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ON THE q-DERIVATIVES OF A NEW SEQUENCE OF OPERATORS

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In this paper we obtain moment estimates for a new sequence of q-operators very recently introduced by Aral and Gupta [1]. We obtain degree of approximation by the q-derivatives of these operators. We show that for a fixed q, these operators do not possess simultaneous approximation properties.

1. Introduction

The q-Bernstein polynomials

$$B_{n,q}(f,x) = \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right) p_{n,k}(q;x),\ f \in C[0,1],$$

where $p_{n,k}(q;x) = \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{r=0}^{n-k-1}(1-q^r x)$ proposed by Phillips [19] have been extensively studied by several authors (cf.[12], [19]-[25]). Since the summation type operators are not suitable to approximate Lebesgue integrable functions, two modifications of the Bernstein polynomials were given by Durrmeyer [6], and Kantorovich[15]. In a similar way the q-Bernstein polynomials have been modified by Derriennic [4] which is q-analogue of the Durrmeyer operators [6]. Subsequently, q-analogue of some well known positive linear operators e.g.

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Meyer-König and Zeller, Baskakov and Szász operators, based on q-integers have been introduced and studied by several authors ([2], [5], [9]-[11], [16]). Very recently, Aral and Gupta [1] introduced a Durrmeyer type generalization of q-Baskakov type operators as follows

$$\mathcal{M}_n(f,q,x) = [n-1]_q \sum_{k=0}^{\infty} p_{n,k}(q;x) \int_0^{\infty/A} p_{n,k}(q;t) f(t)\, d_q t,$$

where $p_{n,k}(q;x) = \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q \frac{q^{k^2/2} x^k}{(1+x)_q^{n+k}}$.

In [1] it has been observed that these operators satisfy the conditions of the Bohmann-Korovkin theorem for a fixed q in $(0,1)$. Therefore, they converge to any arbitrary real function defined on $[0,\infty)$. We shall show that the q-derivatives of the operators do not converge to the corresponding q-derivatives of the function for a fixed q. Let $C_B[0,\infty)$ be the space of all real valued continuous and bounded functions on $[0,\infty)$. The space $C_B[0,\infty)$ is endowed with the uniform norm $\|f\| = \sup\{|f(x)| : x \in [0,\infty)\}$. By

$$\omega(f,\delta) = \sup_{0<h\leqslant\delta} \sup_{x\in[0,\infty)} |f(x+h) - f(x)|$$

we denote the usual modulus of continuity of $f \in C_B[0,\infty)$. We use the notation $C_\lambda^r[0,\infty)$ for the class of the functions f for which $D_q^i f$, $i = 1(1)r$ are continuously differentiable and $D_q^i f(t) = O(t^\lambda)$ as $t \to \infty$ for some $\lambda \geqslant 0$. In what follows, we shall use the notations $\varphi^2(x) = x(1+x)$ and throughout this paper M is a constant different at each occurrence. Moreover, we simply write $[n]$ in stead of $[n]_q$ unless otherwise stated.

2. Moments

Remark 2.1. Applying the product rule for q-differentiation we obtain the relation

$$q^k \varphi^2(x) D_q[p_{n,k}(q;x)] = \Big([k] - q^k [n] x\Big) p_{n,k}(q;qx), \tag{1}$$

where D_q denotes the q-derivative operator.

Lemma 2.2. *For the functions* $T_{n,m}(x) = \mathcal{M}_n((t-x)^m, q, x)$ *we have* $T_{n,0}(x) = 1$, $T_{n,1}(x) = \frac{[2]x+q}{q^2[n-2]}, n > 2$ *and for* $n > m+2$, *there holds the recurrence relation*

$$\begin{aligned} &q^{m+2}[n-m-2] T_{n,m+1}(qx) \\ &\quad = q\varphi^2(x)\left(D_q T_{n,m}(x) + [m] T_{n,m-1}(qx)\right) \\ &\quad + [m+1](1+2x) T_{n,m}(qx) + [m]\varphi^2(x) T_{n,m-1}(qx). \end{aligned} \tag{2}$$

Proof. The expressions for $T_{n,0}(x)$ and $T_{n,1}(x)$ are immediate from Lemma 2[1]. Now, using product formula for q-differentiation together with (1), we get

$$\begin{aligned}
&D_q(t-x)^m p_{n,k}(q;x)\\
&=-[m](t-x)^{m-1}p_{n,k}(q;qx)+(t-x)^m D_q p_{n,k}(x)\\
&=-[m](t-x)^{m-1}p_{n,k}(q;qx)+(t-qx)^m\frac{([k]-q^k[n]x)}{q^k\varphi^2(x)}p_{n,k}(q;qx)
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\varphi^2(x)\left(D_qT_{n,m}(x)+[m]T_{n,m-1}(qx)\right)\\
&=[n-1]\sum_{k=0}^{\infty}\frac{([k]-q^k[n]t/q)}{q^k}p_{n,k}(q;qx)\int_0^{\infty/A}q^k p_{n,k}(q;t)(t-qx)^m\,d_qt\\
&+[n-1]\sum_{k=0}^{\infty}\frac{(q^k[n]t/q-q^k[n]x)}{q^k}p_{n,k}(q;qx)\int_0^{\infty/A}q^k p_{n,k}(q;t)(t-qx)^m\,d_qt\\
&=E_1+E_2,\ \text{say.}
\end{aligned}$$

