ON A $q$-DUNKL SONINE TRANSFORM

LASSAD BENNASR - RYM HASNA BETTAIEB - FERJANI NOURI

In this paper, we introduce and study the $q$-Dunkl Sonine transform and we establish a Plancherel formula for its dual. Furthermore we give many inversion formulas.

1. Introduction

The transmutation operators allow the transfer of some well known results related to a well known operator to those related to a new one, they play a central role in many areas of Mathematics and mathematical physics such as spectral theory, harmonic analysis, special functions and fractional calculus. This theory, introduced firstly by Delsart and Lions (see [5]), was extended by many authors (see for instance [3, 4, 10, 12, 13, 16–18, 22, 23, 25]), and recently was extended to quantum calculus in [1–3, 9, 10].

In this paper we introduce and study the so-called $q$-Dunkl Sonine operator and its dual, we show that these operators are transmutation operators between two different $q$-Dunkl operators that generalize the $q$-Dunkl intertwining and its dual introduced in [1]. Furthermore, we give various inversion formulas for these operators.

This paper is organized as follows: in Section 2, we present some new preliminaries that we need. In Section 3, we collect some elements of $q$-Dunkl har-
monic analysis ($q$-Dunkl kernel, $q$-Dunkl transform, $q$-Dunkl convolution,...).

In section 4 we introduce the $q$-Dunkl-Sonine transform by

$$S_{\alpha,\beta}^q(f)(x) = \frac{(1+q)\Gamma_q^2(\beta+1)}{2\Gamma_q^2(\alpha+1)\Gamma_q^2(\beta-\alpha)} \int_{-1}^{1} f(xt)W_{\beta-\alpha-\frac{1}{2}}(t; q^2)(1+t)^{2\alpha+1}d_qt,$$

where $\beta > \alpha \geq -1/2$ and

$$W_{\alpha}(t,q^2) = \frac{(t^2q^2;q^2)_\infty}{(t^2q^{2\alpha+1}; q^2)_\infty},$$

is the $q$-analogue of the kernel $W_{\alpha}(t) = (1-t^2)^{\alpha-1/2}$, $t \in ]-1,1[$. We show that $S_{\alpha,\beta}^q$ and its dual $S_{\alpha,\beta}^q$ are transmutation operators between the $q$-Dunkl operators $\Lambda_{\alpha,q}$ and $\Lambda_{\beta,q}$. We also deal with the relations between these transforms and the $q$-Dunkl intertwining operator and its dual. In Sections 5 we give many formulations of the inversion formulas of the $q$-Dunkl Sonine transform and its dual using $q$-pseudo-differential operators in some $q$-analogues of the Lizorkin spaces and we give Plancherel formula for the dual $q$-Dunkl-Sonine transform.

Section 6 is devoted to inversions formulas of the $q$-Dunkl Sonine transform and its dual by using $q$-Dunkl wavelets.

2. Notations and preliminaries

Throughout this paper, we assume $q \in ]0,1[$. We write $\mathbb{R}_q = \{\pm q^n : n \in \mathbb{Z}\}$ and we use the convention $\mathbb{N} = \{0,1,2,\ldots\}$. The $q$-shifted factorials of $a \in \mathbb{C}$ are defined as

$$(a;q)_0 = 1; \quad (a;q)_n = \prod_{k=0}^{n-1} (1-aq^k), \quad n = 1,2,\ldots; \quad (a;q)_\infty = \prod_{k=0}^{\infty} (1-aq^k).$$

The $q$-Factoriel of $n \in \mathbb{N}$ is $[n]_q! = \frac{(q;q)_n}{(1-q)^n}$ and more generally, the $q$-Gamma function is defined for $x \in \mathbb{C}$ by (see [11])

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty} (1-q)^{1-x}, \quad x \neq 0,-1,-2,\ldots.$$

The normalized third Jackson’s $q$-Bessel function is defined for $x \in \mathbb{C}$ by (see [9, 14])

$$j_\alpha(x;q^2) = \Gamma_q^2(\alpha+1) \sum_{n=0}^{+\infty} \frac{(-1)^n q^n(n+1)}{\Gamma_q^2(\alpha+n+1)\Gamma_q^2(n+1)} \left(\frac{x}{1+q}\right)^{2n}. \quad (1)$$
The $q$-trigonometric functions $\cos(x; q^2)$ and $\sin(x; q^2)$ are defined for $x \in \mathbb{C}$ by

$$
\cos(x; q^2) = j_{-\frac{1}{2}}(x; q^2) = \sum_{n=0}^{+\infty} (-1)^n q^{n(n+1)} \frac{x^{2n}}{[2n]_q!},
$$

$$
\sin(x; q^2) = x j_{\frac{1}{2}}(x; q^2) = \sum_{n=0}^{+\infty} (-1)^n q^{n(n+1)} \frac{x^{2n+1}}{[2n+1]_q!}.
$$

In connection with $q^2$-analogue Fourier analysis, R. Rubin [20, 21] constructed a $q^2$-analogue of the exponential function, $e(x; q^2)$, and a $q^2$-derivative $\partial_q$ as follows:

$$
e(ix; q^2) = \cos(x, q^2) + i \sin(x; q^2), \quad x \in \mathbb{C},
$$

and

$$
\partial_q(f)(x) = \begin{cases} 
\frac{f(q^{-1}x) + f(-q^{-1}x) - f(qx) + f(-qx) - 2f(-x)}{2(1-q)x} & \text{if } x \neq 0; \\
\lim_{x \to 0} \partial_q(f)(x) & \text{(in } \mathbb{R}_q) \end{cases}
$$

so that for every $\lambda \in \mathbb{C}$ the $q^2$-exponential function satisfies

$$
\partial_q(e(i\lambda x; q^2)) = i\lambda e(i\lambda x; q^2), \quad x \in \mathbb{C}.
$$

The $q$-Bessel function satisfies

$$
\partial_q j_{\alpha}(x; q^2) = -\frac{x}{[2\alpha + 2]_q} j_{\alpha + 1}(x; q^2).
$$

The $q$-integrals of Jackson are defined by (see [15])

$$
\int_0^a f(x) dq x = (1-q) a \sum_{n=0}^{\infty} q^n f(a q^n), \quad \int_a^b f(x) dq x = \int_0^b f(x) dq x - \int_0^a f(x) dq x,
$$

$$
\int_0^\infty f(x) dq x = (1-q) \sum_{n=-\infty}^{\infty} q^n f(q^n),
$$

and

$$
\int_{-\infty}^\infty f(x) dq x = (1-q) \sum_{n=-\infty}^{\infty} q^n f(q^n) + (1-q) \sum_{n=-\infty}^{\infty} q^n f(-q^n),
$$

provided that the series converge absolutely.

The operator $\partial_q$ and the Jakson’s $q$-integral allow us to introduce the following useful $q$-functional spaces:
The space $E_q(\mathbb{R}_q)$ of all functions $f$ defined on $\mathbb{R}_q$ such that, for all $n \in \mathbb{N}$, the limit $\lim_{x \to 0} \partial_q^n f(x)$ (in $\mathbb{R}_q$) exists. We provide the space $E_q(\mathbb{R}_q)$ with the semi-norms $P_{n,a,q}$ defined for $n \in \mathbb{N}$ and $a > 0$ by
\[
P_{n,a,q}(f) = \sup \left\{ |\partial_q^k f(x)| ; 0 \leq k \leq n; x \in [-a, a] \cap \mathbb{R}_q \right\}, \quad f \in E_q(\mathbb{R}_q).
\]

- The space $I_q(\mathbb{R}_q)$ of all functions $f \in E_q(\mathbb{R}_q)$ satisfying
\[
Q_{n,m,q}(f) = \sup_{x \in \mathbb{R}_q} |x^m \partial_q^n f(x)| < +\infty, \quad n,m \in \mathbb{N}.
\]

We provide $I_q(\mathbb{R}_q)$ with the topology defined by the semi-norms $Q_{n,m,q}$.
- $P_q(\mathbb{R}_q)$ the subspace of $E_q(\mathbb{R}_q)$ of functions with compact supports.
- $L^\infty(\mathbb{R}_q)$ the space of all bounded functions on $\mathbb{R}_q$ endowed with the norm
\[
\|f\|_\infty = \sup_{x \in \mathbb{R}_q} |f(x)|, \quad f \in L^\infty(\mathbb{R}_q).
\]

- $L^p(\mathbb{R}_q, |x|^{2\alpha+1}d_qx)$ the space of all functions $f$ defined on $\mathbb{R}_q$ and satisfying
\[
\|f\|_{p,\alpha,q} = \left( \int_{-\infty}^{\infty} |f(x)|^p |x|^{2\alpha+1}d_qx \right)^{\frac{1}{p}} < \infty
\]
provided with the norm $\| \cdot \|_{p,\alpha,q}$.

### 3. Elements of $q$-Dunkl Harmonic analysis

In this section, we collect some facts regarding some elements of $q$-Dunkl harmonic analysis introduced and studied in [1] and [2]. Throughout this section, unless otherwise stated, we assume $\alpha \geq -1/2$.

