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ON A q-DUNKL SONINE TRANSFORM

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In this paper, we introduce and study the q-Dunkl Sonine transform and we establish a Plancherel formula for its dual. Furthermore we give many inversion formulas.

1. Introduction

The transmutation operators allow the transfer of some well known results related to a well known operator to those related to a new one, they play a central role in many areas of Mathematics and mathematical physics such as spectral theory, harmonic analysis, special functions and fractional calculus. This theory, introduced firstly by Delsart and Lions (see [5]), was extended by many authors (see for instance [3, 4, 10, 12, 13, 16–18, 22, 23, 25]), and recently was extended to quantum calculus in [1–3, 9, 10].

In this paper we introduce and study the so-called q-Dunkl Sonine operator and its dual, we show that these operators are transmutation operators between two different q-Dunkl operators that generalize the q-Dunkl intertwining and its dual introduced in [1]. Furthermore, we give various inversion formulas for these operators.

This paper is organized as follows: in Section 2, we present some new preliminaries that we need. In Section 3, we collect some elements of q-Dunkl har-

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monic analysis (q-Dunkl kernel, q-Dunkl transform, q-Dunkl convolution,...). In section 4 we introduce the q-Dunkl-Sonine transform by

$$\mathcal{S}^q_{\alpha,\beta}(f)(x) = \frac{(1+q)\Gamma_{q^2}(\beta+1)}{2\Gamma_{q^2}(\alpha+1)\Gamma_{q^2}(\beta-\alpha)}\int_{-1}^{1} f(xt)W_{\beta-\alpha-\frac{1}{2}}(t;q^2)(1+t)|t|^{2\alpha+1}d_qt,$$

where $\beta > \alpha \geq -1/2$ and

$$W_\alpha(t,q^2) = \frac{(t^2q^2;q^2)_\infty}{(t^2q^{2\alpha+1};q^2)_\infty},$$

is the q-analogue of the kernel $W_\alpha(t) = (1-t^2)^{\alpha-1/2}, \quad t \in]-1,1[$. We show that $\mathcal{S}^q_{\alpha,\beta}$ and its dual ${}^t\mathcal{S}^q_{\alpha,\beta}$ are transmutation operators between the q-Dunkl operators $\Lambda_{\alpha,q}$ and $\Lambda_{\beta,q}$. We also deal with the relations between these transforms and the q-Dunkl intertwining operator and its dual. In Sections 5 we give many formulations of the inversion formulas of the q-Dunkl Sonine transform and its dual using q-pseudo-differential operators in some q-analogues of the Lizorkin spaces and we give Plancherel formula for the dual q-Dunkl-Sonine transform. Section 6 is devoted to inversions formulas of the q-Dunkl Sonine transform and its dual by using q-Dunkl wavelets.

2. Notations and preliminaries

Throughout this paper, we assume $q \in]0,1[$. We write $\mathbb{R}_q = \{\pm q^n : n \in \mathbb{Z}\}$ and we use the convention $\mathbb{N} = \{0,1,2,\dots\}$. The q-shifted factorials of $a \in \mathbb{C}$ are defined as

$$(a;q)_0 = 1; \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \quad n = 1,2,\dots; \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

The q-Factoriel of $n \in \mathbb{N}$ is $[n]_q! = \dfrac{(q;q)_n}{(1-q)^n}$ and more generally, the q-Gamma function is defined for $x \in \mathbb{C}$ by (see [11])

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \qquad x \neq 0,-1,-2,\dots.$$

The normalized third Jackson's q-Bessel function is defined for $x \in \mathbb{C}$ by (see [9, 14])

$$j_\alpha(x;q^2) = \Gamma_{q^2}(\alpha+1)\sum_{n=0}^{+\infty}\frac{(-1)^n q^{n(n+1)}}{\Gamma_{q^2}(\alpha+n+1)\Gamma_{q^2}(n+1)}\left(\frac{x}{1+q}\right)^{2n}. \tag{1}$$

The q-trigonometric functions $\cos(x;q^2)$ and $\sin(x;q^2)$ are defined for $x \in \mathbb{C}$ by

$$\cos(x;q^2) = j_{-\frac{1}{2}}(x;q^2) = \sum_{n=0}^{+\infty} (-1)^n q^{n(n+1)} \frac{x^{2n}}{[2n]_q!},$$
$$\sin(x;q^2) = x j_{\frac{1}{2}}(x;q^2) = \sum_{n=0}^{+\infty} (-1)^n q^{n(n+1)} \frac{x^{2n+1}}{[2n+1]_q!}.$$