We make use of the linear transformation $t\to qu$ which is valid in q-calculus. Thus, we obtain

$$\begin{aligned}
E_1&=[n-1]\sum_{k=0}^{\infty}p_{n,k}(q;qx)\int_0^{\infty/A}\frac{([k]-q^k[n]u)}{q^k}p_{n,k}(q;qu)q^{m+1}(u-x)^m\,d_qu\\
&=[n-1]\sum_{k=0}^{\infty}p_{n,k}(q;qx)\int_0^{\infty/A}\varphi^2(u)\left(D_qp_{n,k}(q;u)\right)q^{m+1}(u-x)^m\,d_qu\\
&=[n-1]\sum_{k=0}^{\infty}p_{n,k}(q;qx)\int_0^{\infty/A}\Big[(u-x)^{m+2}+(1+2x)(u-x)^{m+1}\\
&+\varphi^2(x)(u-x)^m\Big]\left(D_qp_{n,k}(q;u)\right)q^{m+1}\,d_qu\\
&=F_1+F_2+F_3,\ \text{say.}
\end{aligned}$$

Integration by parts and then the transformation $u\to t/u$ gives

$$\begin{aligned}
F_1&=-[m+2][n-1]\sum_{k=0}^{\infty}p_{n,k}(q;qx)\int_0^{\infty/A}q^{m+1}p_{n,k}(q;qu)(u-x)^{m+1}\,d_qu\\
&=[n-1]\sum_{k=0}^{\infty}p_{n,k}(q;qx)q^{m+1}\times
\end{aligned}$$

$$\times \left(p_{n,k}(q;u)(qu-x)^{m+1}\Big|_0^{\infty/A} - \int_0^{\infty/A} p_{n,k}(q;qu)\left(D_q(u-x)^{m+2}\right) d_qu \right)$$

$$= -[m+2]q^{m+1}[n-1]\sum_{k=0}^{\infty} p_{n,k}(q;qx)\int_0^{\infty/A} p_{n,k}(q;qu)(u-x)^{m+1}\, d_qu$$

$$= -[m+2]q^{-1}[n-1]\sum_{k=0}^{\infty} p_{n,k}(q;qx)\int_0^{\infty/A} p_{n,k}(q;t)(t-qx)^{m+1}\, d_qt$$

$$= -[m+2]q^{-1}T_{n,m+1}(qx),\ n > m+1.$$

Similarly, we obtain $F_2 = -[m+1]q^{-1}(1+2x)T_{n,m}(qx)$ and $F_3 = -\varphi^2(x)q[m]$ $q^{-1}T_{n,m-1}(qx)$. Next,

$$E_2 = [n-1]\sum_{k=0}^{\infty} p_{n,k}(q;qx)\int_0^{\infty/A} [n](t/q-x)p_{n,k}(q;t)(t-qx)^m\, d_qt$$
$$= [n]q^{-1}T_{n,m+1}(qx).$$

Combining the estimates $E_1 - E_2$, the proof of the lemma completes. □

Corollary 2.3. *For the functions $T_{n,m}(x)$ we have*
(i) $T_{n,m}(x)$ are polynomials in x of degree exactly m;
(ii) there holds $T_{n,m}(x) = O\big([n]^{-m}\big)$, $\forall x \in [0,\infty)$.

Proof. We write $T_{n,m}(x) = \sum_{i=0}^{m} a_m^i x^i$ and substitute in (2). This gives

$$q^{m+2}[n-m-2]\sum_{i=0}^{m+1} a_{m+1}^i x^i$$
$$= q(x+x^2)\left(D_q\sum_{i=0}^{m} a_m^i x^i + [m]\sum_{i=0}^{m-1} a_{m-1}^i (qx)^i\right)$$
$$+ [m+1](1+2x)\sum_{i=0}^{m} a_m^i (qx)^i + [m](x+x^2)\sum_{i=0}^{m-1} a_{m-1}^i (qx)^i$$
$$= \sum_{i=0}^{m}\Big(q(x+x^2)[i]x^{i-1} + [m+1](1+2x)(qx)^i\Big)a_m^i$$
$$+ \sum_{i=0}^{m-1}[m](x+x^2)(1+q)(qx)^i a_{m-1}^i.$$

It is easily observed that the largest coefficient in $T_{n,m}(x)$ with respect to $[n]$ is the highest power coefficient a_m^m. Therefore, equating the coefficient of the highest power terms, we get

$$q^{m+2}[n-m-2]a_{m+1}^{m+1} = q(1+2q^{m-1}[m+1])a_m^m + q^{m-1}(1+q)[m]a_{m-1}^{m-1}.$$

Now, the proof is easily completed by an induction on m and in view of $a_0^0 = 1$, $a_1^1 = \frac{2}{q^2[n-2]}$. □

Following is a q-analogue of the Lorentz type lemma :

Lemma 2.4. *For the functions $p_{n,k}(q;x)$ there holds*

$$x^r(1+q^{n+k}x)^{(r)}D_q^r\left(p_{n,k}(q;x)\right) = \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} \alpha_j(x)[n]^i\Big([k]-q^k[n]x\Big)^j p_{n,k}(q;x), \quad (3)$$

where $\alpha_i(x)$ are polynomials in x independent of $[n]$.

