LE MATEMATICHE
Vol. LXVII (2012) – Fasc. I, pp. 103–117
doi: 10.4418/2012.67.1.10

REGULAR SEQUENCES OF POWER SUMS AND COMPLETE SYMMETRIC POLYNOMIALS

NEERAJ KUMAR - IVAN MARTINO

In this article, we carry out the investigation for regular sequences of symmetric polynomials in the polynomial ring in three and four variable. Any two power sum element in $\mathbb{C}[x_1,x_2,\dots,x_n]$ for $n \geq 3$ always form a regular sequence and we state the conjecture when p_a, p_b, p_c for given positive integers $a < b < c$ forms a regular sequence in $\mathbb{C}[x_1,x_2,x_3,x_4]$. We also provide evidence for this conjecture by proving it in special instances. We also prove that any sequence of power sums of the form $p_a, p_{a+1},\dots, p_{a+m-1}, p_b$ with $m < n-1$ forms a regular sequence in $\mathbb{C}[x_1,x_2,\dots,x_n]$. We also provide a partial evidence in support of conjecture's given by Conca, Krattenthaler and Watanbe in [1] on regular sequences of symmetric polynomials.

1. Introduction

The work in this article is inspired by the work of Conca, Krattenthaler and Watanabe on regular sequences of symmetric polynomials [1].

We introduce some basic definitions, notation and well known results which we will use in the sequel. We denote by $p_m(x_1,x_2,\dots,x_n), h_m(x_1,x_2,\dots,x_n)$ and $e_m(x_1,x_2,\dots,x_n)$ the power sum symmetric polynomials, complete homogeneous symmetric polynomials and the elementary symmetric polynomials of

Entrato in redazione: 7 ottobre 2011

AMS 2010 Subject Classification: 05E05, 13P10, 11C08.
Keywords: Regular sequences, Symmetric polynomials.

degree m in $\mathbb{C}[x_1, x_2, \ldots, x_n]$ respectively, that is:

$$p_m(x_1, x_2, \ldots, x_n) := \sum_{i=1}^{n} x_i^m,$$
$$h_m(x_1, x_2, \ldots, x_n) := \sum_{1 \le i_1 \le i_2 \le \cdots \le i_m \le n} x_{i_1} x_{i_2} \cdots x_{i_m},$$
$$e_m(x_1, x_2, \ldots, x_n) := \sum_{1 \le i_1 < i_2 < \cdots < i_m \le n} x_{i_1} x_{i_2} \cdots x_{i_m}.$$

We will also denote by $p_m(n)$, $h_m(n)$, and $e_m(n)$, the power sum symmetric polynomials, complete homogeneous symmetric polynomials, and the elementary symmetric polynomials respectively. When n is clear from the context, we may simply denote them by p_m, h_m and e_m respectively.

Regular Sequence: A set of k homogeneous polynomials $f_1, f_2, \ldots, f_k$ in $\mathbb{C}[x_1, x_2, \ldots, x_n]$ is a regular sequence if f_i is not a zero divisor on

$$\mathbb{C}[x_1, x_2, \ldots, x_n]/(f_1, f_2, \ldots, f_{i-1}) \text{ for } i = 1, 2, \ldots, k.$$

Convention: An expression of the form $f_{i_1}, f_{i_2}, \ldots, f_{i_k}$ or $(f_{i_1}, f_{i_2}, \ldots f_{i_k})$ for power sum and complete symmetric polynomials will always mean $i_1 < i_2 < \cdots < i_k$.

We have used Newton's formulas for p_n, h_n and e_n, (see equation $2.6', 2.11'$) from Macdonald [3]. These relations together with the Theorem 2.2 are very helpful in investigating regular sequences.

We have used the Serre Criterion (see section 18.3, Theorem 18.15 [2]) for proving primeness for power sum polynomials in the polynomial ring. Once we know that p_a, p_b generates a prime ideal in $\mathbb{C}[x_1, x_2, x_3, x_4]$, we can add one more polynomial f and conclude that p_a, p_b, f forms a regular sequence for all $f \notin (p_a, p_b)$. We prove p_1, p_{2m} generates a prime ideal, where $m \in \mathbb{N}$, see Proposition 4.3. We also prove this in the case of consecutive integres $a, a+1$. In fact we prove a more general statement that any consecutive power sum $p_a, p_{a+1}, \ldots, p_{a+m-1}$ with $m < n-1$ generates a prime ideal in $\mathbb{C}[x_1, x_2, \ldots, x_n]$, see Theorem 4.3.

In general, it turns out to be difficult to find conditions on $\{a, b\}$ such that (p_a, p_b) is a prime ideal. We did several computations in CoCoA and found some conditions on $\{a, b\}$ such that (p_a, p_b) is a prime ideal, see Conjecture 4.6. For example when a is prime number, $a \geq 5$ and $b = a + m + 6d$ with $m \in \{1, 5\}$ then (p_a, p_b) is a prime ideal.

However, a very nice introduction to regular sequences of symmetric polynomials is given by Conca, Krattenthaler and Watanable in [1]. So, we refer the reader for detailed introduction to [1].

