots \to \tilde{H}_i(lk_\Delta(v)) \to \tilde{H}_i(del_\Delta(v)) \to \tilde{H}_i(\Delta) \to \tilde{H}_{i-1}(lk_\Delta(v)) \to \ldots \\ \ldots \to \tilde{H}_{-1}(\Delta) \to 0 \quad . \end{aligned}$$

Proof. We apply Theorem 4.2 to $A = \{\tau, \tau \cup \{v\} \mid v \notin \tau \text{ and } \tau \cup \{v\} \in \Delta\}$, the *star* of v in Δ, and $B = del_\Delta(v)$. For those we have $A \cup B = \Delta$ and $A \cap B = lk_\Delta(v)$. Since v is a cone vertex in A, the complex A is contractible. □

A simplicial sphere is Gorenstein. Hence, it is possible to revert to Gorenstein duality, see Chapter I.12 in [5]:

Theorem 4.4. *Let Δ be a Gorenstein simplicial complex of dimension $d-1$ on n vertices and let σ be a subset of its set of vertices. Then*

$$\beta_{i,\sigma}(K[\Delta]) = \beta_{n-d-i,\overline{\sigma}}(K[\Delta]).$$

We start with the computation of the reduced homology of Δ_n^k. For $i, j \in \mathbb{N}$, let $\delta_{i,j}$ denote the Kronecker delta.

Lemma 4.5. *For natural numbers $n \geq 1, k \geq 0$ it holds*

$$\dim \tilde{H}_i(\Delta_n^k) = \delta_{i,k-1}\binom{n-1}{k}.$$

Proof. Since the dimension of Δ_n^k is $k-1$, the i-th reduced homology of Δ_n^k is trivial for $i \geq k$. It coincides with the i-th reduced homology of the full simplicial complex Δ_n for all $i < k-1$, as it depends on the cells up to dimension $i+1$. Thus, $\tilde{H}_i(\Delta_n^k) = 0$ if $i \neq k-1$.

The case $i = k-1$ is proved by induction on n.

If $k > 0$, then the k-skeleton of Δ_1 is Δ_1, whose reduced homologies in all degrees are trivial. The 0-skeleton of Δ_1 is the irrelevant complex $\{\emptyset\}$ with $\dim \tilde{H}_{-1}(\{\emptyset\}) = 1$. This implies $\dim \tilde{H}_{k-1}(\Delta_1^k) = \binom{0}{k}$.

Suppose $n > 1$. We apply the Mayer-Vietoris sequence in the form of Corollary 4.3 on the vertex n. Since the complexes $del_{\Delta_n^k}(n)$ and Δ_{n-1}^k, as well as $lk_{\Delta_n^k}(n)$ and Δ_{n-1}^{k-1}, consist of the same families of faces, they have the same homologies. Together with the observations above, this yields the short exact sequence

$$0 \to \tilde{H}_{k-1}(\Delta_{n-1}^k) \to \tilde{H}_{k-1}(\Delta_n^k) \to \tilde{H}_{k-2}(\Delta_{n-1}^{k-1}) \to 0$$

and in particular $\dim \tilde{H}_{k-1}(\Delta_n^k) = \dim \tilde{H}_{k-1}(\Delta_{n-1}^k) + \dim \tilde{H}_{k-2}(\Delta_{n-1}^{k-1})$. Therefore, it holds $\dim \tilde{H}_{k-1}(\Delta_n^k) = \binom{n-2}{k} + \binom{n-2}{k-1} = \binom{n-1}{k}$. □

Recall that $Bier(\Delta_n^k)$ is defined as $\Delta_n^k \tilde{*} \Delta_n^l$, where $l = n-k-1$. By replacing Δ_n^k by its Alexander dual, we can achieve that $k \leq l$. This assumption simplifies the computation of the Betti numbers of $Bier(\Delta_n^k)$.