Proof. Using $q^k\varphi^2(x)p_{n,k}(q;x) = x(1+q^{n+k}x)p_{n,k}(q;qx)$ in (1), we get

$$x(1+q^{n+k}x)D_q p_{n,k}(q;x) = \Big([k]-q^k[n]x\Big)\, p_{n,k}(q;x).$$

The result holds for $r=1$, where $\alpha_0(x)=0$ and $\alpha_1(x)=1$. Suppose (3) is true for a certain r. Then, the product formula for q-differentiation gives

$$\begin{aligned}(qx)^r(1+q^{n+k+1}x)^{(r)}D_q^{r+1}p_{n,k}(q;x) + D_q^r p_{n,k}(q;x)D_q\Big(x^r(1+q^{n+k}x)^{(r)}\Big)\\ = \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}}[n]^i\Big[\alpha_j(qx)\Big([k]-q^{k+1}[n]x\Big)^j D_q p_{n,k}(q;x)\\ + p_{n,k}(q;x)D_q\Big(\alpha_j(x)\Big([k]-q^k[n]x\Big)^j\Big)\Big]. \quad (4)\end{aligned}$$

We write $\Big([k]-q^{k+1}[n]x\Big)^j = \sum_{s=0}^{j}\binom{j}{s}\Big([k]-q^k[n]x\Big)^s\Big(q^k[n]x(1-q)\Big)^{j-s}$ and use in the first term of the r.h.s. of (4). By rearrangement of the terms the proof follows. □

Lemma 2.5. *For the functions $Q_{i,l}(x)$ defined by*

$$Q_{i,l}(x) = \sum_{k=0}^{\infty}\Big([k]-q^{k+l}[n]x\Big)^i p_{n,k}(q;x),$$

there holds the order $Q_{2i,l}(x) = O\left([n]^{2i}\right)$, where l is fixed positive integer.

Proof. We apply an induction on i. For $i=1$ we get three terms E_1, E_2 and E_3 corresponding to the three terms $[k]^2$, $-2[k]q^{k+l}[n]x$ and $q^{2k+2l}[n]^2x^2$ respectively in $\left([k]-q^{k+l}[n]x\right)^2$. Thus, using $[k]=1+q[k-1]$ we get

$$E_1=\sum_{k=0}^{\infty}\frac{q^{k(k+1)/2}x^{k+1}[n+k]!}{[k]![n-1]!(1+x)^{(n+k+1)}}+q\sum_{k=0}^{\infty}\frac{q^{(k+1)(k+2)/2}x^{k+2}}{[k]![n-1]!(1+x)^{(n+k+2)}}[n+k+1]!$$
$$=F_1+F_2,\text{ say.}$$

Now, in view of the identity $\sum_{k=0}^{\infty}q^{k(k-1)/2}\begin{bmatrix}n+k\\k\end{bmatrix}_q\frac{z^k}{(1+z)^{n+k+1}}=1$ and $0<q<1$, we get

$$|F_1|=\left|\frac{[n]x}{1+x}\sum_{k=0}^{\infty}q^{k(k-1)/2}\begin{bmatrix}n+k\\k\end{bmatrix}_q\frac{(qx)^k}{(1+qx)^{(n+k+1)}}(1+q^{n+k+1}x)\right|$$
$$\leqslant\frac{[n]x}{1+x}(1+x)=[n]x.$$

Similarly,

$$F_2=\frac{q^2x^2}{(1+x)(1+qx)}\sum_{k=0}^{\infty}\frac{q^{k(k-1)/2}(q^2x)^k[n+k+1]!}{[k]![n-1]!(1+q^2x)^{(n+k)}}$$
$$=\frac{q^2x^2[n+1][n]}{(1+x)(1+qx)}\sum_{k=0}^{\infty}\frac{q^{k(k-1)/2}(q^2x)^k[n+k+1]!}{[k]![n+1]!(1+q^2x)^{(n+k+2)}}(1+q^{n+k+2}x)(1+q^{n+k+3}x)$$

so that

$$|F_2|\leqslant\frac{q^2x^2[n+1][n]}{(1+x)(1+qx)}(1+x)^2=\frac{q^2x^2[n+1][n]}{(1+qx)}(1+x).$$

Next,

$$E_2=-2[n]q^lx\sum_{k=0}^{\infty}\frac{q^{k+1}q^{k(k+1)/2}x^k[n+k]!}{[k]![n-1]!(1+x)^{(n+k+1)}}$$
$$=\frac{-2[n]^2q^{l+3/2}x}{(1+x)}\sum_{k=0}^{\infty}q^{k(k-1)/2}\frac{[n+k]!}{[k]![n]!}\frac{(qx)^k}{(1+qx)^{(n+k+1)}}(1+q^{n+k+1}x).$$