2. Some results on regular sequences

Let us recall some well known results about regular sequences.

Lemma 2.1. *The sequence of homogeneous polynomials* $f_1, f_2, \dots, f_k \in S = \mathbb{C}[x_1, x_2, \dots, x_n]$ *is a regular sequence in* S *if and only if*

$$H_{S/I}(z) = \frac{\prod_{i=1}^{k}(1-z^{d_i})}{(1-z)^n}.$$

where $d_i = \deg f_i$ *and* $I = (f_1, f_2, \dots, f_k)$.

We will use the following characterization very often for proving regular sequence for the power sums and complete symmetric polynomials:

Theorem 2.2. *Let* $f_i, f_j, f_k \in S = \mathbb{C}[x_1, x_2, x_3]$. *The sequence* f_i, f_j, f_k *is a regular sequence if and only if* $f_k \notin (f_i, f_j)$ *and for any* f *of degree bigger than* $i+j+k$ *we have* $f \in (f_i, f_j, f_k)$.

Proof. If f_i, f_j, f_k is a regular sequence then f_i is not a zero divisor on S, f_j is not a zero divisor on $S/(f_i)$ and f_k is not a zero divisor on $S/(f_i, f_j)$. This implies $f_k \notin (f_i, f_j)$.

We know that $(0,0,0)$ is the only solution of the system (f_i, f_j, f_k). This means that $(0,0,0)$ has multiplicity $i+j+k$ and (f_i, f_j, f_k) is the $(i+j+k)$-th power of the maximal ideal. So considering f with degree of f bigger than $i+j+k$, this implies $f \in (f_i, f_j, f_k)$. □

Of course there are three possible cases for f_i, f_j, f_k in $\mathbb{C}[x_1, x_2, x_3]$:

1. $f_k \notin (f_i, f_j)$, and f_i, f_j, f_k is a regular sequence;
2. $f_k \notin (f_i, f_j)$, and f_i, f_j, f_k is not a regular sequence,
3. $f_k \in (f_i, f_j)$, then f_i, f_j, f_k is not a regular sequence.

See Example 5.1, where $f_k \notin (f_i, f_j)$ and f_i, f_j, f_k is not a regular sequence.

Notation: For a subset $A \subset \mathbb{N}^*$, we set

$$p_A(n) = \{p_a(n) : a \in A\} \text{ and } h_A(n) = \{h_a(n) : a \in A\}.$$

Proposition 2.1. Let $A \subset \mathbb{N}^*$ be a set of n consecutive elements. Then both $p_A(n)$ and $h_A(n)$ are regular sequences in $k[x_1, \dots, x_n]$.

Proof. Refer to Proposition 2.9 [1] for proof. □

We are going to use the Newton's formulas:

Proposition 2.2. Let p_n be the power sum symmetric polynomial of degree n, h_n be the complete homogeneous symmetric polynomial of degree n and let e_n be the elementary symmetric polynomial of degree n. Then

$$ne_n = \sum_{i=1}^{n} (-1)^{i-1} e_{n-i} p_i \text{ for all } n \geq 1.$$

$$\text{and } \sum_{i=0}^{n} (-1)^i e_i h_{n-i} = 0 \text{ for all } n \geq 1.$$

These equations are due to I. Newton, see Macdonald [3] (equation $2.6', 2.11'$).

Next Lemma follows from Eisenstein's Criterion.

Lemma 2.3. *Let R be a unique factorization domain and $b \in R$. Suppose that p but not p^2 divides b for some irreducible $p \in R$. Then $x^m + b$ is irreducible in $R[x]$*

3. Symmetric Polynomials in 3 variables

3.1. Power Sums in 3 variables

Conjecture(Conca, Krattenthaler, Watanabe) Let a, b, c be positive integers with $a < b < c$ and $\gcd(a, b, c) = 1$. Then p_a, p_b, p_c is a regular sequence if and only if $abc \equiv 0 \pmod 6$.

Remark: For this conjecture, the "only if" part has been proved in [1], they provide partial result in support of the "if" part. We have also tried to prove this in some special cases, here the only difference is in approach, we provide a nice expression for $p_c \mod (p_a, p_b)$.

Proposition 3.1. Consider the power sum sequence p_1, p_2, p_n, then

$$p_n = \begin{cases} 3e_3^k \mod (p_1, p_2), & \text{if } n = 3k; \\ 0 \mod (p_1, p_2), & \text{otherwise.} \end{cases}$$

Proof. As $p_0 = 3$, we use Newton's formula, see Proposition 2.2, to write p_n,

$$\begin{aligned} p_1 =& e_1 = 0, \\ p_2 =& e_1 p_1 - 2e_2 = 0 \implies e_2 = 0, \\ p_3 =& 3e_3 + e_1 p_2 - e_2 p_1 = 3e_3, \\ p_4 =& e_1 p_3 - e_2 p_2 + e_3 p_1 = 0 \text{ similarly } p_5 = 0, \\ p_6 =& 3e_3^2. \text{ And so on, we continue in this way.} \end{aligned}$$

Hence we get $p_n = 3e_3^k \mod (p_1, p_2)$ if $n = 3k$.