The $q$-Dunkl operator $\Lambda_{\alpha,q}$ is defined for a complex function $f$, defined on $\mathbb{R}_q$, by
\[
\Lambda_{\alpha,q}(f)(x) = \partial_q^2 f + q^{2\alpha+1} f_o(x) + [2\alpha+1]_q \frac{f(x) - f(-x)}{x},
\]
(3)

where $f_e$ and $f_o$ are respectively the even and the odd parts of $f$.

For any complex number $\lambda$, the $q$-Dunkl Kernel $\psi_{\lambda,q}^{\alpha}$ is defined on $\mathbb{C}$ by
\[
\psi_{\lambda,q}^{\alpha}(x) = j\alpha(\lambda x; q^2) + \frac{i\lambda x}{[2\alpha+1]_q} j_{\alpha+1}(\lambda x; q^2),
\]
(4)

where $j\alpha(\cdot; q^2)$ is the $q$-Bessel function given by (1). Note that for $\alpha = -1/2$ we have $\psi_{\lambda,q}^{\alpha}(x) = e(i\lambda x; q^2)$ and $\Lambda_{\alpha,q} = \partial_q$. 
Proposition 3.1. (see [1]) For every $\lambda \in \mathbb{C}$, the $q$-Dunkl kernel $\psi_{\alpha,q}^x$ is the unique analytic solution of the $q$-differential-difference equation

$$\begin{align*}
\Lambda_{\alpha,q}(f) &= i\lambda f; \\
f(0) &= 1.
\end{align*}$$

When $\alpha > -1/2$, for every $\lambda \in \mathbb{C}$, the $q$-Dunkl kernel $\psi_{\alpha,q}^x$ possesses the following $q$-integral representation of Mehler type:

$$\psi_{\alpha,q}^x(x) = \frac{(1+q)^{\Gamma_q^2(\alpha+1)}\Gamma_q^2(\alpha+1)}{2\Gamma_q^2(\alpha+\frac{1}{2})} \int_{-1}^{1} \frac{(t^2 q^2; q^2)_{\infty}}{(t^2 q^2 \alpha+1; q^2)_{\infty}} (1+t)e(i\lambda xt; q^2)d_qt,$$

for all $x \in \mathbb{R}_q$. This formula gives rise to the $q$-Dunkl intertwining operator $V_{\alpha,q}$ defined for $f \in \mathcal{E}_q(\mathbb{R}_q)$ by

$$V_{\alpha,q}(f)(x) = \frac{(1+q)^{\Gamma_q^2(\alpha+1)}\Gamma_q^2(\alpha+1)}{2\Gamma_q^2(\alpha+\frac{1}{2})} \int_{-1}^{1} W_{\alpha}(t,q^2)(1+t)f(xt)d_qt, \quad x \in \mathbb{R}_q, \quad (5)$$

and the dual $q$-Dunkl intertwining operator $^{t}V_{\alpha,q}$ defined for $f \in \mathcal{D}_q(\mathbb{R}_q)$ by

$$(^{t}V_{\alpha,q})(f)(x) = \frac{(1+q)^{-\alpha+\frac{1}{2}}\Gamma_q^2(\alpha+1)}{2\Gamma_q^2(\alpha+\frac{1}{2})} \int_{|y|\geq q|x|} W_{\alpha}\left(\frac{x}{y}; q^2\right)\left(1+\frac{x}{y}\right)f(y)|y|^{2\alpha}d_qy,$$

for all $x \in \mathbb{R}_q$, where

$$W_{\alpha}(t,q^2) = \frac{(t^2 q^2; q^2)_{\infty}}{(t^2 q^2 \alpha+1; q^2)_{\infty}}. \quad (6)$$

Note that for every $\lambda \in \mathbb{C}$, we have $V_{\alpha,q}(e(-i\lambda \cdot ; q^2))(x) = \psi_{\alpha,q}^x(x)$ for all $x \in \mathbb{R}_q$.

The operators $V_{\alpha,q}$ and $^{t}V_{\alpha,q}$ are linked to each other by the duality relation

$$\int_{-\infty}^{+\infty} V_{\alpha,q}(f)(x)g(x)|x|^{2\alpha+1}d_qx = \frac{(1+q)^{\alpha+\frac{1}{2}}\Gamma_q^2(\alpha+1)}{\Gamma_q^2(\alpha+\frac{1}{2})} \int_{-\infty}^{+\infty} f(t)(^{t}V_{\alpha,q})(g)(t)d_qt, \quad (7)$$

for all $f \in \mathcal{E}_q(\mathbb{R}_q)$ and $g \in \mathcal{D}_q(\mathbb{R}_q)$. Moreover, it has been shown in[1] that the $q$-Dunkl intertwining operator $V_{\alpha,q}$ is a transmutation operator between $\Lambda_{\alpha,q}$ and $\partial_q$ on $\mathcal{E}_q(\mathbb{R}_q)$, that is, a topological isomorphism from $\mathcal{E}_q(\mathbb{R}_q)$ into itself satisfying the following transmutation relation:

$$\Lambda_{\alpha,q}V_{\alpha,q}(f) = V_{\alpha,q}(\partial_q f), \quad f \in \mathcal{E}_q(\mathbb{R}_q), \quad (8)$$

whereas, for the dual $q$-Dunkl intertwining operator $^{t}V_{\alpha,q}$, only the transmutation relation

$$\partial_q(^{t}V_{\alpha,q}(f)) = ^{t}V_{\alpha,q}(\Lambda_{\alpha,q} f), \quad f \in \mathcal{D}_q(\mathbb{R}_q).$$
have been shown. In the next section we will extend the operator $iV_{\alpha,q}$ to the space $L^1(\mathbb{R}_q, |x|^{2\alpha+1}d_qx)$ and we will show that it is a transmutation operator between $\partial_q$ and $\Lambda_{\alpha,q}$ on $\mathcal{S}_q(\mathbb{R}_q)$.

In the remainder of this paper, we assume, as in [1, 2, 20, 21], that

$$\log(1-q) \log q \in 2\mathbb{Z}.$$ 

The $q$-Dunkl kernel $\psi^{\alpha,q}_\lambda$, $\lambda \in \mathbb{C}$, gives rise to the $q$-Dunkl transform $F^\alpha_q$ defined for $f \in L^1(\mathbb{R}_q, |x|^{2\alpha+1}d_qx)$ by

$$F^\alpha_q(f)(\lambda) = \frac{(1+q)^{-\alpha}}{2\Gamma_q^2(\alpha+1)} \int_{-\infty}^{+\infty} f(x) \psi^{\alpha,q}_\lambda(x)|x|^{2\alpha+1}d_qx, \quad \lambda \in \mathbb{R}_q,$$

which is a $q$-analogue of the classical Dunkl transform studied in [6–8, 19].

**Remark 3.2.** For $\alpha = -1/2$, the $q$-Dunkl transform is the $q^2$-analogue Fourier transform (see [20, 21]) given by

$$F_q(f)(\lambda) = \frac{(1+q)^{1/2}}{2\Gamma_q^2(1/2)} \int_{-\infty}^{+\infty} f(x) e(-i\lambda x; q^2)d_qx, \quad \lambda \in \mathbb{R}_q,$$  

(9)

and on the space of even functions, $F^\alpha_q$ coincides with the $q$-Bessel transform given by (see [1, 2, 9])

$$F_{\alpha,q}(f)(\lambda) = \frac{(1+q)^{-\alpha}}{\Gamma_q^2(\alpha+1)} \int_{0}^{+\infty} f(x) j_{\alpha}(\lambda x; q^2)x^{2\alpha+1}d_qx, \quad \lambda \in \mathbb{R}_q.$$  

(10)

**Proposition 3.3.**

(a) The $q$-Dunkl transform $F^\alpha_q$ is a topological automorphism of $\mathcal{S}_q(\mathbb{R}_q)$.

(b) For $f \in \mathcal{S}_q(\mathbb{R}_q)$, we have $F^\alpha_q(\Lambda_{\alpha,q}f)(\lambda) = i\lambda F^\alpha_q(\lambda)$ for all $\lambda \in \mathbb{R}_q$.

