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FIXED POINTS OF WEAKLY COMPATIBLE MAPPINGS USING COMMON (E. A) LIKE PROPERTY

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The aim of this paper is to prove common fixed point theorems in Menger spaces using implicit relation and common (E.A) like property. An example is derived to support our main result. We extend our result to four finite families of self mappings. As an application of our main result, we prove an integral type common fixed point theorem satisfying ψ-contraction condition in Menger space. Our results improve some recent results in Menger spaces.

1. Introduction

In 1991, Mishra [23] introduced the notion of compatible mappings in probabilistic metric spaces (shortly PM-spaces). It is seen that most of the common fixed point theorems for contraction mappings invariably require a compatibility condition besides assuming continuity of at least one of the mappings. Later on, Singh and Jain [34] introduced the notion of weakly compatible mappings and proved fixed point theorems in Menger spaces. In 2008, Kubiaczyk and Sharma [15] proved some common fixed point theorems under strict contractive conditions for weakly compatible mappings satisfying the property (E.A) due to Aamri and El Moutawakil [1]. Further, Ali et al. [2] introduced the notion of common property (E.A) in Menger space and improved

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the results of Kubiaczyk and Sharma [15]. Subsequently, there are a number of results which contained the notions of property (E.A) and common property (E.A) in Menger spaces (see [3, 10, 11]). Inspired by Sintunavarat and Kumam [35, 38], Wadhwa et al. [41] defined the notion of (E.A) like property and common (E.A) like property in fuzzy metric spaces and improved the results of Kumar [16] as the conditions on containment of ranges amongst the involved mappings and closedness of the underlying subspaces are completely relaxed. Many mathematicians proved various fixed point theorems in Menger spaces (see [8, 12, 21, 22, 24, 25, 27, 28, 30, 33]).

In fixed point theory, implicit relations are used to find the common fixed point of the involved mappings. Subsequently several authors (see [6, 17–19, 26, 32]) have studied existence of fixed points in Menger spaces satisfying implicit relations. Recently, Imdad et al. [13] defined a class of implicit relations for the existence of fixed points in Menger spaces.

In 2002, Branciari [7] obtained a fixed point result for a mapping satisfying an integral analogue of Banach contraction principle. Many authors proved a host of fixed point theorems involving relatively more general integral type contractive conditions (see [4, 9, 29, 36, 37, 39, 40]).

In this paper, we prove a common fixed point theorem for two pairs of weakly compatible mappings using implicit relation and common (E.A) like property. An illustrative example is furnished to support our main result. We also prove a fixed point theorem for six self mappings in Menger spaces by using the notion of pairwise commuting. As an application to our result, we present an integral type fixed point theorem in Menger space.

2. Preliminaries

Definition 2.1. [31] A mapping $\triangle : [0,1] \times [0,1] \to [0,1]$ is t-norm if $\triangle$ is satisfying the following conditions:

1. $\triangle$ is commutative and associative;

2. $\triangle(a,1) = a$ for all $a \in [0,1]$;

3. $\triangle(a,b) \leq \triangle(c,d)$ whenever $a \leq c$ and $b \leq d$, for all $a,b,c,d \in [0,1]$.

Example 2.2. [31] The following are the four basic t-norms.

1. The minimum t-norm: $\triangle_M(a,b) = \min\{a,b\}$;

2. The product t-norm: $\triangle_P(a,b) = ab$;

3. The Lukasiewicz t-norm: $\triangle_L(a,b) = \max\{a+b-1,0\}$;

4. The weakest t-norm, the drastic product:

$$\triangle_D(a,b) = \begin{cases} \min\{a,b\}, & \text{if } \max\{a,b\} = 1; \\ 0, & \text{otherwise.} \end{cases}$$

In respect of above mentioned t-norms, we have the following ordering:

$$\triangle_D(a,b) < \triangle_L(a,b) < \triangle_P(a,b) < \triangle_M(a,b).$$

Throughout this paper, $\triangle$ stands for an arbitrary continuous t-norm.

Definition 2.3. [31] A mapping $F : \mathbb{R} \to \mathbb{R}^+$ is called a distribution function if it is non-decreasing and left continuous with $\inf\{F(t) : t \in \mathbb{R}\} = 0$ and $\sup\{F(t) : t \in \mathbb{R}\} = 1$.

We shall denote by $\Im$ the set of all distribution functions defined on $(-\infty, \infty)$ while $H(t)$ will always denote the specific distribution function defined by

$$H(t) = \begin{cases} 0, & \text{if } t \leq 0; \\ 1, & \text{if } t > 0. \end{cases}$$

If X is a non-empty set, $\mathcal{F} : X \times X \to \Im$ is called a probabilistic distance on X and the value of $\mathcal{F}$ at $(x,y) \in X \times X$ is represented by $F_{x,y}$.