In connection with q^2-analogue Fourier analysis, R. Rubin [20, 21] constructed a q^2-analogue of the exponential function, $e(x;q^2)$, and a q^2-derivative ∂_q as follows :

$$e(ix;q^2) = \cos(x,q^2) + i\sin(x;q^2), \qquad x \in \mathbb{C},$$

and

$$\partial_q(f)(x) = \begin{cases} \dfrac{f\left(q^{-1}x\right) + f\left(-q^{-1}x\right) - f\left(qx\right) + f\left(-qx\right) - 2f(-x)}{2(1-q)x} & \text{if } x \neq 0; \\ \lim\limits_{x\to 0} \partial_q(f)(x) \qquad (\text{in } \mathbb{R}_q) & \text{if } x = 0 \end{cases}$$

so that for every $\lambda \in \mathbb{C}$ the q^2-exponential function satisfies

$$\partial_q(e(i\lambda x;q^2) = i\lambda e(i\lambda x;q^2), \qquad x \in \mathbb{C}.$$

The q-Bessel function satisfies

$$\partial_q j_\alpha(x;q^2) = -\frac{x}{[2\alpha+2]_q} j_{\alpha+1}(x;q^2). \tag{2}$$

The q-integrals of Jackson are defined by (see [15])

$$\int_0^a f(x)d_qx = (1-q)a\sum_{n=0}^{\infty} q^n f(aq^n), \quad \int_a^b f(x)d_qx = \int_0^b f(x)d_qx - \int_0^a f(x)d_qx,$$

$$\int_0^{\infty} f(x)d_qx = (1-q)\sum_{n=-\infty}^{\infty} q^n f(q^n),$$

and

$$\int_{-\infty}^{\infty} f(x)d_qx = (1-q)\sum_{n=-\infty}^{\infty} q^n f(q^n) + (1-q)\sum_{n=-\infty}^{\infty} q^n f(-q^n),$$

provided that the series converge absolutely.

The operator ∂_q and the Jakson's q-integral allow us to introduce the following useful q-functional spaces :

• The space $\mathscr{E}_q(\mathbb{R}_q)$ of all functions f defined on $\mathbb{R}_q$ such that, for all $n \in \mathbb{N}$, the limit $\lim_{x\to 0} \partial_q^n f(x)$ (in $\mathbb{R}_q$) exists. We provide the space $\mathscr{E}_q(\mathbb{R}_q)$ with the semi-norms $P_{n,a,q}$ defined for $n \in \mathbb{N}$ and $a > 0$ by

$$P_{n,a,q}(f) = \sup\left\{|\partial_q^k f(x)|; 0 \le k \le n\,; x \in [-a\,,a] \cap \mathbb{R}_q\right\}, \qquad f \in \mathscr{E}_q(\mathbb{R}_q).$$

• The space $\mathscr{S}_q(\mathbb{R}_q)$ of all functions $f \in \mathscr{E}_q(\mathbb{R}_q)$ satisfying

$$Q_{n,m,q}(f) = \sup_{x\in\mathbb{R}_q} |x^m \partial_q^n f(x)| < +\infty, \qquad n,m \in \mathbb{N}.$$

We provide $\mathscr{S}_q(\mathbb{R}_q)$ with the topology defined by the semi-norms $Q_{n,m,q}$.

• $\mathscr{D}_q(\mathbb{R}_q)$ the subspace of $\mathscr{E}_q(\mathbb{R}_q)$ of functions with compact supports.

• $L_q^\infty(\mathbb{R}_q)$ the space of all bounded functions on $\mathbb{R}_q$ endowed with the norm

$$\|f\|_{\infty,q} = \sup_{x\in\mathbb{R}_q} |f(x)|, \qquad f \in L_q^\infty(\mathbb{R}_q).$$

• $L^p(\mathbb{R}_q, |x|^{2\alpha+1} d_q x)$ the space of all functions f defined on $\mathbb{R}_q$ and satisfying

$$\|f\|_{p,\alpha,q} = \left(\int_{-\infty}^{\infty} |f(x)|^p |x|^{2\alpha+1} d_q x\right)^{\frac{1}{p}} < \infty$$

provided with the norm $\|\,.\,\|_{p,\alpha,q}$.

3. Elements of q-Dunkl Harmonic analysis

In this section, we collect some facts regarding some elements of q-Dunkl harmonic analysis introduced and studied in [1] and [2]. Throughout this section, unless otherwise stated, we assume $\alpha \ge -1/2$.