(c) Inversion formula: For $f \in \mathcal{S}_q(\mathbb{R}_q)$ we have

$$f(x) = \frac{(1+q)^{-\alpha}}{2\Gamma_q^2(\alpha+1)} \int_{-\infty}^{+\infty} F^\alpha_q(f)(\lambda) \psi^{\alpha,q}_\lambda(x)|\lambda|^{2\alpha+1}d_q\lambda, \quad x \in \mathbb{R}_q.$$  

(11)

(d) Plancherel formula: For all $f \in \mathcal{S}_q(\mathbb{R}_q)$ we have

$$\|F^\alpha_q(f)\|_{2,q} = \|f\|_{2,q}. \quad (12)$$

For any $y \in \mathbb{R}_q \cup \{0\}$ we define the generalized $q$-Dunkl translation operator $T^\alpha_{y,q}$ for $f \in \mathcal{S}_q(\mathbb{R}_q)$ by

$$T^\alpha_{y,q}(f)(x) = \frac{(1+q)^{-\alpha}}{2\Gamma_q^2(\alpha+1)} \int_{-\infty}^{+\infty} F^\alpha_q(f)(\lambda) \psi^{\alpha,q}_\lambda(x) \psi^{\alpha,q}_\lambda(y)|\lambda|^{2\alpha+1}d_q\lambda, \quad (13)$$
for all $x \in \mathbb{R}_q$. The q-Dunkl translation operators allow us to define a q-Dunkl convolution product $\ast_{\alpha,q}$ on $\mathcal{S}_q(\mathbb{R}_q)$ as follows (see [2]): for $f, g \in \mathcal{S}_q(\mathbb{R}_q)$,

$$f \ast_{\alpha,q} g(x) = \frac{(1 + q)^{-\alpha}}{2\Gamma_q^2(\alpha + 1)} \int_{-\infty}^{\infty} T^\alpha_q f(-y)g(y)|y|^{2\alpha + 1}d_qy, \quad x \in \mathbb{R}_q.$$ 

This convolution product is commutative and associative. Moreover, for $f, g$ in $\mathcal{S}_q(\mathbb{R}_q)$ we have $f \ast_{\alpha,q} g \in \mathcal{S}_q(\mathbb{R}_q)$, and $F^\alpha_q(f \ast_{\alpha,q} g) = F^\alpha_q(f) \cdot F^\alpha_q(g)$.

4. The q-Dunkl-Sonine transform

From now on, unless otherwise stated, we assume that $\beta > \alpha \geq -1/2$.

**Proposition 4.1.** For every $\lambda \in \mathbb{C}$, the function $j_\alpha(\lambda \cdot; q^2)$ admits the q-integral representation of Sonine type:

$$j_\beta(\lambda x; q^2) = \frac{(1 + q)\Gamma_q^2(\beta + 1)}{\Gamma_q^2(\alpha + 1)\Gamma_q^2(\beta - \alpha)} \int_0^1 j_\alpha(\lambda xt; q^2)W_{\beta - \alpha - \frac{1}{2}}(t; q^2)t^{2\alpha + 1}d_qt, \quad x \in \mathbb{C}, \quad (14)$$

where $W_{\beta - \alpha - \frac{1}{2}}(t; q^2)$ is given by (6).

**Proof.** It follows from (1) that

$$\int_0^1 j_\alpha(\lambda xt; q^2)W_{\beta - \alpha - \frac{1}{2}}(t; q^2)t^{2\alpha + 1}d_qt =$$

$$\sum_{n=0}^{+\infty} \frac{(-1)^nq^n(n+1)\Gamma_q^2(\alpha + 1)I_n}{\Gamma_q^2(n + \alpha + 1)\Gamma_q^2(n + 1)} \left( \frac{\lambda x}{1+q} \right)^{2n},$$

for all $x \in \mathbb{C}$, where

$$I_n = (1 + q)\int_0^1 \frac{(t^2q^2;q^2)^\infty_0}{(t^2q^{2\alpha-\beta};q^2)^\infty_0}t^{2\alpha + 2n + 1}d_qt.$$ 

Using the q-Beta integral (see the proof of Theorem 1 in [9])

$$\frac{\Gamma_q^2(x)\Gamma_q^2(y)}{\Gamma_q^2(x+y)} = (1 + q)\int_0^1 \frac{(t^2q^2;q^2)^\infty_0}{(t^2q^{2\alpha-\beta};q^2)^\infty_0}t^{2x-1}d_qt, \quad x, y > 0,$$

we get the q-Sonine formula (14). □

In the following proposition, we extend (14) to the q-Dunkl setting.

**Proposition 4.2.** For every $\lambda \in \mathbb{C}$, the q-Dunkl-Sonine formula

$$\psi_\lambda^\beta,q(x) = \frac{(1 + q)\Gamma_q^2(\beta + 1)}{2\Gamma_q^2(\alpha + 1)\Gamma_q^2(\beta - \alpha)} \int_{-\infty}^{\infty} \psi_\lambda^\alpha,q(xt)W_{\beta - \alpha - \frac{1}{2}}(t; q^2)(1 + t)|t|^{2\alpha + 1}d_qt, \quad (15)$$

holds for all $x \in \mathbb{C}$. 


Proof. Set \( a^q_{\alpha, \beta} = \frac{(1 + q)\Gamma_q(\beta + 1)}{\Gamma_q(\beta - \alpha)\Gamma_q(\beta + 1)/\Gamma_q(\beta - \alpha)} \). By parity argument, formula (14) can be written as

\[
j_\beta(\lambda x; q^2) = a^q_{\alpha, \beta} \int_{-1}^{1} j_\alpha(\lambda xt; q^2) W_{\beta - \alpha - \frac{1}{2}}(t; q^2)(1 + t)|t|^{2\alpha + 1} d_q t.
\]  

(16)

Hence,

\[
\lambda \partial_q (j_\beta(\cdot; q^2))(\lambda x) = a^q_{\alpha, \beta} \int_{-1}^{1} \lambda t \partial_q (j_\alpha(\cdot; q^2))(\lambda xt) W_{\beta - \alpha - \frac{1}{2}}(t; q^2)(1 + t)|t|^{2\alpha + 1} d_q t.
\]

Using (2), we get

\[
\frac{\lambda x}{[2\beta + 2]^q} j_{\beta + 1}(\lambda x; q^2) = a^q_{\alpha, \beta} \int_{-1}^{1} \frac{\lambda xt}{[2\alpha + 2]^q} j_{\alpha + 1}(\lambda xt; q^2) W_{\beta - \alpha - \frac{1}{2}}(t; q^2)(1 + t)|t|^{2\alpha + 1}.
\]

Combining this with (16) yields the \( q \)-Dunkl-Sonine formula (15). \( \square \)

**Definition 4.3.** The \( q \)-Dunkl Sonine transform \( \mathcal{S}^q_{\alpha, \beta} \) is defined, for \( f \in \mathcal{E}_q(\mathbb{R}_q) \), by

\[
\mathcal{S}^q_{\alpha, \beta}(f)(x) = \frac{(1 + q)\Gamma_q(\beta + 1)}{2\Gamma_q(\alpha + 1)\Gamma_q(\beta - \alpha)} \int_{-1}^{1} f(xt) W_{\beta - \alpha - \frac{1}{2}}(t; q^2)(1 + t)|t|^{2\alpha + 1} d_q t,
\]

(17)

for all \( x \in \mathbb{R}_q \), which can be written as

\[
\mathcal{S}^q_{\alpha, \beta}(f)(x) = \frac{(1 + q)\Gamma_q(\beta + 1)}{2\Gamma_q(\alpha + 1)\Gamma_q(\beta - \alpha)|x|^{2\alpha + 2}} \int_{-|x|}^{|x|} f(y) W_{\beta - \alpha - \frac{1}{2}}\left(\frac{y}{x} q^2\right)(1 + \frac{y}{x})|y|^{2\alpha + 1} d_q y,
\]

(18)

for all \( x \in \mathbb{R}_q \).

**Remark 4.4.**

(a) For every \( \lambda \in \mathbb{C} \), we have \( q_\lambda^{\beta, q} = \mathcal{S}^q_{\alpha, \beta}(\psi^{\alpha, q}_\lambda) \).

(b) If \( \alpha = -\frac{1}{2} \), then the \( q \)-Dunkl Sonine transform \( \mathcal{S}^q_{\alpha, \beta} \) reduces to the \( q \)-Dunkl intertwining operator \( V_{\beta, q} \) given by (5).

**Definition 4.5.** The dual \( q \)-Dunkl-Sonine transform \( \mathcal{S}^q_{\alpha, \beta} \) is defined for suitable function \( f \), by

\[
\mathcal{S}^q_{\alpha, \beta}(f)(x) = \frac{(1 + q)^{\alpha - \beta + 1}}{2\Gamma_q(\beta - \alpha)} \int_{|y| \geq |x|} f(y) W_{\beta - \alpha - \frac{1}{2}}\left(\frac{x}{y} q^2\right)(1 + \frac{x}{y})|y|^{2(\beta - \alpha) - 1} d_q y,
\]

(19)

for all \( x \in \mathbb{R}_q \).
Proposition 4.6.  (a) The dual $q$-Dunkl-Sonine transform $\mathcal{S}^q_{\alpha,\beta}$ is a continuous linear mapping from $L^1(\mathbb{R}_q, |x|^{2\beta+1}d_qx)$ into $L^1(\mathbb{R}_q, |x|^{2\alpha+1}d_qx)$.