Definition 2.4. [20] A PM-space is an ordered pair $(X, \mathcal{F})$, where X is a non-empty set of elements and $\mathcal{F}$ is a probabilistic distance satisfying the following conditions: for all $x,y,z \in X$ and $t,s > 0$,

1. $F_{x,y}(t) = H(t)$ for all $t > 0$ if and only $x = y$;
2. $F_{x,y}(t) = F_{y,x}(t)$;
3. if $F_{x,y}(t) = 1$ and $F_{y,z}(s) = 1$ then $F_{x,z}(t+s) = 1$.

Every metric space (X,d) can always be realized as a probabilistic metric space by considering $\mathcal{F} : X \times X \to \Im$ defined by $F_{x,y}(t) = H(t - d(x,y))$ for all $x,y \in X$. So probabilistic metric spaces offer a wider framework than that of metric spaces and are better suited to cover even wider statistical situations.

Definition 2.5. [31] A Menger space $(X, \mathcal{F}, \triangle)$ is a triplet where $(X, \mathcal{F})$ is a PM-space and $\triangle$ is a t-norm satisfying the following condition:

$$F_{x,y}(t+s) \geq \triangle(F_{x,z}(t), F_{z,y}(s)),$$

for all $x,y,z \in X$ and $t,s > 0$.

Definition 2.6. [15] A pair (A,S) of self mappings of a Menger space $(X,\mathcal{F},\triangle)$ is said to satisfy the property (E.A) if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z,$$

for some $z \in X$.

Definition 2.7. [3] Two pairs (A,S) and (B,T) of self mappings of a Menger space $(X,\mathcal{F},\triangle)$ are said to satisfy the common property (E.A) if there exist two sequences $\{x_n\}$, $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z,$$

for some $z \in X$.

Inspired by Wadhwa et al. [41], we adopt the following:

Definition 2.8. A pair (A,S) of self mappings of a Menger space $(X,\mathcal{F},\triangle)$ is said to satisfy the (E.A) like property if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z,$$

for some $z \in A(X)$ or $z \in S(X)$.

Example 2.9. Let $X = [0,2]$ with the usual metric d, that is, $d(x,y) = |x-y|$ and for each $t \in [0,1]$ define

$$F_{x,y}(t) = \begin{cases} \dfrac{t}{t+|x-y|}, & \text{if } t > 0; \\ 0, & \text{if } t = 0. \end{cases}$$

for all $x,y \in X$. Then $(X,\mathcal{F},\triangle)$ be a Menger space, where $\triangle$ is a continuous t-norm. Let A,B,S and T be self mappings of X defined by

$$A(X) = \begin{cases} 1, & \text{if } x \in [0,1]; \\ \frac{3-x}{2}, & \text{if } x \in (1,2]. \end{cases} \quad S(X) = \begin{cases} 1, & \text{if } x \in [0,1]; \\ 2-x, & \text{if } x \in (1,2]. \end{cases}$$

Then we get $A(X) = \left[\frac{1}{2}, 1\right]$ and $S(X) = [0,1]$. Consider a sequence $\{x_n\} = \{1+\frac{1}{n}\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} \left\{1 - \frac{1}{2n}\right\} = 1 \in S(X),$$

and

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} \left\{1 - \frac{1}{n}\right\} = 1 \in A(X).$$

Hence, the pair (A,S) satisfies the (E.A) like property.

Example 2.10. In the setting of Example 2.9, replace the self mappings A and S by the following, besides retaining the rest:

$$A(X)=\begin{cases} 2, & \text{if } x\in[0,1];\\ \frac{3-x}{2}, & \text{if } x\in(1,2]. \end{cases} \quad S(X)=\begin{cases} 0, & \text{if } x\in[0,1];\\ 2-x, & \text{if } x\in(1,2]. \end{cases}$$

Then we get $A(X)=\left[\frac{1}{2},1\right)\cup\{2\}$ and $S(X)=[0,1)$. Define a sequence $\{x_n\}=\{1+\frac{1}{n}\}$ in X, we have

$$\lim_{n\to\infty} Ax_n=\lim_{n\to\infty} A\left(1+\frac{1}{n}\right)=\lim_{n\to\infty}\left\{1-\frac{1}{2n}\right\}=1\in X,$$

and

$$\lim_{n\to\infty} Sx_n=\lim_{n\to\infty} S\left(1+\frac{1}{n}\right)=\lim_{n\to\infty}\left\{1-\frac{1}{n}\right\}=1\in X.$$

Here it is noted that 1 does not belong to $A(X)$ and $S(X)$. But the pair (A,S) satisfies the property (E.A).

From the Examples 2.9-2.10, it is evident that a pair (A,S) satisfying the (E.A) like property always enjoys the property (E.A) but the implication is not reversible.

Definition 2.11. Two pairs (A,S) and (B,T) of self mappings of a Menger space $(X,\mathcal{F},\triangle)$ are said to satisfy the common (E.A) like property if there exist two sequences $\{x_n\}$, $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n=\lim_{n\to\infty} Sx_n=\lim_{n\to\infty} By_n=\lim_{n\to\infty} Ty_n=z,$$

where $z\in S(X)\cap T(X)$ or $z\in A(X)\cap B(X)$.