$\square$

Corollary 3.1. *p_1, p_2, p_n is a regular sequence if and only if $n = 3k, k \in \mathbb{N}$.*

Proof. We only need to verify the cases of the form p_1, p_2, p_n where $n = 3k, k \in \mathbb{N}$. Choose any $m > 1+2+3k = 3(k+1)$, we observe that $p_m \in (p_1, p_2, p_n)$. Hence p_1, p_2, p_n is a regular sequence for $n = 3k, k \in \mathbb{N}$. $\square$

Proposition 3.2. Consider the sequence p_1, p_3, p_n, then

$$p_n = \begin{cases} (-1)^k e_2^k \mod (p_1, p_3), & \text{if } n = 2k; \\ 0 \mod (p_1, p_3), & \text{otherwise.} \end{cases}$$

Proof. Similar to Proposition 3.1. $\square$

Corollary 3.2. *p_1, p_3, p_n is a regular sequence if and only if $n = 2k, k \in \mathbb{N}$.*

Proof. Similar to Corollary 3.1. $\square$

Remark: p_2, p_3, p_n is a regular sequence for all n , see Theorem 2.11 [1]. In the paper [1], they have given a complete proof. We present here the slightly tricky argument from their paper, they managed to reduce the problem and concluded that it is enough to prove this for the case $n = 4$. They did computer experiments to show this for $n = 4$ case. But it follows directly from Proposition 2.9 [1] as $2, 3, 4$ are consecutive integers.

3.2. Complete symmetric polynomials in 3 variables

Conjecture(Conca, Krattenthaler, Watanabe) Let $A = \{a, b, c\}$ with $a < b < c$. Then h_a, h_b, h_c is a regular sequence if and only if the following conditions are satisfied:

1 $abc \equiv 0(\mod 6)$.

2 $\gcd(a+1, b+1, c+1) = 1$.

3 For all $t \in \mathbb{N}$ with $t > 2$ there exist $d \in A$ such that $d + 2 \not\equiv 0, 1(\mod t)$.

Remark: For this conjecture, the “only if” part has been proved by authors, the “if’ part is still open. We are able to give partial proof of this conjecture under some special choice of a,b and for any c, both the ”if“ and the ”only if“ part.

Proposition 3.3. Consider the sequence h_1, h_2, h_n, then

$$h_n = \begin{cases} -e_3^k \mod (h_1, h_2), & \text{if } n = 3k; \\ 0 \mod (h_1, h_2), & \text{otherwise.} \end{cases}$$

Proof. We know by the Proposition 2.2 that

$$h_n = e_1 h_{n-1} - e_2 h_{n-2} + \cdots + (-1)^n e_n h_0.$$

Now as in our case $n = 3$, So $e_n = 0$ for $n > 4$. Hence

$$\begin{aligned} h_0 =& 1, \\ h_1 =& e_1 h_0 = e_1 = 0 \\ h_2 =& e_1 h_1 - e_2 h_0 = 0 \text{ which means } e_2 = 0, \\ h_3 =& e_1 h_2 - e_2 h_1 + e_3 h_0 = -e_3 \mod (h_1, h_2). \end{aligned}$$

In this way, we carry out the simplification for h_n, $n \geq 4$ and we arrive at the following expression:

$$h_n = \begin{cases} -e_3^k \mod (h_1, h_2), & \text{if } n = 3k; \\ 0 \mod (h_1, h_2), & \text{otherwise.} \end{cases}$$

□

Corollary 3.3. *h_1, h_2, h_n is a regular sequence if and only if $n = 3k, k \in \mathbb{N}$.*

Proof. Clearly the cases $n = 3k+1$ and $n = 3k+2$ are ruled out. For the case $n = 3k$ choose $m > 1+2+3k = 3(k+1)$, clearly $h_m \in (h_1, h_2, h_{3k})$. Hence (h_1, h_2, h_n) is a regular sequence for $n = 3k, k \in \mathbb{N}$. □

Proposition 3.4. Consider the sequence h_1, h_3, h_n, then

$$h_n = \begin{cases} (-1)^{\frac{n}{2}-1} e_2^{\frac{n}{2}} \mod (h_1, h_3), & \text{if } n = 2k; \\ 0 \mod (h_1, h_3), & \text{if } n = 2k+1. \end{cases}$$

Proof. Similar to Proposition 3.3. □

Corollary 3.4. *h_1, h_3, h_n is a regular sequence if and only if $n = 2k, k \in \mathbb{N}$.*

Proof. Similar to Corollary 3.3. □

Proposition 3.5. Consider the sequence h_1, h_4, h_n, then

$$h_n = \begin{cases} e_3^k \mod (h_1, h_4), & \text{if } n = 3k; \\ 0 \mod (h_1, h_4), & \text{if } n = 3k+1; \\ -(k+1)e_2 e_3^k \mod (h_1, h_4), & \text{if } n = 3k+2. \end{cases}$$