Hence, we obtain $|E_2|\leqslant 2[n]^2q^{l+3/2}x$. In a similar way we find

$$E_3=\left(q^l[n]x\right)^2\sum_{k=0}^{\infty}q^{2k}p_{n,k}(q;x)$$
$$=\left(q^l[n]x\right)^2\sum_{k=0}^{\infty}\frac{[n+k-1]!}{[k]![n-1]!}\frac{(q^2x)^k}{(1+q^2x)^{(n+k)}}(1+q^{n+k}x)(1+q^{n+k+1}x).$$

Therefore, $|E_3| \leqslant \frac{(q^l[n]x)^2(1+x)}{(1+qx)}$. Combining these estimates, it follows $Q_{2,l}(x) = O\left([n]^2\right)$. Hence the lemma is true for $i=1$. Suppose it holds for a certain i. Then, by q-differentiation, we get

$$D_qQ_{2i,l}(x) = -[2i][n]q^l\sum_{k=0}^{\infty} q^k\left([k]-q^{k+l}[n]x\right)^{2i-1}p_{n,k}(q;qx) + \sum_{k=0}^{\infty}\frac{\left([k]-q^{k+l}[n]x\right)^{2i}}{q^k\varphi^2(x)}p_{n,k}(q;qx)\left([k]-q^k[n]x\right)$$

Rearrangement of the terms gives

$$\begin{aligned}\sum_{k=0}^{\infty}&\frac{\left([k]-q^{k+l}[n]x\right)^{2i+1}}{q^k}p_{n,k}(q;qx)\\ &= \varphi^2(x)D_qQ_{2i,l}(x)\\ &+\varphi^2(x)[2i][n]q^l\sum_{k=0}^{\infty}q^k\left([k]-q^{k+l}[n]x\right)^{2i-1}p_{n,k}(q;qx)\\ &-[n]x\left(q^l-1\right)\sum_{k=0}^{\infty}\left([k]-q^{k+l}[n]x\right)^{2i}p_{n,k}(q;qx). \qquad (5)\end{aligned}$$

Therefore from the definition of $Q_{2i+1,l}(x)$ and (5) we get

$$\begin{aligned}|Q_{2i+1,l}(x)| &\leqslant \left|\sum_{k=0}^{\infty}\frac{\left([k]-q^{k+l}[n]x\right)^{2i+1}}{q^k}p_{n,k}(q;qx)\right|\\ &\leqslant M[n]^{2i}+M'[n]^{2i+1} = M[n]^{2i+1}.\end{aligned}$$

This completes the proof. □

Lemma 2.6. *For $f \in C_\lambda^r[0,\infty)$, there holds the relation*

$$D_q^r\left(\mathcal{M}_n(f,q,x)\right) = [n-1]\prod_{i=1}^{r}\frac{[n+i-1]}{[n-i]}q^{-\frac{r^2}{2}}\sum_{k=0}^{\infty}p_{n+r,k}(q;x)\times \times\int_0^{\infty/A}p_{n-r,k+r}(q;q^rt)D_q^r\left(f(t)\right)d_qt. \qquad (6)$$

Proof. We prove the lemma by induction on r. For $r=1$ we use

$$D_q\left(p_{n,k}(q;x)\right) = [n]q^k\left(q^{-1/2}p_{n+1,k-1}(q;x)-p_{n+1,k}(q;x)\right).$$

This gives

$$\begin{aligned}
&D_q\left(\mathcal{M}_n(f,q,x)\right)\\
&=[n-1]\sum_{k=0}^{\infty}[n]q^{k-1/2}\left(p_{n+1,k-1}(q;x)-q^{1/2}p_{n+1,k}(q;x)\right)\int_0^{\infty/A}p_{n,k}(q;t)f(t)\,d_qt\\
&=[n-1][n]\sum_{k=0}^{\infty}p_{n+1,k}(q;t)\int_0^{\infty/A}q^{k+1/2}\left(p_{n,k+1}(q;x)-q^{-1/2}p_{n,k}(q;t)\right)f(t)\,d_qt\\
&=-[n-1][n]q^{-1/2}\sum_{k=0}^{\infty}p_{n+1,k}(q;x)\int_0^{\infty/A}\frac{D_q\left(p_{n-1,k+1}(q;t)\right)}{[n-1]}f(t)\,d_qt.
\end{aligned}$$

Integration by parts gives

$$\begin{aligned}
D_q\left(\mathcal{M}_n(f,q,x)\right)&=-[n]q^{-1/2}\sum_{k=0}^{\infty}p_{n+1,k}(q;x)\times\\
&\times\left(p_{n-1,k+1}(q;t)f(t)\Big|_0^{\infty/A}-\int_0^{\infty/A}p_{n-1,k+1}(q;qt)D_q\left(f(t)\right)\,d_qt\right)\\
&=[n]q^{-1/2}\sum_{k=0}^{\infty}p_{n+1,k}(q;x)\int_0^{\infty/A}p_{n-1,k+1}(q;qt)D_q\left(f(t)\right)\,d_qt.
\end{aligned}$$