The q-Dunkl operator $\Lambda_{\alpha,q}$ is defined for a complex function f, defined on $\mathbb{R}_q$, by

$$\Lambda_{\alpha,q}(f)(x) = \partial_q[f_e + q^{2\alpha+1} f_o](x) + [2\alpha+1]_q \frac{f(x) - f(-x)}{x}, \tag{3}$$

where f_e and f_o are respectively the even and the odd parts of f.

For any complex number λ, the q-Dunkl Kernel $\psi_\lambda^{\alpha,q}$ is defined on $\mathbb{C}$ by

$$\psi_\lambda^{\alpha,q}(x) = j_\alpha(\lambda x; q^2) + \frac{i\lambda x}{[2\alpha+1]_q} j_{\alpha+1}(\lambda x; q^2), \tag{4}$$

where $j_\alpha(.\,; q^2)$ is the q-Bessel function given by (1). Note that for $\alpha = -1/2$ we have $\psi_\lambda^{\alpha,q}(x) = e(i\lambda x; q^2)$ and $\Lambda_{\alpha,q} = \partial_q$.

Proposition 3.1. (see [1]) *For every $\lambda \in \mathbb{C}$, the q-Dunkl kernel $\psi_\lambda^{\alpha,q}$ is the unique analytic solution of the q-differential-difference equation*

$$\begin{cases} \Lambda_{\alpha,q}(f) = i\lambda f; \\ f(0) = 1. \end{cases}$$

When $\alpha > -1/2$, for every $\lambda \in \mathbb{C}$, the q-Dunkl kernel $\psi_\lambda^{\alpha,q}$ possesses the following q-integral representation of Mehler type:

$$\psi_\lambda^{\alpha,q}(x) = \frac{(1+q)\Gamma_{q^2}(\alpha+1)}{2\Gamma_{q^2}(\alpha+\frac{1}{2})} \int_{-1}^{1} \frac{(t^2q^2;q^2)_\infty}{(t^2q^{2\alpha+1};q^2)_\infty}(1+t)e(i\lambda xt;q^2)d_qt,$$

for all $x \in \mathbb{R}_q$. This formula gives rise to the q-Dunkl intertwining operator $V_{\alpha,q}$ defined for $f \in \mathscr{E}_q(\mathbb{R}_q)$ by

$$V_{\alpha,q}(f)(x) = \frac{(1+q)\Gamma_{q^2}(\alpha+1)}{2\Gamma_{q^2}(\frac{1}{2})\Gamma_{q^2}(\alpha+\frac{1}{2})} \int_{-1}^{1} W_\alpha(t,q^2)(1+t)f(xt)d_qt, \quad x \in \mathbb{R}_q, \quad (5)$$

and the dual q-Dunkl intertwining operator ${}^tV_{\alpha,q}$ defined for $f \in \mathscr{D}_q(\mathbb{R}_q)$ by

$$({}^tV_{\alpha,q})(f)(x) = \frac{(1+q)^{-\alpha+\frac{1}{2}}}{2\Gamma_{q^2}(\alpha+\frac{1}{2})} \int_{|y|\geq q|x|} W_\alpha\left(\frac{x}{y};q^2\right)\left(1+\frac{x}{y}\right) f(y)|y|^{2\alpha}d_qy,$$

for all $x \in \mathbb{R}_q$, where

$$W_\alpha(t,q^2) = \frac{(t^2q^2;q^2)_\infty}{(t^2q^{2\alpha+1};q^2)_\infty}. \quad (6)$$

Note that for every $\lambda \in \mathbb{C}$, we have $V_{\alpha,q}(e(-i\lambda\,\cdot\,;q^2))(x) = \psi_{-\lambda}^{\alpha,q}(x)$ for all $x \in \mathbb{R}_q$.

The operators $V_{\alpha,q}$ and ${}^tV_{\alpha,q}$ are linked to each other by the duality relation

$$\int_{-\infty}^{+\infty} V_{\alpha,q}(f)(x)g(x)|x|^{2\alpha+1}d_qx = \frac{(1+q)^{\alpha+\frac{1}{2}}\Gamma_{q^2}(\alpha+1)}{\Gamma_{q^2}(\frac{1}{2})} \int_{-\infty}^{+\infty} f(t)({}^tV_{\alpha,q})(g)(t)d_qt, \quad (7)$$