Definition 2.12. [14] A pair (A,S) of self mappings of a non-empty set X is said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, that is, if $Ax=Sx$ for some $x\in X$, then $ASx=SAx$.

Definition 2.13. [12] Two families of self mappings $\{A_i\}_{i=1}^{m}$ and $\{S_k\}_{k=1}^{n}$ are said to be pairwise commuting if

1. $A_iA_j=A_jA_i$, $i,j\in\{1,2,\ldots,m\}$,
2. $S_kS_l=S_lS_k$, $k,l\in\{1,2,\ldots,n\}$,
3. $A_iS_k=S_kA_i$, $i\in\{1,2,\ldots,m\}$, $k\in\{1,2,\ldots,n\}$.

Lemma 2.14. [23] *Let $(X,\mathcal{F},\triangle)$ be a Menger space. If there exists a constant $k\in(0,1)$ such that*

$$F_{x,y}(kt)\geq F_{x,y}(t),$$

for all $t>0$ with fixed $x,y\in X$ then $x=y$.

3. Implicit Relation

Following Imdad et al. [13], let Θ be the set of all continuous functions $\varphi(t_1,t_2,t_3,t_4,t_5,t_6) : [0,1]^6 \to \mathbb{R}$ satisfying the following conditions:

(φ_1) $\varphi(u,1,u,1,1,u) < 0$, for all $u \in (0,1)$,

(φ_2) $\varphi(u,1,1,u,u,1) < 0$, for all $u \in (0,1)$,

(φ_3) $\varphi(u,u,1,1,u,u) < 0$, for all $u \in (0,1)$.

Example 3.1. [13] Define $\varphi(t_1,t_2,t_3,t_4,t_5,t_6) : [0,1]^6 \to \mathbb{R}$ as

$$\varphi(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - \psi(\min\{t_2,t_3,t_4,t_5,t_6\}), \tag{1}$$

where $\psi : [0,1] \to [0,1]$ is increasing and continuous function such that $\psi(t) > t$ for all $t \in (0,1)$. Notice that

(φ_1) $\varphi(u,1,u,1,1,u) = u - \psi(u) < 0$, for all $u \in (0,1)$,

(φ_2) $\varphi(u,1,1,u,u,1) = u - \psi(u) < 0$, for all $u \in (0,1)$,

(φ_3) $\varphi(u,u,1,1,u,u) = u - \psi(u) < 0$, for all $u \in (0,1)$.

Example 3.2. [13] Define $\varphi(t_1,t_2,t_3,t_4,t_5,t_6) : [0,1]^6 \to \mathbb{R}$ as

$$\varphi(t_1,t_2,t_3,t_4,t_5,t_6) = \int_0^{t_1} \phi(t)dt - \psi\left(\int_0^{\min\{t_2,t_3,t_4,t_5,t_6\}} \phi(t)dt\right), \tag{2}$$

where $\psi : [0,1] \to [0,1]$ is increasing and continuous function such that $\psi(t) > t$ for all $t \in (0,1)$ and $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesgue integrable function which is summable and satisfies

$$0 < \int_0^{\varepsilon} \phi(s)ds < 1, \text{ for all } 0 < \varepsilon < 1, \int_0^1 \phi(s)ds = 1.$$

We observe that

(φ_1) $\varphi(u,1,u,1,1,u) = \int_0^u \phi(t)dt - \psi\left(\int_0^u \phi(t)dt\right) < 0$, for all $u \in (0,1)$,

(φ_2) $\varphi(u,1,1,u,u,1) = \int_0^u \phi(t)dt - \psi\left(\int_0^u \phi(t)dt\right) < 0$, for all $u \in (0,1)$,

(φ_3) $\varphi(u,u,1,1,u,u) = \int_0^u \phi(t)dt - \psi\left(\int_0^u \phi(t)dt\right) < 0$, for all $u \in (0,1)$.

Example 3.3. [13] Define $\varphi(t_1,t_2,t_3,t_4,t_5,t_6) : [0,1]^6 \to \mathbb{R}$ as

$$\begin{aligned}\varphi(t_1,t_2,t_3,t_4,t_5,t_6) &= \int_0^{t_1}\phi(t)dt - \psi(\min\{\int_0^{t_2}\phi(t)dt, \int_0^{t_3}\phi(t)dt, \\ &\quad \int_0^{t_4}\phi(t)dt, \int_0^{t_5}\phi(t)dt, \int_0^{t_6}\phi(t)dt\}), \qquad (3)\end{aligned}$$

where $\psi : [0,1] \to [0,1]$ is increasing and continuous function such that $\psi(t) > t$ for all $t \in (0,1)$ and $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesgue integrable function which is summable and satisfies

$$0 < \int_0^{\varepsilon}\phi(s)ds < 1, \text{ for all } 0 < \varepsilon < 1, \int_0^1 \phi(s)ds = 1.$$

We observe that

(φ_1) $\varphi(u,1,u,1,1,u) = \int_0^u \phi(t)dt - \psi\left(\int_0^u \phi(t)dt\right) < 0$, for all $u \in (0,1)$,

(φ_2) $\varphi(u,1,1,u,u,1) = \int_0^u \phi(t)dt - \psi\left(\int_0^u \phi(t)dt\right) < 0$, for all $u \in (0,1)$,

(φ_3) $\varphi(u,u,1,1,u,u) = \int_0^u \phi(t)dt - \psi\left(\int_0^u \phi(t)dt\right) < 0$, for all $u \in (0,1)$.