(b) For $f \in \mathcal{E}_q(\mathbb{R}_q) \cap L^\infty_q(\mathbb{R}_q)$ and $g \in L^1(\mathbb{R}_q, |x|^{2\beta+1}d_qx)$, we have the duality relation

$$\int_{-\infty}^{+\infty} \mathcal{S}^q_{\alpha,\beta}(f)(x)g(x)|x|^{2\beta+1}d_qx = \frac{(1+q)^{\beta} \Gamma_q^{2}(\beta+1)}{(1+q)^{\alpha} \Gamma_q^{2}(\alpha+1)} \int_{-\infty}^{+\infty} f(x)\mathcal{S}^q_{\alpha,\beta}(g)(x)|x|^{2\alpha+1}d_qx. \quad (20)$$

Proof. To prove (a), let $f \in L^1(\mathbb{R}_q, |x|^{2\beta+1})$. Using Tonelly’s theorem and series manipulations, we obtain

$$\int_{-\infty}^{+\infty} \left( \int_{|y| \geq q|x|} |f(y)| W_{\beta-\alpha-\frac{1}{2}} \left( \frac{x}{y}; q^2 \right) (1 + \frac{x}{y}) |y|^{2(\beta-\alpha)-1}d_qy \right) |x|^{2\beta+1}d_qx = \int_{-\infty}^{+\infty} \left( \frac{1}{|y|^{2\alpha+2}} \int_{|y|} |W_{\beta-\alpha-\frac{1}{2}} \left( \frac{x}{y}; q^2 \right) (1 + \frac{x}{y}) |y|^{2\alpha+1}d_qy \right) |f(y)||y|^{2\beta+1}d_qy.$$

Since by (18) together with Remark 4.4 (a), we have

$$\frac{(1+q)^{\beta} \Gamma_q^{2}(\beta+1)}{2 \Gamma_q^{2}(\alpha+1) \Gamma_q^{2}(\beta-\alpha)|y|^{2\alpha+2}} \int_{|y|} |W_{\beta-\alpha-\frac{1}{2}} \left( \frac{x}{y}; q^2 \right) (1 + \frac{x}{y}) |y|^{2\alpha+1}d_qy = \psi_0^{a,q}(y) = 1,$$

for all $y \in \mathbb{R}_q$, it follows that

$$\int_{-\infty}^{+\infty} \left( \int_{|y| \geq q|x|} |f(y)| W_{\beta-\alpha-\frac{1}{2}} \left( \frac{x}{y}; q^2 \right) (1 + \frac{x}{y}) |y|^{2(\beta-\alpha)-1}d_qy \right) |x|^{2\beta+1}d_qx = \frac{2 \Gamma_q^{2}(\alpha+1) \Gamma_q^{2}(\beta-\alpha)}{(1+q)^{\beta} \Gamma_q^{2}(\beta+1)} \|f\|_{1,\beta,q}.$$

Hence, by the Fubini theorem, the function $\mathcal{S}^q_{\alpha,\beta}(f)$ is defined on $\mathbb{R}_q$, belongs to $L^1(\mathbb{R}_q, |x|^{2\alpha+1}d_qx)$ and satisfies

$$\|\mathcal{S}^q_{\alpha,\beta}(f)\|_{1,\alpha,q} \leq \frac{(1-q)^{\alpha} \Gamma_q^{2}(\alpha+1)}{(1-q)^{\beta} \Gamma_q^{2}(\beta+1)} \|f\|_{1,\beta,q}.$$

Thus, $\mathcal{S}^q_{\alpha,\beta}$ maps continuously $L^1(\mathbb{R}_q, |x|^{2\beta+1})$ into $L^1(\mathbb{R}_q, |x|^{2\alpha+1})$.

Part (b) can be proved using the Fubini’s theorem and series manipulations. \qed

Remark 4.7. The relation (20) holds also for $f \in \mathcal{E}_q(\mathbb{R}_q)$ and $g \in \mathcal{D}_q(\mathbb{R}_q)$.

Corollary 4.8. The $q$-Dunkl transform $F^q_{D}$ admits the factorization

$$F^q_{D} = F^q_{D} \circ \mathcal{S}^q_{\alpha,\beta} \quad (21)$$

on $L^1(\mathbb{R}_q, |x|^{2\beta+1}d_qx)$. 

Since, by Proposition 6 of [1], we have \[ |\psi_{\lambda}^{q,a}(x)| \leq 4/(q,q)_\infty, \] for all \( x \in \mathbb{R}_q \), it follows that \( \psi_{\lambda}^{q,a} \in \mathcal{S}_q(\mathbb{R}_q) \cap L^1_q(\mathbb{R}_q) \). Hence, using the duality relation (20), we obtain

\[
F_D^{\beta,q}(f)(\lambda) = \frac{(1 + q)\beta}{2\Gamma q^2(\beta + 1)} \int_{-\infty}^{+\infty} f(x) S_{\alpha,\beta}^{q,a}(\psi_{\lambda}^{q,a}(x)) x^{2\beta+1} d_q x.
\]

Thus, \( F_D^{\beta,q}(f) = F_D^{a,q} \circ S_{\alpha,\beta}^{q,a}(f) \) for all \( f \in L^1(\mathbb{R}_q, |x|^{2\beta+1} d_q x) \).

Next, we extend the definition of the dual \( q \)-Dunkl intertwining operator, \( ^tV_\alpha \), to the space \( L^1(\mathbb{R}_q, |x|^{2\alpha+1} d_q x) \).

**Definition 4.9.** Let \( \alpha > -\frac{1}{2} \). We define the dual \( q \)-Dunkl intertwining operator \( ^tV_{\alpha,a} \), for \( f \in L^1(\mathbb{R}_q, |x|^{2\alpha+1} d_q x) \), by

\[
(^tV_{\alpha,a})(f)(x) = \frac{(1 + q)\alpha + \frac{1}{2}}{2\Gamma q^2(\alpha + \frac{1}{2})} \int_{|y| \geq |x|} W_\alpha \left( \frac{x}{y}; q^2 \right) \left( 1 + \frac{x}{y} \right) f(y) |y|^{2\alpha} d_q y, \quad x \in \mathbb{R}_q.
\]

**Remark 4.10.**

(a) For \( \alpha > -\frac{1}{2} \), we have \( ^tV_{\alpha,a} = ^tS_{\alpha+\frac{1}{2},\alpha}^{q,a} \).

(b) For \( \beta > \alpha \geq -\frac{1}{2} \), we have \( ^tS_{\alpha,\beta}^{q,a} = ^tV_{\beta-\alpha-\frac{1}{2},q} \).

**Proposition 4.11.**

(a) The dual \( q \)-Dunkl intertwining operator \( ^tV_{\alpha,a} \) is a continuous linear mapping from \( L^1(\mathbb{R}_q, |x|^{2\alpha+1} d_q x) \) into \( L^1(\mathbb{R}_q, d_q x) \).

(b) The \( q \)-Dunkl transform \( F_D^{a,q} \) is linked to the \( q^2 \)-analogue Fourier transform \( F_q \) by

\[
F_D^{a,q}(f) = (F_q \circ V_{\alpha,a})(f), \quad f \in L^1(\mathbb{R}_q, |x|^{2\alpha+1} d_q x).
\]

(c) The dual \( q \)-Dunkl intertwining operator \( ^tV_{\alpha,a} \) is a topological automorphism of \( \mathcal{S}_q(\mathbb{R}_q) \) and satisfies the transmutation relation

\[
\partial_q(^tV_{\alpha,a}(f)) = ^tV_{\alpha,a}(\Lambda_{\alpha,a} f), \quad f \in \mathcal{S}_q(\mathbb{R}_q).
\]
Proof. Part (a) easily follows from Proposition 4.6 (a) together with Remark 4.10. Part (b) follows from Corollary 4.8 and Remark 4.10. For part (c), it follows from part (b) that $V_{\alpha,q} = F_{q}^{-1} \circ F_{D}^{\alpha,q}$ on $\mathcal{S}_{q}(\mathbb{R}_{q})$. So, by Proposition 3.3 (a), $V_{\alpha,q}$ is a topological automorphism of $\mathcal{S}_{q}(\mathbb{R}_{q})$. Moreover, using part (b) of Proposition 3.3 together with (22), we obtain

$$F_{q}(\partial_{q}(V_{\alpha,q}(f))) = i\lambda F_{q}(V_{\alpha,q}(f))$$

$$= i\lambda F_{D}^{\alpha,q}(f) = F_{D}^{\alpha,q}(\Lambda_{\alpha,q}f) = F_{q}(\Lambda_{\alpha,q}f),$$

for all $f \in \mathcal{S}_{q}(\mathbb{R}_{q})$. Hence, (23) follows from the injectivity of $F_{q}$.