for all $f \in \mathcal{E}_q(\mathbb{R}_q)$ and $g \in \mathscr{D}_q(\mathbb{R}_q)$. Moreover, it has been shown in[1] that the q-Dunkl intertwining operator $V_{\alpha,q}$ is a transmutation operator between $\Lambda_{\alpha,q}$ and ∂_q on $\mathscr{E}_q(\mathbb{R}_q)$, that is, a topological isomorphism from $\mathscr{E}_q(\mathbb{R}_q)$ into itself satisfying the following transmutation relation:

$$\Lambda_{\alpha,q}V_{\alpha,q}(f) = V_{\alpha,q}(\partial_q f), \qquad f \in \mathscr{E}_q(\mathbb{R}_q), \quad (8)$$

whereas, for the dual q-Dunkl intertwining operator ${}^tV_{\alpha,q}$, only the transmutation relation

$$\partial_q({}^tV_{\alpha,q}(f)) = {}^tV_{\alpha,q}(\Lambda_{\alpha,q}f), \qquad f \in \mathscr{D}_q(\mathbb{R}_q).$$

have been shown. In the next section we will extend the operator ${}^tV_{\alpha,q}$ to the space $L^1(\mathbb{R}_q,|x|^{2\alpha+1}d_qx)$ and we will show that it is a transmutation operator between ∂_q and $\Lambda_{\alpha,q}$ on $\mathscr{S}_q(\mathbb{R}_q)$.

In the remainder of this paper, we assume, as in [1, 2, 20, 21], that

$$\frac{\log(1-q)}{\log q} \in 2\mathbb{Z}.$$

The q-Dunkl kernel $\psi_\lambda^{\alpha,q}$, $\lambda \in \mathbb{C}$, gives rise to the q-Dunkl transform $F_D^{\alpha,q}$ defined for $f \in L^1(\mathbb{R}_q,|x|^{2\alpha+1}d_qx)$ by

$$F_D^{\alpha,q}(f)(\lambda) = \frac{(1+q)^{-\alpha}}{2\Gamma_{q^2}(\alpha+1)}\int_{-\infty}^{+\infty} f(x)\psi_{-\lambda}^{\alpha,q}(x)|x|^{2\alpha+1}d_qx, \qquad \lambda \in \mathbb{R}_q,$$

which is a q-analogue of the classical Dunkl transform studied in [6–8, 19].

Remark 3.2. For $\alpha = -1/2$, the q-Dunkl transform is the q^2-analogue Fourier transform (see [20, 21]) given by

$$F_q(f)(\lambda) = \frac{(1+q)^{\frac{1}{2}}}{2\Gamma_{q^2}(\frac{1}{2})}\int_{-\infty}^{+\infty} f(x)e(-i\lambda x;q^2)d_qx, \qquad \lambda \in \mathbb{R}_q, \tag{9}$$

and on the space of even functions, $F_D^{\alpha,q}$ coincides with the q-Bessel transform given by (see [1, 2, 9])

$$F_{\alpha,q}(f)(\lambda) = \frac{(1+q)^{-\alpha}}{\Gamma_{q^2}(\alpha+1)}\int_0^{+\infty} f(x)j_\alpha(\lambda x;q^2)x^{2\alpha+1}d_qx, \qquad \lambda \in \mathbb{R}_q. \tag{10}$$

Proposition 3.3.

(a) *The q-Dunkl transform $F_D^{\alpha,q}$ is a topological automorphism of $\mathscr{S}_q(\mathbb{R}_q)$.*

(b) *For $f \in \mathscr{S}_q(\mathbb{R}_q)$, we have $F_D^{\alpha,q}(\Lambda_{\alpha,q}f)(\lambda) = i\lambda F_D^{\alpha,q}(\lambda)$ for all $\lambda \in \mathbb{R}_q$.*

(c) Inversion formula : *For $f \in \mathscr{S}_q(\mathbb{R}_q)$ we have*

$$f(x) = \frac{(1+q)^{-\alpha}}{2\Gamma_{q^2}(\alpha+1)}\int_{-\infty}^{+\infty} F_D^{\alpha,q}(f)(\lambda)\psi_\lambda^{\alpha,q}(x)|\lambda|^{2\alpha+1}d_q\lambda, \qquad x \in \mathbb{R}_q. \tag{11}$$