Hence the result is true for $r=1$. Suppose the result holds for a certain r. Then, by q-differentiation of (6), we get

$$\begin{aligned}
&D_q^{r+1}\left(\mathcal{M}_n(f,q,x)\right)\\
&=[n-1]\prod_{i=1}^{r}\frac{[n+i-1]}{[n-i]}q^{-\frac{r^2}{2}}\sum_{k=0}^{\infty}[n+r]q^k\times\\
&\times\left(q^{-1/2}p_{n+r+1,k-1}(q;x)-p_{n+r+1,k}(q;x)\right)\int_0^{\infty/A}p_{n-r,k+r}(q;q^rt)D_q^r\left(f(t)\right)\,d_qt\\
&=-[n-1]\prod_{i=1}^{r}\frac{[n+i-1]}{[n-i]}\sum_{k=0}^{\infty}q^{k+1/2}p_{n+r+1,k}(q;x)[n+r]\times\\
&\times\int_0^{\infty/A}\left(q^{-1/2}p_{n-r,k+r}(q;t)-p_{n-r,k+r+1}(q;t)\right)D_q^r\left(f(t)\right)\,d_qt
\end{aligned}$$

$$= -[n-1]\prod_{i=1}^{r}\frac{[n+i-1]}{[n-i]}\sum_{k=0}^{\infty} q^{k+1/2} p_{n+r+1,k}(q;x)[n+r]\times$$
$$\times \int_0^{\infty/A} \frac{D_q\left(p_{n-r-1,k+r+1}(q;t)\right)}{[n-r-1]q^{k+r+1}} D_q^r\left(f(t)\right) d_q t.$$

Again integration by parts gives

$$D_q^{r+1}\left(\mathcal{M}_n(f,q,x)\right) = [n-1]\prod_{i=1}^{r+1}\frac{[n+i-1]}{[n-i]} q^{-\frac{(r+1)^2}{2}} \sum_{k=0}^{\infty} p_{n+r+1,k}(q;x)\times$$
$$\times \int_0^{\infty/A} p_{n-r-1,k+r+1}(q;q^{r+1}t) D_q^{r+1}\left(f(t)\right) d_q t.$$

This completes the proof. □

For $f \in C_\lambda^r[0,\infty)$ we define the operators $\overline{\mathcal{M}}_{n,q,r}(f,x)$ as follows

$$\overline{\mathcal{M}}_{n,q,r}(f,x) = \sum_{k=0}^{\infty} p_{n+r,k}(q;x) \int_0^{\infty/A} p_{n-r,k+r}(q;q^r t) f(t)\, d_q t.$$

Let us write $V_{n,m,r}(x)$ for the functions $\overline{\mathcal{M}}_{n,q,r}(t^m,x)$.

Lemma 2.7. *For the functions $V_{n,m,r}(x)$, we have*

$$V_{n,0,r}(x) = \frac{1}{q^{3r/2}[n-r-1]}, V_{n,1,r}(x) = \frac{[n+r]xq^{-1}+q^{r-1}+[r]}{q^{2r-1}\left([n-r]q-[2]\right)} V_{n,0,r}(x)$$

and there holds the recurrence relation

$$q^{2r-1}\left([n-r]q-[m+2]\right) V_{n,m+1,r}(qx)$$
$$= \varphi^2(x) D_q V_{n,m,r}(x) + \left([n+r]x + [m+1]q^{r-1} + [r]\right) V_{n,m,r}(qx). \quad (7)$$

Proof. We have

$$
\begin{aligned}
&\varphi^2(x)D_qV_{n,m,r}(x)+\big([n+r]x+[r]\big)V_{n,m,r}(qx)\\
&=\sum_{k=0}^{\infty}\left(q^{-k}[k]\right)p_{n+r,k}(q;qx)\int_0^{\infty/A}p_{n-r,k+r}(q;q^rt)t^m\,d_qt+[r]V_{n,m,r}(qx)\\
&=\sum_{k=0}^{\infty}p_{n+r,k}(q;qx)\int_0^{\infty/A}q^r\left(\left(q^{-k-r}[k+r]-[n-r]tq^r\right)+[n-r]tq^{2r}\right)\times\\
&\times p_{n-r,k+r}(q;q^rt)t^m\,d_qt\\
&=\sum_{k=0}^{\infty}p_{n+r,k}(q;qx)\int_0^{\infty/A}\left(q^{-k-r}[k+r]-[n-r]u\right)p_{n-r,k+r}(q;u)\left(\frac{u}{q^r}\right)^m d_qu\\
&=\frac{1}{q^{mr}}\sum_{k=0}^{\infty}p_{n+r,k}(q;qx)\int_0^{\infty/A}\varphi^2(u/q)D_q\left(p_{n-r,k+r}(q;u/q)\right)u^m\,d_qu\\
&=A_1+A_2,\ \text{say,}
\end{aligned}
$$

where A_1 and A_2 are the two terms of r.h.s. corresponding to $\varphi^2(u/q)=(u/q)+(u/q)^2$. Using $u\to qy$ we get

$$
\begin{aligned}
A_1&=\frac{q^m}{q^{mr}}\sum_{k=0}^{\infty}p_{n+r,k}(q;qx)\Bigg(p_{n-r,k+r}(q;y)y^{m+1}\Big|_0^{\infty/A}\\
&\quad-[m+1]\int_0^{\infty/A}p_{n-r,k+r}(q;qy)y^m\,d_qy\Bigg)\\
&=\frac{-[m+1]}{q^{mr+1}}\sum_{k=0}^{\infty}p_{n+r,k}(q;qx)q^{mr+r}\int_0^{\infty/A}p_{n-r,k+r}(q;q^rt)t^m\,d_qt\\
&=-[m+1]q^{r-1}V_{n,m,r}(qx).
\end{aligned}
$$