Proof. Similar to Proposition 3.3. □

Corollary 3.5. *The sequence h_1, h_4, h_n is a regular sequence if and only if $n = 3k$, $k \in \mathbb{N}$.*

Proof. Similar to Corollary 3.3. □

Proposition 3.6. Consider the sequence h_2, h_3, h_n, then

$$h_n = \begin{cases} e_1^{2k-2} e_2^{k+1} \mod (h_2, h_3), & \text{if } n = 4k; \\ e_1^{2k-1} e_2^{k+1} \mod (h_2, h_3), & \text{if } n = 4k+1; \\ 0 \mod (h_1, h_2), & \text{if } n = 4k+2, 4k+3. \end{cases}$$

Proof. Similar to Proposition 3.3. □

Corollary 3.6. *The sequence h_2, h_3, h_n is a regular sequence if and only if $n = 4k, 4k+1$, where $k \in \mathbb{N}$.*

Proof. Clearly $n = 4k+2, 4k+3$ is ruled out. Now let $m_1 > 2+3+4k = 4(k+1)+1$ and $m_2 > 2+3+4k+1 = 4(k+1)+2$ then $h_{m_1} \in (h_2, h_3, h_{4k})$ and $h_{m_2} \in (h_2, h_3, h_{4k+1})$. Hence (h_2, h_3, h_n) for all $n = 4k, 4k+1$, $k \in \mathbb{N}$ is a regular sequence. □

4. Symmetric Polynomials in 4 variables

4.1. Power sums in 4 variables

Theorem 4.1. *Let p_i be the power sum symmetric polynomials of degree i in the polynomial ring $S = \mathbb{C}[x_1, x_2, \ldots, x_n]$. Let $n \geq 3$, then p_a, p_b is a regular sequence.*

Proof. We know $p_a(n)$ is reducible for $n = 1$ and $p_2(n)$ is reducible for $n = 2$. For $n \geq 3$, we will show $p_a(n)$ is an irreducible element. We prove this by induction on n. For $n = 3$, we can write $p_a(3) = x_3^a + g$, where $g = x_1^a + x_2^a \in \mathbb{C}[x_1, x_2]$, g is a homogeneous and monic polynomial in both variable, of degree a. So proving factorization of $g(x_1, x_2)$ is same as proving factorization of $g(x_1,1)$. Since $g(x_1,1)$ has simple roots, g is a product of a linear forms. Thus Lemma 2.3 shows that $p_a(3)$ is irreducible. Then, if $n>3$, $p_a(n)=x_n^a+p_a(n-1)$ is irreducible by Lemma 2.3 by induction. Therefore the ideal generated by p_a is a prime ideal. Hence $S/(p_a)$ is a domain. Now p_b being an irreducible element in S, can not be factored into lower degree power sum polynomials p_a. So p_b is a non zero divisor on $S/(p_a)$ for $b>a$. Hence p_a,p_b is a regular sequence. □

Note 4.2. If the characteristic of base field $\mathbb{K}$ is not zero, then above result does not hold. Consider the field with $\operatorname{char}(\mathbb{K})=2$, then one has $p_4=p_2^2$.

In particular,

Proposition 4.1. Let p_i be the power sum symmetric polynomials of degree i in the polynomial ring $S=\mathbb{C}[x_1,x_2,x_3,x_4]$. Then p_a,p_b is a regular sequence.

We know that a subset of a regular sequence is a regular sequence. So by Proposition 2.1, $p_a,p_{a+1},\dots,p_{a+m-1}$ is a regular sequence. Let $R=S/I$, where $S=\mathbb{C}[x_1,x_2,\dots,x_n]$, $I=\langle p_a,p_{a+1},\dots,p_{a+m-1}\rangle$ with $m<n-1$. Hence R is Cohen Macaulay. Now we are going to use the Serre Criterion (see section 18.3, Theorem 18.15 [2]) for proving m consecutive power sums polynomials generates a prime ideal in the polynomial rings S. Once we know that I is a prime ideal in S, we can add one more power sum element p_c and conclude that $p_a,p_{a+1},\dots,p_{a+m-1},p_c$ forms a regular sequence provided $p_c\notin I$.

Theorem 4.3. *Let p_i be the power sum symmetric polynomials of degree i in the polynomial ring $S=\mathbb{C}[x_1,x_2,\dots,x_n]$, with $n\geq 4$.*
Then $I=\langle p_a,p_{a+1},\dots,p_{a+m-1}\rangle$ with $m<n-1$ is a prime ideal in S. In particular, $p_a,p_{a+1},\dots,p_{a+m-1},p_c$ forms a regular sequence provided $p_c\notin I$.