4. Results

Theorem 4.1. *Let A, B, S and T be self mappings of a Menger space $(X, \mathcal{F}, \triangle)$ satisfying the following conditions:*

1. *the pairs (A,S) and (B,T) share the common (E.A) like property;*
2. *there exists $\varphi \in \Theta$ such that*

$$\varphi(F_{Ax,By}(t), F_{Sx,Ty}(t), F_{Ax,Sx}(t), F_{By,Ty}(t), F_{By,Sx}(t), F_{Ax,Ty}(t)) \geq 0, \qquad (4)$$

for all $x, y \in X$ and $t > 0$. Then the pairs (A,S) and (B,T) have a coincidence point each. Moreover, A, B, S and T have a unique common fixed point if both the pairs (A,S) and (B,T) are weakly compatible.

Proof. Since (A,S) and (B,T) share the common (E.A) like property, there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z,$$

where $z \in S(X) \cap T(X)$ or $z \in A(X) \cap B(X)$.

Case I Suppose that $z \in S(X) \cap T(X)$. Since $z \in S(X)$, we have $\lim_{n\to\infty} Ax_n = z \in S(X)$, that is, $z = Su$ for some $u \in X$. Now we assert that $Au = Su$. Suppose that $Au \neq Su$, then using inequality (4) with $x = u$, $y = y_n$, we get

$$\varphi\left(\begin{array}{c} F_{Au,By_n}(t), F_{Su,Ty_n}(t), F_{Au,Su}(t), \\ F_{By_n,Ty_n}(t), F_{By_n,Su}(t), F_{Au,Ty_n}(t) \end{array}\right) \geq 0.$$

Taking limit as $n \to \infty$, we obtain

$$\varphi\left(F_{Au,z}(t), F_{z,z}(t), F_{Au,z}(t), F_{z,z}(t), F_{z,z}(t), F_{Au,z}(t)\right) \geq 0,$$

and so

$$\varphi\left(F_{Au,z}(kt), 1, F_{Au,z}(t), 1, 1, F_{Au,z}(t)\right) \geq 0,$$

which contradicts (φ_1). Hence $Au = z = Su$ which shows that u is a coincidence point of the pair (A, S).

Since $z \in T(X)$, we have $\lim_{n\to\infty} By_n = z \in T(X)$, that is, $z = Tv$ for some $v \in X$. We show that $Bv = Tv$. If it is not so, then using inequality (4) with $x = x_n$, $y = v$, we have

$$\varphi\left(\begin{array}{c} F_{Ax_n,Bv}(t), F_{Sx_n,Tv}(t), F_{Ax_n,Sx_n}(t), \\ F_{Bv,Tv}(t), F_{Bv,Sx_n}(t), F_{Ax_n,Tv}(t) \end{array}\right) \geq 0.$$

Taking limit $n \to \infty$, we obtain

$$\varphi\left(F_{z,Bv}(t), F_{z,z}(t), F_{z,z}(t), F_{Bv,z}(t), F_{Bv,z}(t), F_{z,z}(t)\right) \geq 0,$$

or

$$\varphi\left(F_{z,Bv}(t), 1, 1, F_{Bv,z}(t), F_{Bv,z}(t), 1\right) \geq 0,$$

which contradicts (φ_2). Hence $Bv = z = Tv$ which shows that v is a coincidence point of the pair (B, T).

Since the pairs (A, S) and (B, T) are weakly compatible and $Au = Su$, $Bv = Tv$, therefore $Az = ASu = SAu = Sz$ and $Bz = BTv = TBv = Tz$. We assert that z is a common fixed point of the mappings A and S. Let, on the contrary, $z \neq Az$. On using inequality (4) with $x = z$, $y = v$, we get

$$\varphi\left(\begin{array}{c} F_{Az,Bv}(t), F_{Sz,Tv}(t), F_{Az,Sz}(t), \\ F_{Bv,Tv}(t), F_{Bv,Sz}(t), F_{Az,Tv}(t) \end{array}\right) \geq 0,$$

and so

$$\varphi\left(F_{Az,z}(t), F_{Az,z}(t), F_{Az,Az}(t), F_{z,z}(t), F_{z,Az}(t), F_{Az,z}(t)\right) \geq 0,$$