**Theorem 4.12.** Let $\alpha, \beta \in \frac{1}{2}, +\infty$ such that $\beta > \alpha$.

(a) The dual $q$-Dunkl-Sonne transform $^t\mathcal{S}_{q}(\alpha, \beta)$ is a topological automorphism of $\mathcal{S}_{q}(\mathbb{R}_{q})$, and satisfies the following relations:

$$^t\mathcal{S}_{q}(\alpha, \beta)(f) = (V_{\alpha,q})^{-1} \circ ^tV_{\beta,q}(f), \quad f \in \mathcal{S}_{q}(\mathbb{R}_{q}),$$

$$\Lambda_{\alpha,q}(\mathcal{S}_{q}^{\alpha,\beta}(f)) = \mathcal{S}_{q}^{\alpha,\beta}(\Lambda_{\beta,q}f), \quad f \in \mathcal{S}_{q}(\mathbb{R}_{q}).$$

(b) The $q$-Dunkl-Sonne transform $\mathcal{S}_{q}(\alpha, \beta)$ is a topological automorphism of $\mathcal{D}_{q}(\mathbb{R}_{q})$ and satisfies the following relations

$$\mathcal{S}_{q}^{\alpha,\beta}(f) = V_{\beta,q} \circ (V_{\alpha,q})^{-1}(f), \quad f \in \mathcal{D}_{q}(\mathbb{R}_{q}),$$

$$\Lambda_{\beta,q}(\mathcal{S}_{q}^{\alpha,\beta}(f)) = \mathcal{S}_{q}^{\alpha,\beta}(\Lambda_{\alpha,q}f), \quad f \in \mathcal{D}_{q}(\mathbb{R}_{q}).$$

Proof. For part (a), by Corollary 4.8, we have $^t\mathcal{S}_{q}(\alpha, \beta) = (F_{D}^{\alpha,q})^{-1} \circ F_{D}^{\beta,q}$. Hence, part (a) of Proposition 3.3 infers that $^t\mathcal{S}_{q}(\alpha, \beta)$ is a topological automorphism of $\mathcal{S}_{q}(\mathbb{R}_{q})$. Moreover, (24) follows immediately from (22). The proof of (25) runs in a similar way as the proof of (23) using Corollary 4.8 and Proposition 3.3 (b).

For (b), we start by proving (26) which is equivalent to $\mathcal{S}_{q}(\alpha, \beta) \circ V_{\alpha,q} = V_{\beta,q}$. It suffices, therefore, to prove that for all $f \in \mathcal{S}_{q}(\mathbb{R}_{q})$ and $g \in \mathcal{D}_{q}(\mathbb{R}_{q})$ we have

$$\int_{-\infty}^{+\infty} \mathcal{S}_{q}(\alpha, \beta)(V_{\alpha,q}f)(x)g(x)|x|^{2\beta+1}d_{q} = \int_{-\infty}^{+\infty} V_{\beta,q}f(x)g(x)|x|^{2\beta+1}d_{q}.x.$$

Let $f \in \mathcal{S}_{q}(\mathbb{R}_{q})$ and $g \in \mathcal{D}_{q}(\mathbb{R}_{q})$. To simplify the notations, set

$$K_{\alpha,q} = \frac{(1+q)^{\alpha+\frac{1}{2}}\Gamma_{q^{2}}(\alpha+1)}{\Gamma_{q^{2}}(\frac{1}{2})} \quad \text{and} \quad K_{\alpha,\beta}^{q} = \frac{(1+q)^{\beta-\alpha}\Gamma_{q^{2}}(\beta+1)}{\Gamma_{q^{2}}(\alpha+1)}.$$

By Remark 4.6, we can apply the duality relation (20) as follows

$$\int_{-\infty}^{+\infty} \mathcal{S}_{q}^{\alpha,\beta}(V_{\alpha,q}f)(x)g(x)|x|^{2\beta+1}d_{q} = K_{\alpha,\beta}^{q} \int_{-\infty}^{+\infty} V_{\alpha,q}f(x)^{t}\mathcal{S}_{q}^{\alpha,\beta}(g)(x)|x|^{2\alpha+1}d_{q}.$$

Since, obviously, $^t\mathcal{S}_{q}^{\alpha,\beta}(g) \in \mathcal{D}_{q}(\mathbb{R}_{q})$, we can apply the duality relation (7). We obtain
Hence, using the duality relation (7) and the fact that $K^q_{\alpha,\beta}$ is a topological isomorphism of $E_q(\mathbb{R}_q)$, we get
\[
\int_{-\infty}^{+\infty} V_{\alpha,q} f(x) S^q_{\alpha,\beta}(g)(x) |x|^{2\alpha+1} d_q x = K_{\alpha,q} \int_{-\infty}^{+\infty} f(x) \left( V_{\alpha,q} \circ S^q_{\alpha,\beta} \right)(g)(x) d_q x.
\]

Using (24) we get
\[
\int_{-\infty}^{+\infty} V_{\alpha,q} f(x) S^q_{\alpha,\beta}(g)(x) |x|^{2\alpha+1} d_q x = K_{\alpha,q} \int_{-\infty}^{+\infty} f(x) \left( V_{\beta,q} \right)(g)(x) d_q x.
\]

Hence, using the duality relation (7) and the fact that $K^q_{\alpha,\beta} = \frac{K_{\beta,q}}{K_{\alpha,q}}$, we deduce that
\[
\int_{-\infty}^{+\infty} S^q_{\alpha,\beta}(V_{\alpha,q} f)(x) g(x) |x|^{2\beta+1} d_q x = \int_{-\infty}^{+\infty} V_{\beta,q} f(x) g(x) |x|^{2\beta+1} d_q x.
\]

This achieves the proof of (26) which, together with the fact that, as mentioned in Section 2, $V_{\alpha,q}$ is a topological isomorphism of $E_q(\mathbb{R}_q)$, infers that $S^q_{\alpha,\beta}$ is a topological automorphism of $E_q(\mathbb{R}_q)$.

To prove (27), let $f \in E_q(\mathbb{R}_q)$. From the factorization Relation (26) and the transmutation relation (8), we have
\[
\Lambda_{\beta,q}(S^q_{\alpha,\beta}(f)) = \Lambda_{\beta,q} V_{\beta,q}(V^{-1}_{\alpha,q}(f)) = V_{\beta,q} \left( \partial_q V^{-1}_{\alpha,q}(f) \right).
\]

Since (8) implies that $\partial_q (V^{-1}_{\alpha,q} f) = V^{-1}_{\alpha,q} (\Lambda_{\alpha,q} f)$, it follows that
\[
\Lambda_{\beta,q}(S^q_{\alpha,\beta}(f)) = V_{\beta,q} \circ V^{-1}_{\alpha,q}(\Lambda_{\alpha,q} f) = S^q_{\alpha,\beta}(\Lambda_{\alpha,q} f).
\]

This completes the proof of the theorem. \( \square \)

5. Inversion formulas for the $q$-Dunkl-Soneine transform and its dual using $q$-pseudo-differential operators

In this section, we give inversion formulas for the $q$-Dunkl sonine transform $S^q_{\alpha,\beta}$ and its dual $iS^q_{\alpha,\beta}$ using $q$-pseudo-differential operators. Next, we give Plancherel formula for the dual $q$-Dunkl-sonine transform.

We begin by introducing the $q$-analogues of the Lizorkin spaces (see [22]):

- $\Phi_q(\mathbb{R}_q) = \{ f \in \mathcal{S}_q(\mathbb{R}_q) : \int_{-\infty}^{+\infty} f(x) |x|^{2\alpha+k+1} = 0, \; k = 0, 1, \ldots \}$;
- $\Psi_q(\mathbb{R}_q) = \{ f \in \mathcal{S}_q(\mathbb{R}_q) : \partial^k_q f(0) = 0, \; k = 0, 1, \ldots \}$.

**Proposition 5.1.** (see [2]) For every $\alpha \geq -1/2$, the $q$-Dunkl transform $F^q_\alpha$ is an isomorphism from $\Phi_q(\mathbb{R}_q)$ into $\Psi_q(\mathbb{R}_q)$.

**Lemma 5.2.** (see [2, 3]) For every $\lambda \in \mathbb{C}$, the multiplication operator $M_{-\lambda} : f \mapsto |x|^\lambda f$ is a topological automorphism of $\Psi_q(\mathbb{R}_q)$, its inverse operator is $M_{-\lambda}$.
Proposition 5.3. If \( f \in \Phi_\alpha^q(\mathbb{R}_q) \), then for all \( \lambda \in \mathbb{R}_q \), we have

\[
F_D^\beta,q(S_{\alpha,\beta}^q g f)(\lambda) = \frac{(1 + q)^\beta \Gamma_{q^2}(\beta + 1)}{(1 + q)^\alpha \Gamma_{q^2}(\alpha + 1)} \frac{1}{|\lambda|^{2(\beta - \alpha)}} F_D^{\alpha,q}(f)(\lambda).
\]  

Proof. By the inversion formula (11), we have

\[
f(x) = \frac{(1 + q)^{-\alpha}}{2 \Gamma_{q^2}(\alpha + 1)} \int_{-\infty}^{+\infty} F_D^{\alpha,q}(f)(\lambda) \psi_\lambda^{\alpha,q}(x) |\lambda|^{\alpha + 1} d_\alpha \lambda, \quad x \in \mathbb{R}_q.
\]