Proof. Consider $S=\mathbb{C}[x_1,x_2,\dots,x_n]$, $I=(p_a,p_{a+1},\dots,p_{a+m-1})$ with $m<n-1$ and $R=S/I$. Now let us compute the Jacobian of I, say Jacobian:$=J$.

$$J=c\begin{pmatrix} x_1^{a-1} & x_2^{a-1} & \cdots & x_n^{a-1}\\ x_1^{a} & x_2^{a} & \cdots & x_n^{a}\\ \vdots & \vdots & \cdots & \vdots\\ x_1^{a+m-2} & x_2^{a+m-2} & \cdots & x_n^{a+m-2}\end{pmatrix}$$

We have taken the coefficients out from each row, where $c=\prod_{i=0}^{m-1}(a+i)$. We can ignore the coefficients since we are in the field of characteristic zero and c is a unit in $\mathbb{C}$. Let $J'=I_m(J)$, denote's the ideal generated by $m\times m$ minors of Jacobian. Also $m=\operatorname{ht}(I)$, since I is generated by a regular sequence of length

m. The $m \times m$ submatrices of the jacobian are standard Van der monde matrices, we know their determinants. So we can write

$$J' = \langle\, x_{i_1}^{j_1} x_{i_2}^{j_2} \cdots x_{i_m}^{j_m} \prod_{1 \le a < b \le m} (x_{i_a} - x_{i_b}) \,\rangle \text{ for } 1 \le i_1 < i_2 < \cdots < i_m \le n,$$

and for some positive integers $j_1, j_2, \ldots, j_m$. Therefore

$$I + J' = \langle p_a, p_{a+1}, \ldots, p_{a+m-1},\, x_{i_1}^{j_1} x_{i_2}^{j_2} \cdots x_{i_m}^{j_m} \prod_{1 \le a < b \le m} (x_{i_a} - x_{i_b}) \rangle.$$

Claim: $\sqrt{I + J'} = (x_1, x_2, \ldots, x_n)$.
Suppose not, that is there exists $w \in \mathbb{P}^{n-1}$ with $w \in Z(I + J')$. Then the vector w can have at the most $m-1$ distinct non zero coordinates. If w has m or more than m distinct non zero coordinates, then $w \notin Z(J')$. Say w has v distinct non zero coordinates. We can write

$$w = (w_1, \ldots w_1, w_2, \ldots, w_2, \ldots, w_v, \ldots, w_v, 0, 0, \ldots, 0),$$

where w_i appears β_i times and $v \le m-1$. Also w should satisfy p_{a+i} for $i = 0, 1, \ldots, m-1$ i.e.

$$\beta_1 w_1^{a+i} + \beta_2 w_2^{a+i} + \cdots + \beta_v w_v^{a+i} = 0 \text{ for } i = 1, 2, \ldots, m.$$

This is a system of equation, which can be represented in the matrix form with m rows, v column.

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ w_1 & w_2 & \cdots & w_v \\ \vdots & \vdots & \cdots & \vdots \\ w_1^{m-1} & w_2^{m-1} & \cdots & w_v^{m-1} \end{pmatrix} \begin{pmatrix} \beta_1 w_1^{a+i} \\ \beta_2 w_2^{a+i} \\ \vdots \\ \beta_v w_v^{a+i} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

We know that neither $\beta_i = 0$ nor $w_i = 0$ for $i = 1, \ldots, v$. So, $\beta_i w_i^{a+i} \neq 0$ for $i = 1, \ldots, v$. We can choose the matrix say M with first v rows out of m rows and look for the solution. The matrix M is of full rank since $w_i \neq w_j$ for $i \neq j$, so the only possible solution has to be the trivial solution.

Therefore such a w does not exist and hence the claim is proved. This implies $\operatorname{ht}(I + J') = n$ and $\dim \frac{S}{I+J'} = 0$. The co-dimension of J' in S is $n-2$. Hence by Theorem 18.15 in [2], R is a product of normal domain. So, we can write $R = R_1 \times \cdots \times R_k$. Since R is a standard graded $\mathbb{C}$-algebra with $R_0 = \mathbb{C}$, also $R_0 = (R_1)_0 \times \cdots \times (R_k)_0 = \mathbb{C}^k$. Hence $k = 1$. Therefore R is a normal domain and I is a prime ideal in S. $\square$

In particular,

Proposition 4.2. Let p_i be the power sum symmetric polynomials of degree i in the polynomial ring $S = \mathbb{C}[x_1, x_2, x_3, x_4]$; then $I = (p_i, p_{i+1})$ is a prime ideal in S. In particular, p_i, p_{i+1}, p_n is a regular sequence for all $p_n \notin (p_i, p_j)$.

Lemma 4.4. *Let $n \geq 5$ be any natural number. If the sum of four distinct n-th roots of unity is zero, then they must be two pair of opposite sign.*

Furthermore, if n is odd number, then sum of four distinct n-th roots of unity is never zero.