or

$$\varphi\left(F_{Az,z}(t), F_{Az,z}(t), 1, 1, F_{z,Az}(t), F_{Az,z}(t)\right) \geq 0,$$

which contradicts (φ_3). Hence $Az = Sz = z$. Now we show that z is also a common fixed point of the pair (B,T). If $z \neq Bz$, then using inequality (4) with $x = u$, $y = z$, we get

$$\varphi\left(\begin{array}{c} F_{Au,Bz}(t), F_{Su,Tz}(t), F_{Au,Su}(t), \\ F_{Bz,Tz}(t), F_{Bz,Su}(t), F_{Au,Tz}(t) \end{array}\right) \geq 0,$$

and so

$$\varphi\left(F_{z,Bz}(t), F_{z,Bz}(t), F_{z,z}(t), F_{Bz,Bz}(t), F_{Bz,z}(t), F_{z,Bz}(t)\right) \geq 0,$$

or

$$\varphi\left(F_{z,Bz}(t), F_{z,Bz}(t), 1, 1, F_{Bz,z}(t), F_{z,Bz}(t)\right) \geq 0.$$

which contradicts (φ_3). Therefore $Bz = z = Tz$. Thus we conclude that z is a common fixed point of A, B, S and T. Uniqueness of common fixed point is an easy consequence of inequality (4).

Case II If $z \in A(X) \cap B(X)$ then the proof is similar to Case I, hence it is omitted.

□

Remark 4.2. Theorem 4.1 improve the results of Imdad et al. [13, Theorem 3.3, Theorem 3.7, Corollary 3.8] without any requirement on containment of ranges of the involved mappings and closedness of the underlying subspaces.

Example 4.3. Consider $X = [1,15)$ and define $F_{x,y}(t) = \frac{t}{t+|x-y|}$ for all $x, y \in X$ and $t > 0$. Then $(X, \mathcal{F}, \triangle)$ be a Menger space with $\triangle(a,b) = \min\{a,b\}$. Define $\varphi(t_1,t_2,t_3,t_4,t_5,t_6) : [0,1]^6 \to \mathbb{R}$ as

$$\varphi(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - \psi\left(\min\{t_2,t_3,t_4,t_5,t_6\}\right),$$

where $\psi(s) = \sqrt{s}$ for all $s \in [0,1]$. Let A, B, S and T be self mappings defined by

$$A(x) = \begin{cases} 1, & \text{if } x \in \{1\} \cup (10,15); \\ x+3, & \text{if } x \in (1,10]. \end{cases}$$

$$B(x) = \begin{cases} 1, & \text{if } x \in \{1\} \cup [5,15); \\ x+2, & \text{if } x \in (1,5). \end{cases}$$

$$S(x) = \begin{cases} 1, & \text{if } x = 1; \\ 3, & \text{if } x \in (1,10]; \\ \frac{x-5}{5}, & \text{if } x \in (10,15). \end{cases}$$

$$T(x) = \begin{cases} 1, & \text{if } x = 1; \\ 10, & \text{if } x \in (1,5]. \\ x-4, & \text{if } x \in (5,15); \end{cases}$$

Consider the sequences $\{x_n = 10 + \frac{1}{n}\}_{n\in\mathbb{N}}$ and $\{y_n = 5 + \frac{1}{n}\}_{n\in\mathbb{N}}$ in X such that $\lim_{n\to\infty} Ax_n = 1 \in S(X)$, $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} \left\{ \frac{10+\frac{1}{n}-5}{5} \right\} = 1 \in A(X)$, $\lim_{n\to\infty} By_n = 1 \in T(X)$, $\lim_{n\to\infty} Ty_n = \lim_{n\to\infty} \left\{ 5 + \frac{1}{n} - 4 \right\} = 1 \in B(X)$.

Notice that the pairs (A,S) and (B,T) commute at 1 which is their common coincidence point. Thus all the conditions of Theorem 4.1 are satisfied and 1 is a unique common fixed point of self mappings A,B,S and T. All the mappings are even discontinuous at their unique common fixed point 1. It is pointed out that $A(X) = \{1\} \cup (4,14]$, $B(X) = \{1\} \cup (3,7)$, $S(X) = [1,2) \cup \{3\}$ and $T(X) = [1,11)$. Hence $A(X) \not\subseteq T(X)$ and $B(X) \not\subseteq S(X)$. Also $A(X)$, $B(X)$, $S(X)$ and $T(X)$ are not closed subsets of X.

By choosing A,B,S and T suitably, we can derive a multitude of common fixed point theorems for a pair or triod of mappings. Our next result is proved for a pair of mappings by using (E.A) like property under weak compatibility.

