LE MATEMATICHE
Vol. LXVIII (2013) – Fasc. I, pp. 83–86
doi: 10.4418/2013.68.1.7

A COMMENT ON COPOSINORMAL OPERATORS

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Dedicated to Eric Xaver Mader

Hyponormal operators are necessarily posinormal, but they need not be coposinormal. Coposinormality can nevertheless sometimes be an aid in determining hyponormality. This idea is applied to coposinormal factorable matrices whose interrupter is diagonal. Examples are given, including some Toeplitz matrices and some terraced matrices associated with the logistic sequence.

1. Introduction

If $B(H)$ denotes the set of all bounded linear operators on a Hilbert space H, then $M \in B(H)$ is *posinormal* if there exists a positive operator $P \in B(H)$ satisfying $MM^* = M^*PM$. The operator P is referred to as the *interrupter*. From [2], Theorem 2.1, we know that M is posinormal if and only if

$$\lambda^2 M^*M \geq MM^*$$

for some $\lambda \geq 0$. M is *hyponormal* when $\lambda = 1$. M is *coposinormal* if M^* is posinormal. More information on posinormal operators can be found in [1], [2].

Some previous efforts to identify hyponormal operators had been focused on using posinormality in hopes that the interrupter P could be shown to satisfy

Entrato in redazione: 19 maggio 2012

AMS 2010 Subject Classification: 47B99.

Keywords: Posinormal operator, Hyponormal operator, Factorable matrix, Terraced matrix, Toeplitz matrix, Logistic sequence.

$I \geq P$; see [2], [3], [4]. This approach seemed reasonable and natural since all hyponormal operators are posinormal. Although hyponormal operators are not necessarily coposinormal, the following theorem tells us that there will be situations where coposinormality can also help us identify hyponormal operators.

Theorem 1.1. *Suppose that $M \in B(H)$ is coposinormal; that is, M satisfies $M^*M = MQM^*$ for some positive operator $Q \in B(H)$. If P is a positive operator and $Q \geq P \geq I$, then $Y :\equiv \sqrt{P}M\sqrt{P}$ is hyponormal.*

Proof. If $[Y^*, Y] :\equiv Y^*Y - YY^*$, then

$$\langle [Y^*, Y]f, f\rangle = \langle (P-I)M\sqrt{P}f, M\sqrt{P}f\rangle + \langle (Q-P)M^*\sqrt{P}f, M^*\sqrt{P}f\rangle \geq 0$$

for all f in ℓ^2, so Y is hyponormal. □

Corollary 1.2. *If $M \in B(H)$ satisfies $M^*M = MQM^*$ for some positive operator $Q \in B(H)$ such that $Q \geq I$, then M is hyponormal.*

In the next section we apply this theorem and its corollary to lower triangular factorable matrices M acting on ℓ^2.

2. Coposinormal Factorable Matrices with a Diagonal Interrupter

A lower triangular infinite matrix $M = M(\{a_i\}, \{c_j\})$, acting through multiplication to give a bounded linear operator on ℓ^2, is *factorable* if its nonzero entries m_{ij} satisfy $m_{ij} = a_i c_j$ where a_i depends only on i and c_j depends only on j; if $c_j = 1$ for all j, then M is *terraced*. Throughout this section we assume that $M \in B(\ell^2)$ is a lower triangular factorable matrix.

Theorem 2.1. *Suppose M is a lower triangular factorable matrix that acts as a bounded operator on ℓ^2 and the following conditions are satisfied:*
(1) both $\{a_n\}$ and $\{\frac{a_n}{c_n}\}$ are positive decreasing sequences that converge to 0;
(2) the matrix $T = [t_{mn}]$ by

$$t_{mn} = \begin{cases} \frac{a_m}{a_n} & if \quad n = 0; \\ a_m\left(\frac{1}{a_n} - \frac{c_{n-1}}{c_n}\frac{1}{a_{n-1}}\right) & if \quad 0 < n \leq m; \\ -\frac{c_{n-1}}{c_n} & if \quad n = m+1; \\ 0 & if \quad n > m+1. \end{cases}$$

is a bounded operator on ℓ^2;
(3) the sequence $\{-\frac{c_{n-1}a_{n-1}}{c_n} + [\frac{1}{a_n} - \frac{c_{n-1}}{c_n}\frac{1}{a_{n-1}}]\sum_{k=n}^{\infty} a_k^2\}$ is constantly 0; and
(4) the sequence $\{c_n a_n\}$ is nonincreasing.
Then M is coposinormal with a diagonal interrupter, and furthermore, M is hyponormal.