Proof. We have $z^n = 1$. Let us pick four distinct n-th roots of unity, call it z_j for $j = 1, 2, 3, 4$. Each $z_j = x_j + iy_j$, where $x_j, y_j \in \mathbb{R}$ and $|z_j| = 1$.
Claim: If $\sum_{j=1}^4 z_j = 0$, then $z_1,\ z_2,\ z_3,\ z_4$ must be of the form $z_1,\ z_2,\ -z_1,\ -z_2$.
Let $\sum_{j=1}^4 z_j = 0$ i.e. $\sum_{j=1}^4 x_j = 0$ and $\sum_{j=1}^4 y_j = 0$. So, we can write,

$$x_1 + x_2 = -(x_3 + x_4), \text{ and } y_1 + y_2 = -(y_3 + y_4).$$

Now, squaring and adding both the equation, we obtain $2 + 2(x_1x_2 + y_1y_2) = 2 + 2(x_3x_4 + y_3y_4)$. Therefore, we get,

$$|z_1 - z_2|^2 = 2 - 2(x_1x_2 + y_1y_2) = 2 - 2(x_3x_4 + y_3y_4) = |z_3 - z_4|^2.$$

So $|z_1 - z_2| = |z_3 - z_4|$. Similarly, we get $|z_2 - z_3| = |z_1 - z_4|$. So, four distinct z_j form a parellelogram. The diagonals of a parellelogram intersect at mid point. Hence solving $\frac{z_1+z_3}{2} = \frac{z_2+z_4}{2}$ and $\sum_{j=1}^4 z_j = 0$, we conclude that $z_3 = -z_1$ and $z_4 = -z_2$. So, we obtain four distinct roots of unity as $z_1,\ z_2,\ -z_1,\ -z_2$.

Furthermore, z_1 and $-z_1$ both can not be n-th roots of unity for any n odd number. □

Proposition 4.3. Let $I = (p_1, p_{2m})$, where $m \in \mathbb{N}$. Then I is a prime ideal in $\mathbb{C}[x_1, x_2, x_3, x_4]$. Therefore p_1, p_{2m}, p_n form a regular sequence for all $p_n \notin (p_1, p_{2m})$.

Proof. For $m = 1$, it follows from Proposition 4.1. Let $m > 1$. Consider $S = \mathbb{C}[x_1, x_2, x_3, x_4]$, $I = (p_1, p_{2m})$, we know by Theorem 4.3 that $p_1,\ p_{2m}$ is a regular sequence in S. So $\mathrm{ht}(I) = 2$. Let $R = S/I$. Now let us compute the Jacobian of I, say Jacobian:$= J$.

$$J = (2m-1)\begin{pmatrix} 1 & 1 & 1 & 1 \\ x_1^{2m-1} & x_2^{2m-1} & x_3^{2m-1} & x_4^{2m-1} \end{pmatrix}$$

We can ignore the coefficients $2m-1$, since we are in the field of characteristic zero and $2m-1$ is a unit in $\mathbb{C}$. Let $J' = I_2(J)$, denote's the ideal generated by

2×2 minors of J. So, we can write $J' = \langle\, x_j^{2m-1} - x_i^{2m-1} \,\rangle$ for $1 \le i \le j \le 4$. Therefore consider

$$I + J' = \langle\, p_1, p_{2m},\, x_j^{2m-1} - x_i^{2m-1} \,\rangle \text{ for } 1 \le i \le j \le 4.$$

Claim: $\sqrt{I+J'} = (x_1, x_2, x_3, x_4)$.
Suppose not, i.e. there exists $w \in \mathbb{P}^3$ with $w \in Z(I+J)$. Let $w = (w_1, w_2, w_3, w_4)$. If one of w_i is zero, then it is easy to see all the w_i's are zero. So we assume none of w_i is zero. Also assume $w_i \neq w_j$ for $i \neq j$. Since w is in $\mathbb{P}^3$, we can make $w_1 = 1$ if $w_1 \neq 0$. So let $w = (1, x, y, z)$. Since $w \in Z(I+J)$ implies $1 = x^{2m-1} = y^{2m-1} = z^{2m-1}$ and $x^{2m-1} = y^{2m-1} = z^{2m-1}$. Also w satisfies p_1, p_{2m}. Therefore

$$1 + x + y + z = 0 \text{ and } 1 + x^{2m} + y^{2m} + z^{2m} = 0.$$

Both the equation reduces to existence of solution of $1 + x + y + z = 0$. We assumed all the coordinates are distinct, We use the fact that all the x or y or z is $(2m-1)$-th roots of unity, say $1, \zeta_1, \ldots, \zeta_{2m-2}$. Now, it follows from the Lemma 4.4 that $1 + \zeta_i + \zeta_j + \zeta_k \neq 0$ for distinct i, j, k. For ζ_i's to be distinct, one must have $m > 2$. But for $m = 2$, one has cube roots of unity, so one of $\zeta_i = \zeta_j$ for some i, j. In that case it is clear that there is no solution. Now it is easy to verify that if $w = (1, x, y, y)$ or $w = (1, x, x, x)$, then also, there does not exist solution of $p_1(w) = 0$. So, the only possible solution has to be the trivial solution. Hence the claim is proved. This implies $\mathrm{ht}(I + J') = 4$ and $\dim \frac{S}{I+J'} = 0$. The co-dimension of J' in S is 2. Hence by Theorem 18.15 in [2], R is a product of normal domain. So, we can write $R = R_1 \times \cdots \times R_k$. Since R is a standard graded $\mathbb{C}$-algebra with $R_0 = \mathbb{C}$, also $R_0 = (R_1)_0 \times \cdots \times (R_k)_0 = \mathbb{C}^k$. Hence $k = 1$. Therefore R is a normal domain and I is a prime ideal in S. $\square$