Theorem 4.4. *Let A and S be self mappings of a Menger space $(X,\mathcal{F},\triangle)$ satisfying the following conditions:*

1. *the pair (A,S) satisfies the (E.A) like property;*
2. *there exists $\varphi \in \Theta$ such that*

$$\varphi\left(F_{Ax,Ay}(t), F_{Sx,Sy}(t), F_{Ax,Sx}(t), F_{Ay,Sy}(t), F_{Ay,Sx}(t), F_{Ax,Sy}(t)\right) \geq 0, \qquad (5)$$

for all $x,y \in X$ and $t > 0$. Then the pair (A,S) has a coincidence point. Moreover, A and S have a unique common fixed point if the pair (A,S) is weakly compatible.

Proof. Since the pair (A,S) satisfies the (E.A) like property, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z,$$

for some $z \in A(X)$ or $z \in S(X)$.

Case I Suppose that $z \in S(X)$, then we have $\lim_{n\to\infty} Ax_n = z \in S(X)$, that is, $z = Su$ for some $u \in X$. We show that $Au = Su$. Let, on the contrary, $Au \neq Su$. On using inequality (5) with $x = u$, $y = x_n$, we get

$$\varphi\left(\begin{array}{c} F_{Au,Ax_n}(t), F_{Su,Sx_n}(t), F_{Au,Su}(t), \\ F_{Ax_n,Sx_n}(t), F_{Ax_n,Su}(t), F_{Au,Sx_n}(t) \end{array} \right) \geq 0.$$

Letting $n \to \infty$, we have

$$\varphi\left(F_{Au,z}(t), F_{z,z}(t), F_{Au,z}(t), F_{z,z}(t), F_{z,z}(t), F_{Au,z}(t)\right) \geq 0,$$

and so

$$\varphi\left(F_{Au,z}(t), 1, F_{Au,z}(t), 1, 1, F_{Au,z}(t)\right) \geq 0.$$

which is a contradiction to (φ_1). Hence $Au = z = Su$ which shows that u is a coincidence point of the pair (A, S).

Since the pair (A,S) is weakly compatible and $Au = Su$, therefore $Az = ASu = SAu = Sz$. Now we assert that z is a common fixed point of A and S. Suppose that $z \neq Az$. On using inequality (5) with $x = z$, $y = u$, we get

$$\varphi\left(\begin{array}{c} F_{Az,Au}(t), F_{Sz,Su}(t), F_{Az,Sz}(t), \\ F_{Au,Su}(t), F_{Az,Su}(t), F_{Au,Sz}(t) \end{array}\right) \geq 0,$$

and so

$$\varphi\left(F_{Az,z}(t), F_{Az,z}(t), F_{Az,Az}(t), F_{z,z}(t), F_{Az,z}(t), F_{z,Az}(t)\right) \geq 0,$$

or

$$\varphi\left(F_{Az,z}(t), F_{Az,z}(t), 1, 1, F_{Az,z}(t), F_{z,Az}(t)\right) \geq 0.$$

which is a contradiction to (φ_3). Hence $Az = z = Sz$. Therefore z is a common fixed point of A and S. Uniqueness of common fixed point is an easy consequence of inequality (5).

Case II The proof is similar to Case I if it is assumed that $z \in A(X)$. □

Now, we utilize Definition 2.13 (which is indeed a natural extension of commutativity condition to two finite families) to prove a common fixed point theorem for six mappings in Menger space.

Theorem 4.5. *Let A, B, S, R, T and H be self mappings of a Menger space $(X, \mathcal{F}, \triangle)$ satisfying the following conditions:*

1. *the pairs (A, SR) and (B, TH) share the common (E.A) like property;*

2. *there exists $\varphi \in \Theta$ such that*

$$\varphi\left(\begin{array}{c} F_{Ax,By}(t), F_{SRx,THy}(t), F_{Ax,SRx}(t), \\ F_{By,THy}(t), F_{By,SRx}(t), F_{Ax,THy}(t) \end{array}\right) \geq 0, \qquad (6)$$

for all $x, y \in X$ and $t > 0$. Then the pairs (A, SR) and (B, TH) have a coincidence point each. Moreover, A, B, S, R, T and H have a unique common fixed point if both the pairs (A, SR) and (B, TH) commute pairwise (that is, $AS = SA$, $AR = RA$, $SR = RS$, $BT = TB$, $BH = HB$ and $TH = HT$).

Proof. Since (A,SR) and (B,TH) are commuting pairwise, obviously both the pairs are weakly compatible. By Theorem 4.1, A, B, SR and TH have a unique common fixed point z in X. We show that z is a unique common fixed point of the self mappings A,B,R,S,H and T. Let, on the contrary, $z \neq Rz$. On using inequality (6) with $x = Rz$, $y = z$ to get

$$\varphi\left(\begin{array}{c} F_{A(Rz),Bz}(t), F_{SR(Rz),THz}(t), F_{A(Rz),SR(Rz)}(t), \\ F_{Bz,THz}(t), F_{Bz,SR(Rz)}(t), F_{A(Rz),THz}(t) \end{array}\right) \geq 0,$$

and so

$$\varphi\left(\begin{array}{c} F_{Rz,z}(t), F_{Rz,z}(t), F_{Rz,Rz}(t), \\ F_{z,z}(t), F_{z,Rz}(t), F_{Rz,z}(t) \end{array}\right) \geq 0,$$

or, equivalently,

$$\varphi\left(F_{Rz,z}(t), F_{Rz,z}(t), 1, 1, F_{Rz,z}(t), F_{z,Rz}(t)\right) \geq 0.$$

which is a contradiction to (φ_3). Then we obtain $z = Rz$ and so $S(Rz) = Sz = z$. Similarly, one can prove that $z = Hz$, that is, $T(Hz) = Tz = z$. Hence $z = Az = Bz = Sz = Rz = Tz = Hz$, and z is a unique common fixed point of A,B,S,R,T and H. □