Proof. As in [4, Theorems 3, 4], we have $M = TM^*$, so we compute the interrupter $Q = T^*T$; the entries of $Q = [q_{mn}]$ are given by

$$q_{mn} = \begin{cases} \frac{1}{a_0^2}\sum_{k=0}^{\infty} a_k^2 & if\ m=n=0; \\ \frac{c_{n-1}^2}{c_n^2} + (\frac{1}{a_n} - \frac{c_{n-1}}{c_n}\frac{1}{a_{n-1}})^2 \sum_{k=n}^{\infty} a_k^2 & if\ m=n\geq 1; \\ \frac{1}{a_0}(-\frac{c_{m-1}a_{m-1}}{c_m} + [\frac{1}{a_m} - \frac{c_{m-1}}{c_m}\frac{1}{a_{m-1}}]\sum_{k=m}^{\infty} a_k^2) & if\ m>n=0; \\ (\frac{1}{a_n} - \frac{c_{n-1}}{c_n}\frac{1}{a_{n-1}})(-\frac{c_{m-1}a_{m-1}}{c_m} + [\frac{1}{a_m} - \frac{c_{m-1}}{c_m}\frac{1}{a_{m-1}}]\sum_{k=m}^{\infty} a_k^2) & if\ m>n>0; \\ \frac{1}{a_0}(-\frac{c_{n-1}a_{n-1}}{c_n} + [\frac{1}{a_n} - \frac{c_{n-1}}{c_n}\frac{1}{a_{n-1}}]\sum_{k=n}^{\infty} a_k^2) & if\ n>m=0; \\ (\frac{1}{a_m} - \frac{c_{m-1}}{c_m}\frac{1}{a_{m-1}})(-\frac{c_{n-1}a_{n-1}}{c_n} + [\frac{1}{a_n} - \frac{c_{n-1}}{c_n}\frac{1}{a_{n-1}}]\sum_{k=n}^{\infty} a_k^2) & if\ n>m>0. \end{cases}$$

Inspection of the entries reveals that, because condition (3) is satisfied, Q will be a diagonal matrix; Q has the representation

$$Q = diag\left\{ \frac{\sum_{k=0}^{\infty} a_k^2}{a_0^2}, \frac{c_0a_0}{c_1a_1}, \frac{c_1a_1}{c_2a_2}, \ldots, \frac{c_{n-1}a_{n-1}}{c_na_n}, \ldots \right\}.$$

Condition (4) then guarantees that $Q \geq I$, and it follows from Corollary 1.2 that M is hyponormal. $\square$

Example 2.2. (Toeplitz Matrix) (a) Suppose M is the factorable matrix with entries $m_{ij} = a_ic_j$ where $a_i = r^i$, $c_j = \frac{1}{r^j}$ for all i, j where $0 < r < 1$. If

$$Q :\equiv diag\{\frac{1}{1-r^2}, 1, 1, 1, 1, \ldots\},$$

then $MQM^* = M^*M$ is satisfied with $Q \geq I$, so M is hyponormal by Theorem 2.1.

(b) If $D :\equiv diag\{d_n : n \geq 0\}$ where $\frac{1}{1-r^2} \geq d_0 \geq 1$ and $d_n = 1$ for $n \geq 1$, then $\sqrt{D}M\sqrt{D}$ is also a hyponormal factorable matrix by Theorem 1.1.

Example 2.3. (a) For $k > 0$, let M denote the factorable matrix with entries $m_{ij} = a_ic_j$ where $a_i = \frac{1}{\sqrt{(i+k)(i+k+1)}}$ and $c_j = \frac{\sqrt{j+k}}{\sqrt{j+k+1}}$ for all i, j. If

$$Q :\equiv diag\{k+1, \frac{k+2}{k+1}, \frac{k+3}{k+2}, \ldots \frac{k+n+2}{k+n+1}, \ldots\},$$

then $MQM^* = M^*M$ is satisfied with $Q \geq I$, so M is hyponormal.

(b) If $D :\equiv diag\{d_n : n \geq 0\}$ where $k+1 \geq d_0 \geq 1$ and $\frac{k+n+1}{k+n} \geq d_n \geq 1$ for $n \geq 1$, then $\sqrt{D}M\sqrt{D}$ is also a hyponormal factorable matrix.

Example 2.4. (Recursion) (a) We consider the terraced matrix M determined as follows: Choose a_0, a_1 such that $0 < a_0 < 1$ and $0 < a_1 \leq a_0(1-a_0)$; then define $b :\equiv \frac{(a_0-a_1)}{a_0^2}$ and $a_{i+1} = a_i(1-ba_i)$ for each $i \geq 1$. We note that $a_i \leq \frac{1}{i+1}$ for each i. Since $\{a_i\}$ is decreasing, we have $Q :\equiv diag\{\frac{1}{ba_0}, \frac{a_0}{a_1}, \frac{a_1}{a_2}, \dots\} \geq I$; finally, since $MQM^* = M^*M$ is satisfied, it follows that M is a hyponormal operator on ℓ^2. We note that in the special case when $a_1 :\equiv a_0(1-a_0)$, we have $b = 1$, so we obtain the *logistic sequence* $a_{i+1} = a_i(1-a_i)$, all i.

(b) If $D :\equiv diag\{d_n : n \geq 0\}$ where $\frac{1}{ba_0} \geq d_0 \geq 1$ and $\frac{a_{n-1}}{a_n} \geq d_n \geq 1$ for $n \geq 1$, then $\sqrt{D}M\sqrt{D}$ is another hyponormal factorable matrix.

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