(d) Plancherel formula: *For all $f \in \mathscr{S}_q(\mathbb{R}_q)$ we have*

$$\|F_D^{\alpha,q}(f)\|_{2,\alpha,q} = \|f\|_{2,\alpha,q}. \tag{12}$$

For any $y \in \mathbb{R}_q \cup \{0\}$ we define the generalized q-Dunkl translation operator $T_y^{\alpha;q}$ for $f \in \mathscr{S}_q(\mathbb{R}_q)$ by

$$T_y^{\alpha;q}(f)(x) = \frac{(1+q)^{-\alpha}}{2\Gamma_{q^2}(\alpha+1)}\int_{-\infty}^{\infty} F_D^{\alpha,q}(f)(\lambda)\psi_\lambda^{\alpha,q}(x)\psi_\lambda^{\alpha,q}(y)|\lambda|^{2\alpha+1}d_q\lambda, \tag{13}$$

for all $x \in \mathbb{R}_q$. The q-Dunkl translation operators allow us to define a q-Dunkl convolution product $*_{\alpha,q}$ on $\mathscr{S}_q(\mathbb{R}_q)$ as follows (see [2]): for $f, g \in \mathscr{S}_q(\mathbb{R}_q)$,

$$f *_{\alpha,q} g(x) = \frac{(1+q)^{-\alpha}}{2\Gamma_{q^2}(\alpha+1)} \int_{-\infty}^{\infty} T_x^{\alpha;q} f(-y) g(y) |y|^{2\alpha+1} d_q y, \qquad x \in \mathbb{R}_q.$$

This convolution product is commutative and associative. Moreover, for f, g in $\mathscr{S}_q(\mathbb{R}_q)$ we have $f *_{\alpha,q} g \in \mathscr{S}_q(\mathbb{R}_q)$, and $F_D^{\alpha,q}(f *_{\alpha,q} g) = F_D^{\alpha,q}(f) \cdot F_D^{\alpha,q}(g)$.

4. The q-Dunkl-Sonine transform

From now on, unless otherwise stated, we assume that $\beta > \alpha \geq -1/2$.

Proposition 4.1. *For every $\lambda \in \mathbb{C}$, the function $j_\alpha(\lambda \cdot; q^2)$ admits the q-integral representation of Sonine type:*

$$j_\beta(\lambda x; q^2) = \frac{(1+q)\Gamma_{q^2}(\beta+1)}{\Gamma_{q^2}(\alpha+1)\Gamma_{q^2}(\beta-\alpha)} \int_0^1 j_\alpha(\lambda x t; q^2) W_{\beta-\alpha-\frac{1}{2}}(t; q^2) t^{2\alpha+1} d_q t, \quad x \in \mathbb{C}, \quad (14)$$

where $W_{\beta-\alpha-\frac{1}{2}}(t\,; q^2)$ is given by (6).

Proof. It follows from (1) that

$$\int_0^1 j_\alpha(\lambda x t; q^2) W_{\beta-\alpha-\frac{1}{2}}(t; q^2) t^{2\alpha+1} d_q t = \sum_{n=0}^{+\infty} \frac{(-1)^n q^{n(n+1)} \Gamma_{q^2}(\alpha+1) I_n}{\Gamma_{q^2}(n+\alpha+1)\Gamma_{q^2}(n+1)} \left(\frac{\lambda x}{1+q}\right)^{2n},$$

for all $x \in \mathbb{C}$, where

$$I_n = (1+q) \int_0^1 \frac{(t^2 q^2; q^2)_\infty}{(t^2 q^{2(\beta-\alpha)}; q^2)_\infty} t^{2\alpha+2n+1} d_q t.$$

Using the q-Beta integral (see the proof of Theorem 1 in [9])

$$\frac{\Gamma_{q^2}(x)\Gamma_{q^2}(y)}{\Gamma_{q^2}(x+y)} = (1+q) \int_0^1 \frac{(t^2 q^2; q^2)_\infty}{(t^2 q^{2y}; q^2)_\infty} t^{2x-1} d_q t, \qquad x, y > 0,$$

we get the q-Sonine formule (14). □

In the following proposition, we extend (14) to the q-Dunkl setting.

Proposition 4.2. *For every $\lambda \in \mathbb{C}$, the q-Dunkl-Sonine formula*

$$\psi_\lambda^{\beta,q}(x) = \frac{(1+q)\Gamma_{q^2}(\beta+1)}{2\Gamma_{q^2}(\alpha+1)\Gamma_{q^2}(\beta-\alpha)} \int_{-1}^1 \psi_\lambda^{\alpha,q}(xt) W_{\beta-\alpha-\frac{1}{2}}(t; q^2)(1+t)|t|^{2\alpha+1} d_q t, \quad (15)$$

holds for all $x \in \mathbb{C}$.