As an application of Theorem 4.1, we present the following result for four finite families of self mappings.

Theorem 4.6. *Let* $\{A_1, A_2, \ldots, A_m\}$, $\{B_1, B_2, \ldots, B_p\}$, $\{S_1, S_2, \ldots, S_n\}$ *and* $\{T_1, T_2, \ldots, T_q\}$ *be four finite families of self mappings of a Menger space* $(X, \mathcal{F}, \triangle)$ *such that* $A = A_1A_2\ldots A_m$, $B = B_1B_2\ldots B_p$, $S = S_1S_2\ldots S_n$ *and* $T = T_1T_2\ldots T_q$ *which also satisfy conditions (1) and (2) of Theorem 4.1. Then* (A,S) *and* (B,T) *have a point of coincidence each.*

Moreover, if the family $\{A_i\}_{i=1}^m$ *commutes pairwise with the family* $\{S_j\}_{j=1}^n$ *whereas the family* $\{B_r\}_{r=1}^p$ *commutes pairwise with the family* $\{T_s\}_{s=1}^q$, *then (for all* $i \in \{1,2,\ldots,m\}$, $j \in \{1,2,\ldots,n\}$, $r \in \{1,2,\ldots,p\}$ *and* $s \in \{1,2,\ldots,q\}$*)* A_i, B_j, S_r *and* T_s *have a common fixed point.*

Proof. The proof of this theorem is similar to that of Theorem 3.1 contained in [12], hence it is omitted. □

Remark 4.7. Theorem 4.6 improve the result of Imdad et al. [13, Theorem 3.9] without any requirement of closedness of the underlying subspaces.

By setting $A_1 = A_2 = \ldots = A_m = A$, $B_1 = B_2 = \ldots = B_p = B$, $S_1 = S_2 = \ldots = S_n = S$ and $T_1 = T_2 = \ldots = T_q = T$ in Theorem 4.6, we deduce the following:

Corollary 4.8. *Let A,B,S and T be self mappings of a Menger space $(X,\mathcal{F},\triangle)$ satisfying the following conditions:*

1. *the pairs (A^m,S^n) and (B^p,T^q) share the common (E.A) like property;*
2. *there exists $\varphi\in\Theta$ such that*

$$\varphi\left(\begin{array}{c} F_{A^m x,B^p y}(t),F_{S^n x,T^q y}(t),F_{A^m x,S^n x}(t),\\ F_{B^p y,T^q y}(t),F_{B^p y,S^n x}(t),F_{A^m x,T^q y}(t)\end{array}\right)\geq 0, \tag{7}$$

for all $x,y\in X$ and $t>0$. Then (A^m,S^n) and (B^p,T^q) have a coincidence point each. Moreover, A,B,S and T have a unique common fixed point if $AS=SA$ and $BT=TB$.

Now, we prove an integral type fixed point theorem for two pairs of weakly compatible mappings in Menger space satisfying ψ-contraction condition.

Theorem 4.9. *Let A,B,S and T be self mappings of a Menger space $(X,\mathcal{F},\triangle)$ satisfying condition (1) of Theorem 4.1 and*

$$\int_0^{F_{Ax,By}(t)}\phi(t)dt\geq\psi\left(\int_0^{M(x,y)}\phi(t)dt\right), \tag{8}$$

where

$$M(x,y)=\min\{F_{Sx,Ty}(t),F_{Ax,Sx}(t),F_{By,Ty}(t),F_{By,Sx}(t),F_{Ax,Ty}(t)\},$$

for all $x,y\in X$, $t>0$ and $\psi:[0,1]\to[0,1]$ is a lower semicontinuous function such that $\psi(t)>t$, for all $t\in(0,1)$ along with $\psi(0)=0$, $\psi(1)=1$ and $\phi:\mathbb{R}^+\to\mathbb{R}^+$ is a summable Lebesgue integrable function such that

$$0<\int_0^{\varepsilon}\phi(s)ds<1, \text{for all } 0<\varepsilon<1,\ \int_0^1\phi(s)ds=1.$$

Then the pairs (A,S) and (B,T) have a coincidence point each. Moreover, A,B,S and T have a unique common fixed point if both the pairs (A,S) and (B,T) are weakly compatible.