Let σ, τ be subsets of $[n]$. By Hochster's Formula it holds

$$\beta_{i,\sigma\tau'}(K[Bier(\Delta_n^k)]) = \dim \tilde{H}_m(\Delta_n^k \tilde{*} \Delta_n^l|_{\sigma \cup \tau'}),$$

where $m = \#\sigma + \#\tau - i - 1$. Obviously,

$$\Delta_n^k \,\tilde{*}\, \Delta_n^l|_{\sigma\cup\tau'} = \Delta_\sigma^k \,\tilde{*}\, \Delta_\tau^l\,,$$

with the notation Δ_σ^k for the restricted complex $\Delta_n^k|_\sigma$. Hence, the Betti numbers of $Bier(\Delta_n^k)$ rely on the reduced homology of the complexes of the form $\Delta_\sigma^k \,\tilde{*}\, \Delta_\tau^l$.

If τ (or analogously σ) is the empty set, Lemma 4.5 yields

$$\beta_{i,\sigma}(K[Bier(\Delta_n^k)]) = \dim \tilde{H}_m(\Delta_\sigma^k) = \delta_{m,k-1}\binom{\#\sigma-1}{k} = \delta_{i,\#\sigma-k}\binom{\#\sigma-1}{k}.$$

Let us assume σ and τ to be not empty.

First, we consider the case that $\#\tau \leq l$, i.e. the second factor Δ_τ^l is a full simplex on $\#\tau$ vertices. If $\tau\setminus\sigma \neq \emptyset$, then every vertex $j' \in (\tau\setminus\sigma)'$ is a cone vertex in $\Delta_\sigma^k \,\tilde{*}\, \Delta_\tau^l$. Thus, the homology of this complex is trivial. If $\tau \subseteq \sigma$, we apply the Mayer-Vietoris sequence in the form of Corollary 4.3 on a vertex $j \in \tau$. The complexes $lk_{\Delta_\sigma^k \tilde{*} \Delta_\tau^l}(j)$ and $\Delta_{\sigma\setminus\{j\}}^{k-1} \,\tilde{*}\, \Delta_{\tau\setminus\{j\}}^l$ have the same faces and therefore, their homologies coincide. Because j' is a cone vertex, the complex $del_{\Delta_\sigma^k \tilde{*} \Delta_\tau^l}(j)$ is contractible. Hence, one obtains that

$$\dim \tilde{H}_m(\Delta_\sigma^k \,\tilde{*}\, \Delta_\tau^l) = \dim \tilde{H}_{m-1}(\Delta_{\sigma\setminus\{j\}}^{k-1} \,\tilde{*}\, \Delta_{\tau\setminus\{j\}}^l)\,.$$

This step can be iterated until k or $\#\tau$ reaches 0. Depending on which of them does this first, we have to distinguish two cases.

If $k < \#\tau$, it holds

$$\dim \tilde{H}_m(\Delta_\sigma^k \,\tilde{*}\, \Delta_\tau^l) = \dim \tilde{H}_{m-k}(\Delta_{\sigma\setminus\rho}^0 \,\tilde{*}\, \Delta_{\tau\setminus\rho}^l) = \dim \tilde{H}_{m-k}(\Delta_{\tau\setminus\rho}^l) = 0\,,$$

where ρ is an arbitrary subset of τ with k elements. For the last equality, it is required that the set $\tau\setminus\rho$ is non-empty.

For the second case $k \geq \#\tau$, note that $\dim \tilde{H}_{-1}(\{\emptyset\}) = 1$ and 0 in all other degrees. Applying Lemma 4.5 yields