Proof. Set $a^q_{\alpha,\beta} = \dfrac{(1+q)\Gamma_{q^2}(\beta+1)}{2\Gamma_{q^2}(\alpha+1)\Gamma_{q^2}(\beta-\alpha)}$. By parity argument, formula (14) can be written as

$$j_\beta(\lambda x;q^2) = a^q_{\alpha,\beta}\int_{-1}^{1} j_\alpha(\lambda xt;q^2)W_{\beta-\alpha-\frac{1}{2}}(t;q^2)(1+t)|t|^{2\alpha+1}d_qt. \tag{16}$$

Hence,

$$\lambda\partial_q\left(j_\beta(\,.\,;q^2)\right)(\lambda x) = a^q_{\alpha,\beta}\int_{-1}^{1}\lambda t\partial_q\left(j_\alpha(\,.\,;q^2)\right)(\lambda xt)W_{\beta-\alpha-\frac{1}{2}}(t;q^2)(1+t)|t|^{2\alpha+1}d_qt.$$

Using (2), we get

$$\frac{\lambda x}{[2\beta+2]_q}j_{\beta+1}(\lambda x;q^2) = a^q_{\alpha,\beta}\int_{-1}^{1}\frac{\lambda xt}{[2\alpha+2]_q}j_{\alpha+1}(\lambda xt;q^2)W_{\beta-\alpha-\frac{1}{2}}(t;q^2)(1+t)|t|^{2\alpha+1}.$$

Combining this with (16) yields the q-Dunkl-Sonine formula (15). □

Definition 4.3. The q-Dunkl Sonine transform $\mathcal{S}^q_{\alpha,\beta}$ is defined, for $f\in\mathscr{E}_q(\mathbb{R}_q)$, by

$$\mathcal{S}^q_{\alpha,\beta}(f)(x) = \frac{(1+q)\Gamma_{q^2}(\beta+1)}{2\Gamma_{q^2}(\alpha+1)\Gamma_{q^2}(\beta-\alpha)}\int_{-1}^{1} f(xt)W_{\beta-\alpha-\frac{1}{2}}(t;q^2)(1+t)|t|^{2\alpha+1}d_qt, \tag{17}$$

for all $x\in\mathbb{R}_q$, which can be written as

$$\mathcal{S}^q_{\alpha,\beta}(f)(x) = \frac{(1+q)\Gamma_{q^2}(\beta+1)}{2\Gamma_{q^2}(\alpha+1)\Gamma_{q^2}(\beta-\alpha)|x|^{2\alpha+2}}\int_{-|x|}^{|x|} f(y)W_{\beta-\alpha-\frac{1}{2}}\big(\frac{y}{x};q^2\big)\big(1+\frac{y}{x}\big)|y|^{2\alpha+1}d_qy, \tag{18}$$

for all $x\in\mathbb{R}_q$.

Remark 4.4.

(a) For every $\lambda\in\mathbb{C}$, we have $\psi^{\beta,q}_\lambda = \mathcal{S}^q_{\alpha,\beta}(\psi^{\alpha,q}_\lambda)$.

(b) If $\alpha = -\dfrac{1}{2}$, then the q-Dunkl Sonine transform $\mathcal{S}^q_{\alpha,\beta}$ reduces to the q-Dunkl intertwining operator $V_{\beta,q}$ given by (5).

Definition 4.5. The dual q-Dunkl-Sonine transform ${}^t\mathcal{S}^q_{\alpha,\beta}$ is defined for suitable function f, by

$${}^t\mathcal{S}^q_{\alpha,\beta}(f)(x) = \frac{(1+q)^{\alpha-\beta+1}}{2\Gamma_{q^2}(\beta-\alpha)}\int_{|y|\geq q|x|} f(y)W_{\beta-\alpha-\frac{1}{2}}(\frac{x}{y};q^2)(1+\frac{x}{y})|y|^{2(\beta-\alpha)-1}d_qy, \tag{19}$$

for all $x\in\mathbb{R}_q$.

Proposition 4.6. (*a*) *The dual q-Dunkl-Sonine transform* ${}^t\mathcal{S}^q_{\alpha,\beta}$ *is a continuous linear mapping from* $L^1(\mathbb{R}_q,|x|^{2\beta+1}d_qx)$ *into* $L^1(\mathbb{R}_q,|x|^{2\alpha+1}d_qx)$.