Proof. Since the pairs (A,S) and (B,T) enjoy the common (E.A) like property, there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty}Ax_n=\lim_{n\to\infty}Sx_n=\lim_{n\to\infty}By_n=\lim_{n\to\infty}Ty_n=z,$$

where $z\in S(X)\cap T(X)$ or $z\in A(X)\cap B(X)$.

Case I Suppose that $z\in S(X)\cap T(X)$. Since $z\in S(X)$, we have $\lim_{n\to\infty}Ax_n=z\in$

$S(X)$, that is, $z = Su$ for some $u \in X$. We show that $Au = Su$. Let, on the contrary, $Au \neq Su$. On using inequality (8) with $x = u$, $y = y_n$, we have

$$\int_0^{F_{Au,By_n}(t)} \phi(t)dt \geq \psi\left(\int_0^{M(u,y_n)} \phi(t)dt\right), \tag{9}$$

where

$$M(u,y_n) = \min\{F_{Su,Ty_n}(t), F_{Au,Su}(t), F_{By_n,Ty_n}(t), F_{By_n,Su}(t), F_{Au,Ty_n}(t)\}.$$

Taking limit $n \to \infty$ in inequality (9), we obtain

$$\int_0^{F_{Au,z}(t)} \phi(t)dt \geq \psi\left(\lim_{n\to\infty}\int_0^{M(u,y_n)} \phi(t)dt\right), \tag{10}$$

where

$$\begin{aligned}\lim_{n\to\infty} M(u,y_n) &= \min\{F_{z,z}(t), F_{Au,z}(t), F_{z,z}(t), F_{z,z}(t), F_{Au,z}(t)\}\\ &= \min\{1, F_{Au,z}(t), 1, 1, F_{Au,z}(t)\}\\ &= F_{Au,z}(t).\end{aligned}$$

From inequality (10), we get

$$\int_0^{F_{Au,z}(t)} \phi(t)dt \;\geq\; \psi\left(\int_0^{F_{Au,z}(t)} \phi(t)dt\right) > \int_0^{F_{Au,z}(t)} \phi(t)dt,$$

which is a contradiction. Therefore $Au = Su = z$ and hence u is a coincidence point of (A,S).

Since $z \in T(X)$, we have $\lim_{n\to\infty} By_n = z \in T(X)$, that is, $z = Tv$ for some $v \in X$. Now we assert that $Bv = Tv$. If it is not so, then using inequality (8) with $x = x_n$, $y = v$, we have

$$\int_0^{F_{Ax_n,Bv}(t)} \phi(t)dt \geq \psi\left(\int_0^{M(x_n,v)} \phi(t)dt\right), \tag{11}$$

where

$$M(x_n,v) = \min\{F_{Sx_n,Tv}(t), F_{Ax_n,Sx_n}(t), F_{Bv,Tv}(t), F_{Bv,Sx_n}(t), F_{Ax_n,Tv}(t)\}.$$

Taking limit $n \to \infty$ in inequality (11), we obtain

$$\int_0^{F_{z,Bv}(t)} \phi(t)dt \geq \psi\left(\lim_{n\to\infty}\int_0^{M(x_n,v)} \phi(t)dt\right), \tag{12}$$

where

$$\begin{aligned}\lim_{n\to\infty} M(x_n, v) &= \min\{F_{z,z}(t), F_{z,z}(t), F_{Bv,z}(t), F_{Bv,z}(t), F_{z,z}(t)\}\\ &= \min\{1, 1, F_{Bv,z}(t), F_{Bv,z}(t), 1\}\\ &= F_{z,Bv}(t).\end{aligned}$$

From inequality (12), we get

$$\int_0^{F_{z,Bv}(t)} \phi(t)dt \;\geq\; \psi\left(\int_0^{F_{z,Bv}(t)} \phi(t)dt\right) > \int_0^{F_{z,Bv}(t)} \phi(t)dt,$$

which is a contradiction. Therefore $Bv = Tv = z$ and hence v is a coincidence point of (B,T).

Since the pairs (A,S) and (B,T) are weakly compatible and $Au = Su$, $Bv = Tv$, therefore $Az = ASu = SAu = Sz$ and $Bz = BTv = TBv = Tz$. The rest of the proof runs on the lines of the proof of Theorem 4.1, therefore details are omitted. □

Remark 4.10. Theorem 4.9 improve the results of Imdad et al. [13, Theorem 3.13] and Altun et al. [5, Theorem 3.2] without any requirement of closedness of the underlying subspaces.

Remark 4.11. In the setting of Example 4.3, retain the same mappings A,B,S and T and define $\phi(t) = 1$. Then A,B,S and T satisfy all the conditions of Theorem 4.9 and have a unique common fixed point at $x = 1$ which also remains a point of discontinuity.

Remark 4.12. The integral analogue of Theorems 4.4-4.6 and Corollary 4.8 can be outlined easily. But due to the repetition, the details are avoided.

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