Similarly, $A_2=-[m+2]q^{2r-1}V_{n,m+1,r}(qx)$. Therefore, combining these expressions the relation is established. □

Corollary 2.8. *By simple calculations and the recurrence relation (7) we obtain*

$$V_{n,2,r}(x) = \left[\frac{(xq^{-1}+x^2q^{-2})[n+r]q^{-1}}{q^{4r-2}\left([n-r]q-[2]\right)\left([n-r]q-[3]\right)} + \frac{\left([n+r]xq^{-1}+[r]+[2]q^{r-1}\right)\left([n+r]xq^{-1}+[r]+q^{r-1}\right)}{q^{4r-2}\left([n-r]q-[2]\right)\left([n-r]q-[3]\right)} \right] V_{n,0,r}(x).$$

Using $\overline{\mathcal{M}}_{n,q,r}((t-x)^2,x) = V_{n,2,r}(x) - 2xV_{n,1,r}(x) + x^2V_{n,0,r}(x)$, *and then writing* $\overline{\mathcal{M}}_{n,q,r}((t-x)^2,x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2$ *we get*

$$\alpha_0 = \frac{([2]q^{r-1}+[r])(q^{r-1}+[r])}{q^{4r-2}\left([n-r]q-[2]\right)\left([n-r]q-[3]\right)} V_{n,0,r}(x),$$

$$\alpha_1 = \left[\frac{[n+r]q^{-1}\left(q^{-1}+2[r]+([2]+1)q^{r-1}\right)}{q^{4r-2}\left([n-r]q-[2]\right)\left([n-r]q-[3]\right)} - \frac{2\left(q^{r-1}+[r]\right)}{q^{2r-1}\left([n-r]q-[2]\right)} \right] V_{n,0,r}(x)$$

and $\alpha_2 =$

$$= \left[\frac{q^{-2}[n+r](q^{-1}+[n+r])}{q^{4r-2}\left([n-r]q-[2]\right)\left([n-r]q-[3]\right)} - \frac{2[n+r]q^{-1}}{q^{2r-1}\left([n-r]q-[2]\right)} + 1 \right] V_{n,0,r}(x).$$

In α_1 *and* α_2, *we split* $[n+r]$ *as* $1+[2]+q^{n-r+1}[2r-1]+([n-r]q-[2])$ *and* $1+[3]+q^{n-r+1}[2r-1]+([n-r]q-[3])$. *Hence, we obtain*

$$\alpha_0 \leqslant \frac{M_1(r)}{q^{11r/2}[n-r-1]\left([n-r]q-[2]\right)\left([n-r]q-[3]\right)},$$

$$\alpha_1 \leqslant \frac{M_2(r)}{q^{11r/2}[n-r-1]\left([n-r]q-[2]\right)}$$

and

$$\alpha_2 \leqslant \frac{M_3(r)}{q^{11r/2}[n-r-1]} \left(\left(q^{4r-2}-2q^{2r}+1\right) + \frac{1}{([n-r]q-[2])} \right).$$

Consequently, we get the estimate

$$\overline{\mathcal{M}}_{n,q,r}\left((t-x)^2,x\right) \leqslant \frac{M_4(r)}{q^{11r/2}[n-r-1]} \left(\left(q^{4r-2}-2q^{2r}+1\right) + \frac{1}{([n-r]q-[2])} \right) x^2,$$

where $M_i(r)$ *are the constants independent of* q *and* n.

3. Simultaneous Approximation Using q-Moments

Theorem 3.1. *Let $f \in C_\lambda^r[0,\infty)$ and q_n be a sequence in $(0,1)$ such that $q_n \uparrow 1$. Then, there exists a $\hat{q_n}$ such that the sequence $D_{q_n}^r\big(\mathcal{M}_n(f,q,x)\big)$ converges to $D_{q_n}^r\big(f(x)\big)$ point wise for $q_n \in (\hat{q_n},1)$.*

Proof. There exist $\hat{q_n} \in (0,1)$ (see [7]) such that for all $q_n \in (\hat{q_n},1)$ we can write

$$f(t) = \sum_{l=0}^{r} \frac{(D_{q_n}^l f)(x)}{[k]_{q_n}!}(t-x)^{(l)} + \frac{(D_{q_n}^{r+1} f)(\xi)}{[r+1]_{q_n}!}(t-x)^{(r+1)}.$$

Since, $\mathcal{M}_n((t-x)^m,q,x)$ are polynomials of degree exactly m (see Lemma 2.2) we obtain

$$\begin{aligned}
&D_{q_n}^r\big(\mathcal{M}_n(f,q,x)\big) - D_{q_n}^r\big(f(x)\big)\\
&= \frac{1}{[r+1]_{q_n}!}\mathcal{M}_n^{(r)}\Big((D_{q_n}^{r+1} f)(\xi)(t-x)^{(r+1)},q,x\Big)\\
&= [n-1]_{q_n}\sum_{k=0}^{\infty}\sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} \alpha_j(x)[n]_{q_n}^i\Big([k]_{q_n} - q_n^k[n]x\Big)^j p_{n,k}(q_n;x)\times\\
&\times \int_0^{\infty/A} p_{n,k}(q_n;t)\frac{(D_{q_n}^{r+1} f)(\xi)}{[r+1]_{q_n}!}(t-x)^{(r+1)}\,d_qt.
\end{aligned}$$