(*b*) *For* $f\in\mathscr{E}_q(\mathbb{R}_q)\cap L^\infty_q(\mathbb{R}_q)$ *and* $g\in L^1(\mathbb{R}_q,|x|^{2\beta+1}d_qx)$*, we have the duality relation*

$$\int_{-\infty}^{+\infty}\mathcal{S}^q_{\alpha,\beta}(f)(x)g(x)|x|^{2\beta+1}d_qx = \frac{(1+q)^\beta\Gamma_{q^2}(\beta+1)}{(1+q)^\alpha\Gamma_{q^2}(\alpha+1)}\int_{-\infty}^{+\infty}f(x){}^t\mathcal{S}^q_{\alpha,\beta}(g)(x)|x|^{2\alpha+1}d_qx. \quad (20)$$

Proof. To prove (a), let $f\in L^1(\mathbb{R}_q,|x|^{2\beta+1})$. Using Tonelly's Theorem and series manipulations, we obtain

$$\int_{-\infty}^{+\infty}\left(\int_{|y|\geq q|x|}|f(y)|W_{\beta-\alpha-\frac12}\big(\frac{x}{y};q^2\big)\big(1+\frac{x}{y}\big)|y|^{2(\beta-\alpha)-1}d_qy\right)|x|^{2\alpha+1}d_qx = \int_{-\infty}^{+\infty}\left(\frac{1}{|y|^{2\alpha+2}}\int_{-|y|}^{|y|}W_{\beta-\alpha-\frac12}\big(\frac{x}{y};q^2\big)\big(1+\frac{x}{y}\big)|x|^{2\alpha+1}d_qx\right)|f(y)||y|^{2\beta+1}d_qy.$$

Since by (18) together with Remark 4.4 (a), we have

$$\frac{(1+q)\Gamma_{q^2}(\beta+1)}{2\Gamma_{q^2}(\alpha+1)\Gamma_{q^2}(\beta-\alpha)|y|^{2\alpha+2}}\int_{-|y|}^{|y|}W_{\beta-\alpha-\frac12}\big(\frac{x}{y};q^2\big)\big(1+\frac{x}{y}\big)|x|^{2\alpha+1}d_qx = \psi_0^{\alpha,q}(y)=1,$$

for all $y\in\mathbb{R}_q$, it follows that

$$\int_{-\infty}^{+\infty}\left(\int_{|y|\geq q|x|}|f(y)|W_{\beta-\alpha-\frac12}\big(\frac{x}{y};q^2\big)\big(1+\frac{x}{y}\big)|y|^{2(\beta-\alpha)-1}d_qy\right)|x|^{2\alpha+1}d_qx = \frac{2\Gamma_{q^2}(\alpha+1)\Gamma_{q^2}(\beta-\alpha)}{(1+q)\Gamma_{q^2}(\beta+1)}\|f\|_{1,\beta,q}.$$

Hence, by the Fubini theorem, the function ${}^t\mathcal{S}^q_{\alpha,\beta}(f)$ is defined on $\mathbb{R}_q$, belongs to $L^1(\mathbb{R}_q,|x|^{2\alpha+1}d_qx)$ and satisfies

$$\|{}^t\mathcal{S}^q_{\alpha,\beta}(f)\|_{1,\alpha,q}\leq\frac{(1-q)^\alpha\Gamma_{q^2}(\alpha+1)}{(1-q)^\beta\Gamma_{q^2}(\beta+1)}\|f\|_{1,\beta,q}.$$

Thus, ${}^t\mathcal{S}^q_{\alpha,\beta}$ maps continuously $L^1(\mathbb{R}_q,|x|^{2\beta+1})$ into $L^1(\mathbb{R}_q,|x|^{2\alpha+1})$.

Part (b) can be proved using the Fubini's theorem and series manipulations. □

Remark 4.7. The relation (20) holds also for $f\in\mathscr{E}_q(\mathbb{R}_q)$ and $g\in\mathscr{D}_q(\mathbb{R}_q)$.

Corollary 4.8. *The q-Dunkl transform* $F_D^{\beta,q}$ *admits the factorization*

$$F_D^{\beta,q}=F_D^{\alpha,q}\circ{}^t\mathcal{S}^q_{\alpha,\beta} \quad (21)$$

on $L^1(\mathbb{R}_q,|x|^{2\beta+1}d_qx)$.