Computer calculations using CoCoA suggest the following conjecture:

Conjecture 4.5. Let p_i be the power sum symmetric polynomial of degree i in the polynomial ring $S = K[x_1, x_2, x_3, x_4]$. Let $A = \{a, b, n\}$ with $a < b < n$, then $p_A(4)$ is a regular sequence if and only if A satisfies the following conditions:

1 If a is odd and b is even, then for any n.

2 If a is odd and b is odd, then for any n even.

3 If a is even, say $a = 2m$, with m odd, then for all n provided $\lambda \neq 4k, k \in \mathbb{N}$ where $\lambda = b - a$ and If $\lambda = 4k$, then for all n of the form $4l + 2$, with $l \in \mathbb{N}$.

4 If a is even, say $a = 2m$, with m even, then for all n, provided $b \neq 3a$ and $n \neq (2k+1)a$, $k \in \mathbb{N}$.

5 (a,b,n) should not be of the form $(a,2a,5a)$, irrespective of a being odd or even.

We wanted to show $I = (p_a, p_b)$ is a prime ideal for some a,b. We did several computations on computer and found some conditions on a,b. We could not prove these results. We state them as a Conjecture as follows:

Conjecture 4.6. 1. Let $I = (p_a, p_b)$ where a is a prime number, $a \geq 5$ and $b = a+m+6d$ with $m \in \{1,5\}$ and $d \in \mathbb{N} \cup \{0\}$. Then I is a prime ideal in $\mathbb{C}[x_1,x_2,x_3,x_4]$. Therefore p_a, p_b, p_n forms a regular sequence for all $p_n \notin (p_a, p_b)$.

2. Let $I = (p_2, p_{2m})$ with $m \in \mathbb{N}$ and $m \neq 2+3k, 2+4k$, where $k \in \mathbb{N}$. Then I is a prime ideal in $\mathbb{C}[x_1,x_2,x_3,x_4]$. Therefore p_2, p_{2m}, p_n forms a regular sequence for all $p_n \notin (p_2, p_{2m})$.

3. Let $I = (p_3, p_{2m})$ with $m \in \mathbb{N}$ and $m \neq 6+9\lambda$, where $\lambda \in \mathbb{N} \cup \{0\}$. Then I is a prime ideal in $\mathbb{C}[x_1,x_2,x_3,x_4]$. Therefore p_3, p_{2m}, p_n forms a regular sequence for all $p_n \notin (p_3, p_{2m})$.

4. Let $I = (p_4, p_{2m})$ with $m \in \mathbb{N}$ and $m \neq 4+3k, 4+8k$, where $k \in \mathbb{N}$. Then I is a prime ideal in $\mathbb{C}[x_1,x_2,x_3,x_4]$. Therefore p_4, p_{2m}, p_n forms a regular sequence for all $p_n \notin (p_4, p_{2m})$.

Conjecture(Conca, Krattenthaler, Watanabe) Let $A \subset \mathbb{N}^*$ with $|A| = 4$, say $A = \{a_1, a_2, a_3, a_4\}$, and assume $\gcd(A) = 1$. Then $p_A(4)$ is a regular sequence if and only if A satifies the following conditions:

1 At least two of the a_i's are even, at least one is a multiple of 3, and at least one is a multiple of 4,

2 If E is the set of the even elements in A and $d = \gcd(E)$ then the set $\{\frac{a}{d} : a \in E\}$ contains an even number.

3 A does not contain a subset of the form $\{d, 2d, 5d\}$.

We use the Newton's formula for power sum to deduce the following result:

Proposition 4.4. Consider the power sum sequence p_1, p_2, p_3, p_n, then

$$p_n == \begin{cases} (-1)^k 4e_4^k \mod (p_1,p_2,p_3), & \text{if } n = 4k; \\ 0 \mod (p_1,p_2,p_3), & \text{otherwise.} \end{cases}$$

Proof. Similar to Proposition 3.1. The only difference in this case is, $p_0 = 4$. $\square$

Corollary 4.7. *The power sum sequence p_1, p_2, p_3, p_n is a regular sequence if and only if $n = 4k$.*

Proof. Similar to Corollary 3.1. The only difference is, there we use Theorem 2.2 for three variable. Similar result can be deduced for the four variable case also. □

Proposition 4.5. Consider the power sum sequence p_1, p_2, p_4, p_n, then

$$p_n = \begin{cases} 4e_3^k \mod (p_1, p_2, p_4), & \text{if } n = 3k; \\ 0 \mod (p_1, p_2, p_4), & \text{otherwise.} \end{cases}$$

Proof. Similar to Proposition 3.1. □

Corollary 4.8. *The power sum sequence p_1, p_2, p_4, p_n is a regular sequence if and only if $n = 3k$.*

Proof. Similar to Corollary 3.1. □

4.2. Complete symmetric polynomials in 4 variables

Proposition 4.6. Consider the sequence h_1, h_2, h_3, h_n, then

$$h_n = \begin{cases} (-1)^k e_4^k \mod (h_1, h_2, h_3), & \text{if } n = 4k; \\ 0 \mod (h_1, h_2, h_3), & \text{otherwise.} \end{cases}$$