Using Fubini’s theorem and the fact that \( \psi_\lambda^{\beta,q} = S_{\alpha,\beta}^q (\psi_\lambda^{\alpha,q}) \), we obtain

\[
S_{\alpha,\beta}^q f(x) = \frac{(1 + q)^{-\alpha}}{2 \Gamma_{q^2}(\alpha + 1)} \int_{-\infty}^{+\infty} F_D^{\alpha,q}(f)(\lambda) \psi_\lambda^{\beta,q}(x) |\lambda|^{\alpha + 1} d_\alpha \lambda
\]

for all \( x \in \mathbb{R}_q \), where

\[
h_{\alpha,\beta}^q(\lambda) = \frac{(1 + q)^\beta \Gamma_{q^2}(\beta + 1)}{(1 + q)^\alpha \Gamma_{q^2}(\alpha + 1)} \frac{1}{|\lambda|^{2(\beta - \alpha)}} F_D^{\alpha,q}(f)(\lambda), \quad \lambda \in \mathbb{R}_q.
\]

Since \( f \in \Phi_\alpha^q(\mathbb{R}_q) \), it follows from Proposition 5.1 that \( F_D^{\alpha,q}(f) \in \Psi_q(\mathbb{R}_q) \) and hence, Lemma 5.2 infers that \( h_{\alpha,\beta}^q \in \Psi_q(\mathbb{R}_q) \), and the conclusion of the proposition follows from the above inversion formula.

\[\square\]

Definition 5.4. We define the operators \( K_{1,q}, K_{2,q} \) and \( K_{3,q} \) by

\[
K_{1,q}(f) = \frac{(1 + q)^\alpha \Gamma_{q^2}(\alpha + 1)}{(1 + q)^\beta \Gamma_{q^2}(\beta + 1)} (F_D^{\alpha,q})^{-1}(\lambda)^{2(\beta - \alpha)} F_D^{\alpha,q}(f), \quad f \in \Phi_\alpha^q(\mathbb{R}_q);
\]

\[
K_{2,q}(f) = \frac{(1 + q)^\alpha \Gamma_{q^2}(\alpha + 1)}{(1 + q)^\beta \Gamma_{q^2}(\beta + 1)} (F_D^{\beta,q})^{-1}(\lambda)^{2(\beta - \alpha)} F_D^{\beta,q}(f), \quad f \in \Phi_\beta^q(\mathbb{R}_q);
\]

\[
K_{3,q}(f) = (F_D^{\alpha,q})^{-1}(\lambda)^{\beta - \alpha} F_D^{\alpha,q}(f), \quad f \in \Phi_\alpha^q(\mathbb{R}_q).
\]

Using the \( q \)-pseudo-differential operators \( K_{1,q}, K_{2,q} \), we give in the following theorem, inversion formulas for the \( q \)-Dunkl Sonine operator \( S_{\alpha,\beta}^q \) and its dual \( S_{\alpha,\beta}^q \), and, using the \( q \)-pseudo-differential operator \( K_{3,q} \), we give Plancherel formula for \( i S_{\alpha,\beta}^q \).

Theorem 5.5.

(a) Inversion formulas: For all \( f \in \Phi_\alpha^q(\mathbb{R}_q) \) and \( g \in \Phi_\beta^q(\mathbb{R}_q) \), we have the following inversion formulas:

\[
(i) \quad f = (i S_{\alpha,\beta}^q K_{2,q} S_{\alpha,\beta}^q)(f).
\]

\[
(ii) \quad f = (K_{1,q} i S_{\alpha,\beta}^q S_{\alpha,\beta}^q)(f).
\]
Definition 6.1. Let

(iii) \( g = (K_{2,q}S_{\alpha,\beta}^q \ell S_{\alpha,\beta}^q)(g) \).

(iv) \( g = (S_{\alpha,\beta}K_{1,q} \ell S_{\alpha,\beta}^q)(g) \).

(b) Plancherel formula: For all \( f \in \Phi_\beta^q(\mathbb{R}_q) \), we have

\[
\int_{-\infty}^{+\infty} |f(t)|^2 |t|^{2\beta+1} dt = \int_{-\infty}^{+\infty} |K_{3,q}(S_{\alpha,\beta}^q(f))(t)|^2 |t|^{2\alpha+1} dt.
\]

Proof. Part (a) Follows immediately from Proposition 5.3 together with the factorization relation (24).

For part (b), let \( f \in \Phi_\beta^q(\mathbb{R}_q) \). Using the Plancherel formula (12) together with the factorization relation (24), we obtain

\[
\int_{-\infty}^{+\infty} |f(t)|^2 |t|^{2\beta+1} dt = \int_{-\infty}^{+\infty} |F_D^\beta(f)(\lambda)|^2 |\lambda|^{2\beta+1} d\lambda
\]

\[
= \int_{-\infty}^{+\infty} |\lambda|^{(\beta-\alpha)} F_D^{\alpha,q}(S_{\alpha,\beta}^q(f))(\lambda)|^2 |\lambda|^{2\alpha+1} d\lambda
\]

\[
= \int_{-\infty}^{+\infty} |F_D^{\alpha,q}(K_{3,q}(S_{\alpha,\beta}^q(f)))(\lambda)|^2 |\lambda|^{2\alpha+1} d\lambda
\]

\[
= \int_{-\infty}^{+\infty} |K_{3,q}(S_{\alpha,\beta}^q(f))(t)|^2 |t|^{2\alpha+1} dt.\]

This achieves the proof. \( \square \)

6. Inversions of the \( q \)-Dunkl-Sonine transform and its dual by the use of the \( q \)-Dunkl wavelets

We begin this section by summarizing some facts about \( q \)-Dunkl wavelets introduced and studied in [2]. Next, we study the effect of the \( q \)-Dunkl-Sonine transform \( S_{\alpha,\beta}^q \) and its dual \( \ell S_{\alpha,\beta}^q \) on \( q \)-Dunkl wavelets and as applications, we give the inversion formulas for these integral transforms using \( q \)-Dunkl wavelets.

Definition 6.1. Let \( g \in L^2_{\alpha,q}(\mathbb{R}_q) \). We say that \( g \) is a \( q \)-wavelet associated with the \( q \)-Dunkl operator \( \Lambda_{\alpha,q} \), if it satisfies the following admissibility condition

\[
0 < C_{\alpha,q}(g) = \int_{-\infty}^{+\infty} |F_D^{\alpha,q}(g)(a)|^2 \frac{d\alpha}{|a|} < \infty. \quad (29)
\]

Proposition 6.2. Let \( g \neq 0 \) be a function in \( L^2_{\alpha,q}(\mathbb{R}_q) \). If \( g \) satisfies

- The function \( F_D^{\alpha,q}(g) \) is continuous at 0.
- There exists \( \nu > 0 \) such that

\[
F_D^{\alpha,q}(g)(x) - F_D^{\alpha,q}(g)(0) = O(x^\nu) \quad \text{as} \quad x \to 0, \quad x \in \mathbb{R}_q.
\]

Then, the admissibility condition (29) is equivalent to \( F_D^{\alpha,q}(g)(0) = 0 \).
For \( g \in \mathcal{S}^q_\alpha(R_q), a \in R_q \) and \( b \in R_q \cup \{0\} \), we define the function \( g_{a,b}^{\alpha,q} \) by

\[
g_{a,b}^{\alpha,q}(x) = \sqrt{|a|} T_b^{\alpha,q}(g_{a}^{\alpha,q})(x), \quad x \in R_q,
\]

where \( T_b^{\alpha,q} \) is the generalized \( q \)-Dunkl translation operator defined by (13) and

\[
g_a^{\alpha,q}(x) = \frac{1}{|a|^{2\alpha+2}} g\left(\frac{x}{a}\right), \quad x \in R_q.
\]

In the following proposition, we give some properties of function \( g_{a}^{\alpha,q} \), the proof is straightforward.

**Proposition 6.3.** If \( g \in \mathcal{S}^q_\alpha(R_q) \) and \( a \in R_q \) then:

(a) \( g_{a}^{\alpha,q} \in \mathcal{S}^q_\alpha(R_q) \).

(b) \( F_D^{\alpha,q}(g_{a}^{\alpha,q})(\lambda) = F_D^{\alpha,q}(g)(a\lambda) \) and \( \|g_{a}^{\alpha,q}\|_{2,\alpha,q} = \|g\|_{2,\alpha,q} \).

(c) \( K_{1,q}(g_{a}^{\alpha,q}) = \frac{1}{|a|^{2(\beta-\alpha)}} (K_{2,q}(g))^{\alpha,q} \) and

\[
K_{2,q}(g_{a}^{\alpha,q}) = \frac{1}{|a|^{2(\beta-\alpha)}} (K_{2,q}(g))^{\beta,q}
\]

where \( K_{1,q} \) and \( K_{2,q} \) are given by Definition 5.4.

(d) \( \mathcal{S}^q_{\alpha,q}(g_{a}^{\alpha,q}) = (\mathcal{S}^q_{\alpha,\beta}(g))^{\beta,q} \) and \( \mathcal{S}^q_{\alpha,q}(g_{a}^{\alpha,q}) = (\mathcal{S}^q_{\alpha,\beta}(g))^{\alpha,q} \).