Let, $T_{i,j}$ be a typical term of the sum over i,j. Using Hölder's inequality firstly for integration and then for summation we obtain

$$\begin{aligned}
|T_{i,j}| \leqslant M\|D_{q_n}^{r+1} f\|[n-1]_{q_n}\sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}}[n]_{q_n}^i\sum_{k=0}^{\infty}\Big|[k]_{q_n} - q_n^k[n]_{q_n}x\Big|^j p_{n,k}(q_n;x)\times\\
\times\left|\int_0^{\infty/A} p_{n,k}(q_n;t)(t-x)^{(r+1)}\,d_qt\right|
\end{aligned}$$

$$\begin{aligned}
\leqslant M\|D_{q_n}^{r+1} f\|\sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}}[n]_{q_n}^i[n-1]_{q_n}\sum_{k=0}^{\infty}\Big|[k]_{q_n} - q_n^k[n]_{q_n}x\Big|^j p_{n,k}(q_n;x)\times\\
\times\prod_{j=0}^{r}\left(\int_0^{\infty/A} p_{n,k}(q_n;t)|t-q_n^jx|^{r+1}\,d_qt\right)^{\frac{1}{r+1}}
\end{aligned}$$

$$\leqslant M\|D_{q_n}^{r+1}f\| \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} [n]_{q_n}^i \left(\sum_{k=0}^{\infty}\left([k]_{q_n}-q_n^k[n]_{q_n}x\right)^{2j} p_{n,k}(q_n;x)\right)^{1/2} \times$$

$$\times \left([n-1]_{q_n}\sum_{k=0}^{\infty}\prod_{j=0}^{r} p_{n,k}(q_n;x)\left(\int_0^{\infty/A} p_{n,k}(q_n;t)|t-q_n^j x|^{r+1}\,d_qt\right)^{2/r+1}\right)^{1/2}$$

$$\leqslant M\|D_{q_n}^{r+1}f\| \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} [n]_{q_n}^i \left(\sum_{k=0}^{\infty}([k]_{q_n}-q_n^{k+s}[n]_{q_n}x)^{2j} p_{n,k}(q_n;x)\right)^{1/2} \times$$

$$\times \left(\prod_{j=0}^{r}\left([n-1]_{q_n}\sum_{k=0}^{\infty} p_{n,k}(q_n;x)\int_0^{\infty/A} p_{n,k}(q_n;t)(t-q_n^j x)^{2r+2}\,d_qt\right)^{\frac{1}{r+1}}\right)^{1/2}.$$

Using Lemma 2.2, we get

$$\left|\mathcal{M}_n((t-x)^{2r+2},q,x)\right| \leqslant O\left(\frac{1}{[n]_{q_n}}\right)^{r+1}.$$

Therefore, using equality $(t-q_n^j x)^{2r+2}=\sum_{l=0}^{2r+2}\binom{2r+2}{l}(t-x)^l\left(x(1-q_n^j)\right)^{2r+2-l}$ and Hölder's inequality we obtain

$$\left|\mathcal{M}_n((t-q_n^j x)^{2r+2},q_n,x)\right| \leqslant \sum_{l=0}^{2r+2}\binom{2r+2}{l}\left(x(1-q_n^j)\right)^{2r+2-l}\left(\frac{1}{[n]_{q_n}^{l/2}}\right).$$

Let $(1-q_n^j)=O\left(\frac{1}{[n]_{q_n}^{\rho}}\right)$, $\rho\geqslant 0$ for all j. This implies $(1-q_n^r)=O\left(\frac{1}{[n]_{q_n}^{\rho}}\right)$. Consequently, we get

$$\begin{aligned}
|T_{i,j}| &\leqslant M\|D_{q_n}^{r+1}f\|\times\\
&\times \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}}[n]_{q_n}^{i+j}\left(\prod_{j=0}^{r}\left(\sum_{l=0}^{2r+2}\binom{2r+2}{l}\left(x(1-q_n^j)\right)^{2r+2-l}\left(\frac{1}{[n]_{q_n}^{l/2}}\right)\right)^{\frac{1}{r+1}}\right)^{1/2}\\
&\leqslant M\|D_{q_n}^{r+1}f\|\sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}}[n]_{q_n}^{i+j}\left(\prod_{j=0}^{r}\left(\frac{[n]_{q_n}^{(2r+2)(\rho-1/2)}}{[n]_{q_n}^{(2r+2)\rho}}\right)^{\frac{1}{r+1}}\right)^{1/2}\\
&\leqslant M\|D_{q_n}^{r+1}f\|\frac{1}{[n]_{q_n}^{1/2}}.
\end{aligned}$$

Since, this inequality is independent of ρ it follows that $D_{q_n}^r\left(\mathcal{M}_n(f,q_n,x)\right)$ converges point wise to $D_{q_n}^r\left(f(x)\right)$ as $n\to\infty$ for $q_n\in(\hat{q_n},1)$. □