$$\begin{aligned}
\dim \tilde{H}_m(\Delta_\sigma^k \,\tilde{*}\, \Delta_\tau^l) &= \dim \tilde{H}_{m-\#\tau}(\Delta_{\sigma\setminus\tau}^{k-\#\tau} \,\tilde{*}\, \Delta_\emptyset^l) \\
&= \begin{cases} \dim \tilde{H}_{m-\#\tau}(\{\emptyset\}) & \text{if } \tau = \sigma \\ \dim \tilde{H}_{m-\#\tau}(\Delta_{\sigma\setminus\tau}^{k-\#\tau}) & \text{if } \tau \subsetneq \sigma \end{cases} \\
&= \begin{cases} \delta_{m-\#\tau,-1} & \text{if } \tau = \sigma \\ \delta_{m-\#\tau,k-\#\tau-1}\binom{\#\sigma-\#\tau-1}{k-\#\tau} & \text{if } \tau \subsetneq \sigma \end{cases} \\
&= \begin{cases} \delta_{i,\#\tau} & \text{if } \tau = \sigma \\ \delta_{i,\#\sigma+\#\tau-k}\binom{\#\sigma-\#\tau-1}{k-\#\tau} & \text{if } \tau \subsetneq \sigma. \end{cases}
\end{aligned}$$

Finally, consider the case $\#\tau > l$. Using Gorenstein duality and Hochster's Formula, the Betti numbers are computed via

$$\begin{aligned}\beta_{i,\sigma\tau'}(K[Bier(\Delta_n^k)]) &= \beta_{2n-(n-1)-i,\overline{\sigma}\overline{\tau}'}(K[Bier(\Delta_n^k)]) \\ &= \dim \tilde{H}_{n-\#\sigma-\#\tau+i-2}(\Delta_{\overline{\sigma}}^k \tilde{*} \Delta_{\overline{\tau}}^l).\end{aligned}$$

As above, the complex $\Delta_{\overline{\sigma}}^k \tilde{*} \Delta_{\overline{\tau}}^l$ is a cone (in particular, its reduced homology is trivial), if $\overline{\tau}$ is not a subset of $\overline{\sigma}$. Note that $\#\overline{\tau} \leq k$. If $\overline{\tau} \subseteq \overline{\sigma}$, then by the use of Lemma 4.5

$$\begin{aligned}\dim \tilde{H}_{n-\#\sigma-\#\tau+i-2}(\Delta_{\overline{\sigma}}^k \tilde{*} \Delta_{\overline{\tau}}^l) &= \dim \tilde{H}_{i-\#\sigma-2}(\Delta_{\overline{\sigma}\setminus\overline{\tau}}^{k-\#\overline{\tau}} \tilde{*} \Delta_{\emptyset}^l) \\ &= \begin{cases} \delta_{i-\#\sigma-2,-1} & \text{if } \sigma = \tau \\ \delta_{i-\#\sigma-2,k-\#\overline{\tau}-1}\binom{\#\overline{\sigma}-\#\overline{\tau}-1}{k-\#\overline{\tau}} & \text{if } \sigma \subsetneq \tau \end{cases} \\ &= \begin{cases} \delta_{i,\#\tau+1} & \text{if } \sigma = \tau \\ \delta_{i,\#\sigma+\#\tau-(n-k-1)}\binom{\#\overline{\sigma}-\#\overline{\tau}-1}{k-\#\overline{\tau}} & \text{if } \sigma \subsetneq \tau. \end{cases}\end{aligned}$$

We summarize the results in a proposition:

Proposition 4.6. *Let k,n be natural numbers with $k \leq n-k-1$ and $\sigma, \tau \subset [n]$. Then it holds*

$$\beta_{i,\sigma\tau'}(K[Bier(\Delta_n^k)]) = \begin{cases} \binom{\#\sigma-\#\tau-1}{k-\#\tau} & \text{if } \tau \subsetneq \sigma, \#\tau \leq k \text{ and} \\ & i = \#\sigma + \#\tau - k; \\ \binom{\#\overline{\sigma}-\#\overline{\tau}-1}{k-\#\overline{\tau}} & \text{if } \tau \supsetneq \sigma, \#\overline{\tau} \leq k \text{ and} \\ & i = \#\sigma + \#\tau - (n-k-1); \\ 1 & \text{if } \tau = \sigma \text{ and either } i = \#\tau \leq k \\ & \text{or } i-1 = \#\tau \geq n-k; \\ 0 & \text{otherwise.} \end{cases}$$

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