**Proposition 6.4.** If \( g \) is a \( q \)-wavelet associated with the \( q \)-Dunkl operator \( \Lambda_{\alpha,q} \) in \( \mathcal{S}^q_\alpha(R_q) \), then for all \( a \in R_q \) and \( b \in R_q \cup \{0\} \), the function \( g_{a,b}^{\alpha,q} \) is a \( q \)-wavelet associated with the \( q \)-Dunkl operator \( \Lambda_{\alpha,q} \) in \( \mathcal{S}^q_\alpha(R_q) \) and we have

\[
C_{\alpha,q}(g_{a,b}^{\alpha,q}) = |a| \int_{-\infty}^{\infty} \left| \psi_{b}^{\alpha,q}(\frac{x}{a}) \right|^2 |F_D^{\alpha,q}(g)(x)|^2 d_qx.
\]

**Definition 6.5.** Let \( g \) be a \( q \)-wavelet associated with the \( q \)-Dunkl operator \( \Lambda_{\alpha,q} \) in \( \mathcal{S}^q_\alpha(R_q) \). We define the continuous \( q \)-wavelet transform associated with the \( q \)-Dunkl operator \( \Lambda_{\alpha,q} \) for \( f \in \mathcal{S}^q_\alpha(R_q) \) by

\[
\psi_{g}^{\alpha,q}(f)(a,b) = \frac{(1+q)^{-\alpha}}{2\Gamma_q(\alpha+1)} \int_{-\infty}^{\infty} f(x)g_{a,b}^{\alpha,q}(-x)|x|^{2\alpha+1} d_qx, \quad a \in R_q, b \in R_q \cup \{0\},
\]

which can be written as

\[
\psi_{g}^{\alpha,q}(f)(a,b) = \sqrt{|a|} \left( f \ast_{\alpha,q} \overline{g_{a}^{\alpha,q}} \right)(b), \quad a \in R_q, b \in R_q \cup \{0\}.
\]

**Theorem 6.6.** Inversion formula: If \( g \in \mathcal{S}^q_\alpha(R_q) \) is a \( q \)-Dunkl wavelet associated with the \( q \)-Dunkl operator \( \Lambda_{\alpha,q} \), then for all \( f \in \mathcal{S}^q_\alpha(R_q) \) we have

\[
f(x) = \frac{(1+q)^{-\alpha}}{2\Gamma_q(\alpha+1)C_{\alpha,q}(g)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi_{q,g}^{\alpha,q}(f)(a,b)g_{a,b}^{\alpha,q}(-x)|b|^{2\alpha+1} d_qad_qb \frac{d_qx}{|a|^2},
\]

for all \( x \in R_q \).
Note that for the continuous $q$-Dunkl wavelet transform, there are also Parseval and Plancherel formulas (see [2] for more details).

**Lemma 6.7.** (see [2])

(a) If $f \in \Phi^q_\alpha(\mathbb{R}_q)$ and $g \in \mathcal{S}_q(\mathbb{R}_q)$, then $f *_{\alpha,q} g \in \Phi^q_\alpha(\mathbb{R}_q)$ and we have

$$K_{1,q}(f *_{\alpha,q} g) = (K_{1,q}f) *_{\alpha,q} g.$$  \hspace{1cm} (33)

(b) If $f \in \Phi^q_\beta(\mathbb{R}_q)$ and $g \in \mathcal{S}_q(\mathbb{R}_q)$, then $f *_{\beta,q} g \in \Phi^q_\beta(\mathbb{R}_q)$ and we have

$$K_{2,q}(f *_{\beta,q} g) = (K_{2,q}f) *_{\beta,q} g.$$  \hspace{1cm} (34)

**Proposition 6.8.**

(a) If $f, g \in \mathcal{S}_q(\mathbb{R}_q)$, then

$$\mathcal{S}^q_{\alpha,\beta}(f *_{\beta} g) = \mathcal{S}^q_{\alpha,\beta}f *_{\alpha} \mathcal{S}^q_{\alpha,\beta}g.$$  \hspace{1cm} (35)

(b) If $f \in \mathcal{S}_q(\mathbb{R}_q)$ and $g \in \Phi^q_\alpha(\mathbb{R}_q)$, then

$$\mathcal{S}^q_{\alpha,\beta}(\mathcal{S}^q_{\alpha,\beta}f *_{\alpha,q} g) = f *_{\beta,q} \mathcal{S}^q_{\alpha,\beta}g.$$  \hspace{1cm} (36)

**Proof.** Part (a) follows by applying $F^q_{D,\alpha,\beta}$ to both sides of the equation (35).

For part (b), by the inversion formula (ii) of Theorem 5.5, we have

$$\mathcal{S}^q_{\alpha,\beta}(\mathcal{S}^q_{\alpha,\beta}f *_{\alpha,q} g) = \mathcal{S}^q_{\alpha,\beta}\left[\mathcal{S}^q_{\alpha,\beta}f *_{\alpha}(K_{1,q}\mathcal{S}^q_{\alpha,\beta}\mathcal{S}^q_{\alpha,\beta}(g))\right].$$

Using Lemma 6.7 (a), we get

$$\mathcal{S}^q_{\alpha,\beta}(\mathcal{S}^q_{\alpha,\beta}f *_{\alpha,q} g) = \mathcal{S}^q_{\alpha,\beta}(K_{1,q}\mathcal{S}^q_{\alpha,\beta}(\mathcal{S}^q_{\alpha,\beta}(g)))$$

So, by part (a) we obtain

$$\mathcal{S}^q_{\alpha,\beta}(\mathcal{S}^q_{\alpha,\beta}f *_{\alpha,q} g) = \mathcal{S}^q_{\alpha,\beta}(K_{1,q}\mathcal{S}^q_{\alpha,\beta}(f *_{\beta,q} \mathcal{S}^q_{\alpha,\beta}g)),$$

and (36) follows from the inversion formula (iv) of Theorem 5.5.

Now, we are in a situation to discuss the effect of the $q$-Dunkl-Sonine transform $\mathcal{S}^q_{\alpha,\beta}$ and its dual $\mathcal{S}^q_{\alpha,\beta}$ on the $q$-Dunkl wavelets.

**Proposition 6.9.** If $g$ is a $q$-wavelet associated with the $q$-Dunkl operator $\Lambda_{\beta,q}$ in $\mathcal{S}_q(\mathbb{R}_q)$ (respectively in $\Phi^q_\beta(\mathbb{R}_q)$), then $\mathcal{S}^q_{\alpha,\beta}g$ is a $q$-wavelet associated with the $q$-Dunkl operator $\Lambda_{\alpha,q}$ in $\mathcal{S}_q(\mathbb{R}_q)$ (respectively in $\Phi^q_\alpha(\mathbb{R}_q)$) and we have

$$C_{\alpha,q}(\mathcal{S}^q_{\alpha,\beta}g) = C_{\beta,q}(g).$$
Proof. Assume that \( g \) is a \( q \)-wavelet associated with the \( q \)-Dunkl operator \( \Lambda_{\beta,q} \) in \( S_q(\mathbb{R}_q) \). Then Theorem 4.12 (a) implies that \( S_q^{\alpha,\beta}g \in S_q(\mathbb{R}_q) \). Moreover, using the factorization relation (21) together with the Plancherel formula (12) we deduce that \( C_{\alpha,q}(S_q^{\alpha,\beta}) = C_{\beta,q}(g) \). Hence, \( S_q^{\alpha,\beta}g \) is a \( q \)-wavelet associated with the operator \( \Lambda_{\alpha,q} \) in \( S_q(\mathbb{R}_q) \). To complete the proof, it remains only to show that if \( g \in \Phi_\beta^q(\mathbb{R}_q) \) then \( S_q^{\alpha,\beta}g \in \Phi_\alpha^q(\mathbb{R}_q) \). This follows from the fact that \( S_q^{\alpha,\beta}g = (F_{D,q}^{-1} \circ F_{D,q}^\beta) \) on \( S_q(\mathbb{R}_q) \) and Proposition 6.1.

\[
\Psi_{S,q}^{\beta,q}(f)(a,b) = S_{\alpha,\beta}^{\alpha,q} \left[ \Psi_{S,q}^{\alpha,q} \left( (S_{\alpha,\beta}^{-1}f)(a, \cdot) \right) \right] (b), \quad a, b \in \mathbb{R}_q, \tag{37}
\]

and

\[
\Psi_{S,q}^{\alpha,q}(f)(a,b) = S_{\alpha,\beta}^{\alpha,q} \left[ \Psi_{S,q}^{\beta,q} \left( (S_{\alpha,\beta}^{-1}f)(a, \cdot) \right) \right] (b), \quad a, b \in \mathbb{R}_q. \tag{38}
\]