Proof. Similar to Proposition 3.3. □

Corollary 4.9. *The sequence h_1, h_2, h_3, h_n is a regular sequence if and only if $n = 4k, k \in \mathbb{N}$.*

Proof. Similar to Corollary 3.3. The only difference is one has a similar result to Theorem 2.2 in the four variable case. □

Proposition 4.7. Consider the sequence h_1, h_2, h_4, h_n, then

$$h_n = \begin{cases} e_3^k \mod (h_1, h_2, h_4), & \text{if } n = 3k; \\ 0 \mod (h_1, h_2, h_4), & \text{otherwise.} \end{cases}$$

Proof. Similar to Proposition 3.3. □

Corollary 4.10. *The sequence h_1, h_2, h_4, h_n is a regular sequence if and only if $n = 3k, k \in \mathbb{N}$.*

Proof. Similar to Corollary 3.3. □

Proposition 4.8. Consider the sequence h_2, h_3, h_4, h_n, then

$$h_n = \begin{cases} (-1)^k e_1^k e_4^k \mod (h_2, h_3, h_4), & \text{if } n = 5k; \\ (-1)^k e_1^{k+1} e_4^k \mod (h_2, h_3, h_4), & \text{if } n = 5k+1; \\ 0 \mod (h_2, h_2, h_4), & \text{otherwise.} \end{cases}$$

Proof. Similar to Proposition 3.3. □

Corollary 4.11. *The sequence h_2, h_3, h_4, h_n is a regular sequence if and only if $n = 5k, 5k+1, k \in \mathbb{N}$.*

Proof. Similar to Corollary 3.3. □

5. Appendix

5.1. Verification of the Conca, Krattenthaler and Watanabe conjecture

(i) For p_1, p_2, p_n: This follows directly from Corollary 3.1.

(ii) For p_1, p_3, p_n: This follows directly from Corollary 3.2.

(iii) For h_1, h_2, h_n: Let us start with the necessary part.

1 $2c \equiv 0(\mod 6)$ implies that $c = 3k$.
2 $\gcd(2,3) = 1$ (always true).
3 For all $t \in \mathbb{N}$ with $t > 2$ there exist $d \in A$ such that $d + 2 \not\equiv 0, 1(\mod t)$: we know that $1+2$ and $2+2$ are 0 or 1 only modulo 3 and so t=3; these means that this condition is false iff $c + 2 \cong 0, 1(\mod 3)$. Thus, we have that $c = 3k$.

Viceversa, if $n = 3k$, [1] and [3] are fulfilled.

Similarly we can show, (iv) for h_1, h_3, h_n ; (v) for h_1, h_4, h_n, and (vi) for h_2, h_3, h_n respectively.

Example 5.1. This example is a case when $h_k \notin (h_i, h_j)$ and h_i, h_j, h_k is not a regular sequence.

Consider the triple h_1, h_4, h_5. By Proposition 3.5, we have $h_5 = -e_2 e_3 \mod (h_1, h_4)$, hence $h_5 \notin (h_1, h_4)$, still h_1, h_4, h_5 is not a regular sequence.
First we compute the hilbert series of (h_1, h_4, h_5) and we find that

$$H_{S/(h_1,h_4,h_5)}(t) = \frac{1 - t - t^4 + t^6 + t^7 - t^8}{(1-t)^3}.$$

If h_1, h_4, h_5 were a regular sequence, Hilbert series should have been

$$H_{S/(h_1,h_4,h_5)}(t) = \frac{(1-t)(1-t^4)(1-t^5)}{(1-t)^3}, \\ = \frac{1-t-t^4+t^6+t^9-t^{10}}{(1-t)^3}.$$

which is clearly not the case.

Acknowledgements

We thank Professors Ralf Fröberg, Mats Boij and Alexander Engström for the support and the help given during the Pragmatic 2011. We thank Professor Aldo Conca for his valuable suggestions, comments, and writing the CoCoA program for prime ideal test. We also thank Professors Anna M. Bigatti and John Abbott for for their help in CoCoA for the necessary computations.

REFERENCES

[1] A. Conca - C. Krattenthaler - J. Watanabe, *Regular sequences of symmetric polynomials*, Rend. Semin. Math. Univ. Padova 121 (2009), 179–199.

[2] D. Eisenbud, *Commutative Algebra with a view toward Algebraic Geometry*, Graduate Texts in Mathematics 150, Springer-Verlag, New York, 1995.

[3] I. G. Macdonald, *Symmetric functions and Hall polynomials*, Second edition. With contributions by A. Zelevinsky. Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1995.

NEERAJ KUMAR
Dipartimento di Matematica
Università di Genova
Via Dodecaneso 35
16146 Genova, Italy
e-mail: kumar@dima.unige.it

IVAN MARTINO
Matematiska institutionen,
Stockholms Universitet, Stockholm, Sweden
e-mail: martino@math.su.se