Theorem 3.2. *For $f\in C_\lambda^r[0,\infty)$, we have*

$$\left|\frac{q^{r(r+3)/2}[n-r-1]}{[n-1]}\prod_{j=1}^{r}\left(\frac{[n-j]}{[n+j-1]}\right)D_q^r\left(\mathcal{M}_n(f,q,x)\right)-D_q^r\left(f(x)\right)\right|$$
$$\leqslant M(r)[n-r-1]^{1/2}\times$$
$$\times\left(\omega D_q^r f,\sqrt{\frac{x^2}{q^{5r/4}}\left((q^{4r-2}-2q^{2r}+1)+\frac{1}{([n-r]q-[2])}\right)}\right),$$

where M is independent of f, q and n.

Proof. In view of

$$q^{3r/2}[n-r-1]\sum_{k=0}^{\infty}p_{n+r,k}(q;x)\int_0^{\infty/A}p_{n-r,k+r}(q;q^rt)\,d_qt=1,$$

we obtain

$$\left|\frac{q^{r(r+3)/2}[n-r-1]}{[n-1]}\prod_{j=1}^{r}\left(\frac{[n-j]}{[n+j-1]}\right)D_q^r\left(\mathcal{M}_n(f,q,x)\right)-D_q^r\left(f(x)\right)\right|$$
$$\leqslant q^{r(r+1)/2}[n-r-1]\sum_{k=0}^{\infty}p_{n+r,k}(q;x)\int_0^{\infty/A}p_{n-r,k+r}(q;q^rt)\left|D_q^rf(t)-D_q^rf(x)\right|d_qt$$
$$\leqslant q^{3r/2}[n-r-1]\,\omega\left(D_q^rf,\delta\right)\sum_{k=0}^{\infty}p_{n+r,k}(q;x)\times$$
$$\times\int_0^{\infty/A}p_{n-r,k+r}(q;q^rt)\left(1+\frac{|t-x|}{\delta}\right)d_qt=K_1+K_2,\text{ say.}$$

We have $K_1=\omega\left(D_q^rf,\delta\right)$. Using Schwarz's inequality and Corollary 2.8, we get

$$K_2\leqslant\omega\left(D_q^rf,\delta\right)\frac{q^{3r/2}[n-r-1]}{\delta}\left(\sum_{k=0}^{\infty}p_{n+r,k}(q;x)\int_0^{\infty/A}p_{n-r,k+r}(q;q^rt)(t-x)^2\,d_qt\right)^{\frac{1}{2}}$$
$$\leqslant M_5(r)\omega\left(D_q^rf,\delta\right)\frac{\sqrt{[n-r-1]}}{q^{5r/4}\delta}\sqrt{\left((q^{4r-2}-2q^{2r}+1)+\frac{1}{([n-r]q-[2])}\right)x^2}.$$

Finally, choosing $\delta^2 = \frac{x^2}{q^{5r/2}}\left(\left(q^{4r-2}-2q^{2r}+1\right)+\frac{1}{([n-r]q-[2])}\right)$ it follows that

$$\left|\frac{q^{3r/2}[n-r-1]}{[n-1]}\prod_{j=1}^{r}\left(\frac{[n-j]}{[n+j-1]}\right)D_q^r\left(\mathcal{M}_n(f,q,x)\right)-D_q^r\left(f(x)\right)\right|$$
$$\leqslant M(r)[n-r-1]^{1/2}\times$$
$$\times\left(\omega D_q^r f,\sqrt{\frac{x^2}{q^{5r/4}}\left((q^{4r-2}-2q^{2r}+1)+\frac{1}{([n-r]q-[2])}\right)}\right).$$

This completes the proof. □

Remark 3.3. In Theorem 3.1 and Theorem 3.2 we observe that if we take q in place of q_n i.e. for a fixed q the sequence $D_q^r\left(\mathcal{M}_n(f,q,x)\right)$ does not converge to $D_q^r(f(x))$ point wise as $n\to\infty$. In Theorem 3.2 if we take the sequence (q_n) in $(0,1)$ such that $q_n\uparrow 1$ as $n\to\infty$ with the rate $1-q_n=O(1/[n]^{\rho})$, $\rho>0$, then it follows that $\frac{(1+q^{4r-2}-2q^{2r})}{q^{5r/4}}=O(1/[n]^{\rho})$. Consequently, we obtain

$$\left|\frac{q^{3r/2}[n-r-1]}{[n-1]}\prod_{j=1}^{r}\left(\frac{[n-j]}{[n+j-1]}\right)D_q^r\left(\mathcal{M}_n(f,q,x)\right)-D_q^r\left(f(x)\right)\right|$$
$$\leqslant M(r)[n-r-1]^{1/2}\omega\left(D_q^r f,\sqrt{\left(\frac{1}{[n]^{\rho-1}}+\frac{1}{[n]}\right)x^2}\right).$$

Hence, we conclude that for a function $f\in C_\lambda^r[0,\infty)$ such that $D_q^r f$ is uniformly continuous on $[0,\infty)$, r.h.s. tends to zero if $\rho>2$.

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