Proof. It follows, from Definition 6.5 together with Proposition 6.8 (b), that

\[
\Psi_{S,q}^{\beta,q}(f)(a,b) = \sqrt{|a|} \left( \frac{g^\beta_q}{g^\alpha_q} \ast \beta f \right)(b) = \sqrt{|a|} S_{\alpha,\beta} \left( S_{\alpha,\beta}^q \left( g^\beta_q \right) \ast \alpha \left( S_{\alpha,\beta}^q \right)^{-1}f \right)(b) = S_{\alpha,\beta}^q \left( \sqrt{|a|} \left( S_{\alpha,\beta}^q \right)^{-1}f \ast \alpha \left( S_{\alpha,\beta}^q \right) \right)(b).
\]

Since \( S_{\alpha,\beta}^q \left( g^\beta_q \right) = \left( S_{\alpha,\beta}^q \left( g \right) \right)^{\alpha,q} \), as mentioned in Proposition 6.3 (d), and \( S_{\alpha,\beta}^q \) is a \( q \)-Dunkl wavelet, by Proposition 6.9, we can write

\[
\Psi_{S,q}^{\beta,q}(f)(a,b) = S_{\alpha,\beta}^q \left( \sqrt{|a|} \left( S_{\alpha,\beta}^q \right)^{-1}f \ast \alpha \left( S_{\alpha,\beta}^q \right) \right)(b) = S_{\alpha,\beta}^q \left[ \Psi_{S,q}^{\alpha,q} \left( (S_{\alpha,\beta}^{-1}f)(a, \cdot) \right) \right](b).
\]

Thus, we have proved (37). Similarly, (38) is derived using Proposition 6.8 (b) instead of Proposition 6.8 (a).

Lemma 6.11.

(a) If \( g \in \Phi_\beta^q(\mathbb{R}_q) \) is a \( q \)-wavelet associated with the \( q \)-Dunkl operator \( \Lambda_{\alpha,q} \), then \( K_{1,q}g \) is a \( q \)-wavelet associated with the \( q \)-Dunkl operator \( \Lambda_{\alpha,q} \).

(b) If \( g \in \Phi_\beta^q(\mathbb{R}_q) \) is a \( q \)-wavelet associated with the \( q \)-Dunkl operator \( \Lambda_{\beta,q} \), then \( K_{2,q}g \) is a \( q \)-wavelet associated with he \( q \)-Dunkl operator \( \Lambda_{\beta,q} \).
Proof. For part (a), by Proposition 5.1, we have \( K_{1,q} g \in \Psi^q_{\alpha}(\mathbb{R}_q) \) and by Definition 5.4, we have
\[
F_D^{\alpha, q}(K_{1,q} g)(\lambda) = |\lambda|^{2(\beta - \alpha)} F_D^{\alpha, q}(g)(\lambda).
\]
Since \( F_D^{\alpha, q}(g) \) is continuous at the origin, it follows that
\[
F_D^{\alpha, q}(K_{1,q} g)(\lambda) = O(|\lambda|^{2(\beta - \alpha)}) \quad (\lambda \to 0).
\]
Hence, by Proposition 6.2, the function \( K_{1,q} g \) is a \( q \)-wavelet associated with the \( q \)-Dunkl operator \( \Lambda_{\alpha,q} \).

The proof of part (b) is similar to the proof of part (a).

**Proposition 6.12.** If \( g \) is a \( q \)-wavelet associated with the \( q \)-Dunkl operator \( \Lambda_{\beta,q} \) in \( \Phi^q_{\beta}(\mathbb{R}_q) \), then

(a) For all \( f \in \Phi^q_{\beta}(\mathbb{R}_q) \) and all \( a, b \in \mathbb{R}_q \), we have:
\[
\Psi^q_{\alpha,q} \left( (S^q_{\alpha,\beta})^{-1} f \right)(a,b) = \frac{1}{|a|^{2(\beta - \alpha)}} S^q_{\alpha,\beta} \left[ \Psi^q_{\alpha,q} K_{q,1}(S^q_{\alpha,\beta} f)(a,.) \right](b).
\]

(b) For all \( f \in \Phi^q_{\beta}(\mathbb{R}_q) \) and all \( a, b \in \mathbb{R}_q \) we have:
\[
\Psi^q_{\alpha,q} \left( i S^q_{\alpha,\beta} \right)(a,b) = \frac{1}{|a|^{2(\beta - \alpha)}} i S^q_{\alpha,\beta} \left[ \Psi^q_{\alpha,q} (S^q_{\alpha,\beta} f)(a,.) \right](b).
\]

Proof. We prove only part (a), the proof of part (b) is similar. Let \( f \in \Phi^q_{\alpha}(\mathbb{R}_q) \). Since by the inversion formula (ii) of Theorem 5.5 we have
\[
(S^q_{\alpha,\beta})^{-1} f = K_{1,q}(i S^q_{\alpha,\beta})(f),
\]
it follows that
\[
\Psi^q_{i S^q_{\alpha,\beta}} \left( (S^q_{\alpha,\beta})^{-1} f \right)(a,b) = \left( (K_{1,q} i S^q_{\alpha,\beta})(f) *_{\alpha,q} (i S^q_{\alpha,\beta} g)_{a} \right)(b).
\]
Using Lemma 6.7 (a), we obtain
\[
\Psi^q_{i S^q_{\alpha,\beta}} \left( (S^q_{\alpha,\beta})^{-1} f \right)(a,b) = \left( i S^q_{\alpha,\beta} f *_{\alpha,q} K_{1,q} \left( (i S^q_{\alpha,\beta} g)_{a} \right) \right)(b).
\]
By simple computation, we can see that
\[
K_{1,q}(\left( (i S^q_{\alpha,\beta} g)_{a} \right) = K_{1,q}\left( (i S^q_{\alpha,\beta} g)_{a} \right).\]
Using Proposition 6.3 (d), we obtain
\[
\Psi^q_{i S^q_{\alpha,\beta}} \left( (S^q_{\alpha,\beta})^{-1} f \right)(a,b) = \frac{1}{|a|^{2(\beta - \alpha)}} \left( i S^q_{\alpha,\beta} f *_{\alpha,q} (K_{1,q} i S^q_{\alpha,\beta} g)_{a} \right)(b)
\]
\[
= \frac{1}{|a|^{2(\beta - \alpha)}} \Psi^q_{K_{1,q} i S^q_{\alpha,\beta}} (i S^q_{\alpha,\beta} f)(a,b).
\]
Hence, (37) can be written as
\[
\Psi^{\beta, q}_{g}(f)(a, b) = \frac{1}{|a|^{2(\beta - \alpha)}} S^q_{\alpha, \beta} \left[ \Psi^{\alpha, q}_{g}(S^q_{\alpha, \beta} f)(a, \cdot) \right](b),
\]
which achieves the proof. □

Now, using Theorem 6.6 together with Proposition 6.12, we get inversion formulas for the \(q\)-dunkl-sonine transform and its dual. This is the purpose of the following theorem.

**Theorem 6.13.** Let \(g\) be a \(q\)-wavelet associated with the \(q\)-Dunkl operator \(\Lambda_{\beta, q}\) in \(\Phi^q_{\beta}(\mathbb{R}_q)\) and set
\[
\tilde{C}_{\beta, q}(g) = \frac{(1 + q)^{-\beta}}{2\Gamma_q(\beta + 1) C_{\beta, q}(g)}.
\]

(a) If \(f \in \Phi^q_{\alpha}(\mathbb{R}_q)\), then for all \(x \in \mathbb{R}_q\), we have
\[
(S^q_{\alpha, \beta})^{-1}(f)(x) = \tilde{C}_{\beta, q}(g) \times \\
\int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} S^q_{\alpha, \beta} \left[ \Psi^{\alpha, q}_{g}(S^q_{\alpha, \beta} f)(a, \cdot) \right](b) g^{\beta, q}_{a, b}(-x) \frac{|b|^{2\beta + 1}}{|a|^{2(\beta - \alpha) + 1}} dq a \right) dq a.
\]

(b) If \(f \in S_{\beta, q}(\mathbb{R}_q)\), then for all \(x \in \mathbb{R}_q\), we have
\[
S^{-1}_{\alpha, \beta}(f)(x) = \tilde{C}_{\beta, q}(g) \times \\
\int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} S_{\alpha, \beta} \left[ \Psi^{\beta, q}_{g}(f)(a, \cdot) \right](b) S_{\alpha, \beta}(g^{\beta, q}_{a, b})(-x) \frac{|b|^{2\beta + 1}}{|a|^{2(\beta - \alpha) + 1}} dq a \right) dq a.
\]

**REFERENCES**


LASSAD BENNASR
Département de Mathématiques et Informatique
Institut préparatoire aux études d’ingénieur d’Elmanar
Tunis, Tunisie
e-mail: bennasr.lassad@yahoo.fr

RYM HASNA BETTAIEB
Department of Mathematics
Faculty of Sciences - King Faisal University
P.O. Box 400, 31982 Al-Ahasa
Kingdom of Saudi Arabia
e-mail: rym.bettaieb@yahoo.fr

FERJANI NOURI
Département de Mathématiques et Informatique
Institut préparatoire aux études d’ingénieur de Nabeul
8000 Nabeul, Tunisie
e-mail: nouri.ferjani@yahoo.fr