Proof. Let $f \in L^1(\mathbb{R}_q, |x|^{2\beta+1}d_qx)$ and $\lambda \in \mathbb{R}_q$. It follows from Remark 4.4 (a) that

$$F_D^{\beta,q}(f)(\lambda) = \frac{(1+q)^{-\beta}}{2\Gamma_{q^2}(\beta+1)} \int_{-\infty}^{+\infty} f(x)\mathcal{S}_{\alpha,\beta}^q(\psi_{-\lambda}^{\alpha,q})|x|^{2\beta+1}d_qx.$$

Since, by Proposition 6 of [1], we have $|\psi_\lambda^{\alpha,q}(x)| \leq 4/(q,q)_\infty$, for all $x \in \mathbb{R}_q$, it follows that $\psi_\lambda^{\alpha,q} \in \mathscr{E}_q(\mathbb{R}_q) \cap L_q^\infty(\mathbb{R}_q)$. Hence, using the duality relation (20), we obtain

$$\begin{aligned} F_D^{\beta,q}(f)(\lambda) &= \int_{-\infty}^{+\infty} f(x)\mathcal{S}_{\alpha,\beta}^q(\psi_{-\lambda}^{\alpha,q})|x|^{2\beta+1}d_qx \\ &= \frac{(1+q)^{-\alpha}}{2\Gamma_{q^2}(\alpha+1)} \int_{-\infty}^{+\infty} {}^t\mathcal{S}_{\alpha,\beta}^q(f)(x)\psi_{-\lambda}^{\alpha,q}(x)|x|^{2\alpha+1}d_qx \\ &= F_D^{\alpha,q}\big({}^t\mathcal{S}_{\alpha,\beta}^q f\big)(\lambda). \end{aligned}$$

Thus, $F_D^{\beta,q}(f) = F_D^{\alpha,q} \circ {}^t\mathcal{S}_{\alpha,\beta}^q(f)$ for all $f \in L^1(\mathbb{R}_q, |x|^{2\beta+1}d_qx)$. □

Next, we extend the definition of the dual q-Dunkl intertwining operator, ${}^tV_\alpha$, to the space $L^1(\mathbb{R}_q, |x|^{2\alpha+1}d_qx)$.

Definition 4.9. Let $\alpha > -1/2$. We define the dual q-Dunkl intertwining operator ${}^tV_{\alpha,q}$, for $f \in L^1(\mathbb{R}_q, |x|^{2\alpha+1}d_qx)$, by

$$({}^tV_{\alpha,q})(f)(x) = \frac{(1+q)^{-\alpha+\frac{1}{2}}}{2\Gamma_{q^2}(\alpha+\frac{1}{2})} \int_{|y|\geq q|x|} W_\alpha\big(\frac{x}{y};q^2\big)\big(1+\frac{x}{y}\big)f(y)|y|^{2\alpha}d_qy, \quad x \in \mathbb{R}_q.$$

Remark 4.10.

(a) For $\alpha > -1/2$, we have ${}^tV_{\alpha,q} = {}^t\mathcal{S}_{-\frac{1}{2},\alpha}^q$.

(b) For $\beta > \alpha \geq -1/2$, we have ${}^t\mathcal{S}_{\alpha,\beta}^q = {}^tV_{\beta-\alpha-\frac{1}{2},q}$.

Proposition 4.11.

(*a*) *The dual q-Dunkl intertwining operator* ${}^tV_{\alpha,q}$ *is a continuous linear mapping from* $L^1(\mathbb{R}_q, |x|^{2\alpha+1}d_qx)$ *into* $L^1(\mathbb{R}_q, d_qx)$.

(*b*) *The q-Dunkl transform* $F_D^{\alpha,q}$ *is linked to the* q^2*-analogue Fourier transform* F_q *by*

$$F_D^{\alpha,q}(f) = (F_q \circ {}^tV_{\alpha,q})(f), \qquad f \in L^1(\mathbb{R}_q, |x|^{2\alpha+1}d_qx). \tag{22}$$

(*c*) *The dual q-Dunkl intertwining operator* ${}^tV_{\alpha,q}$ *is a topological automorphism of* $\mathscr{S}_q(\mathbb{R}_q)$ *and satisfies the transmutation relation*

$$\partial_q({}^tV_{\alpha,q}(f)) = {}^tV_{\alpha,q}(\Lambda_{\alpha,q}f), \qquad f \in \mathscr{S}_q(\mathbb{R}_q). \tag{23}$$

Proof. Part (a) easily follows from Proposition 4.6 (a) together with Remark 4.10. Part (b) follows from Corollary 4.8 and Remark 4.10. For part (c), it follows from part (b) that ${}^tV_{\alpha,q} = F_q^{-1} \circ F_D^{\alpha,q}$ on $\mathscr{S}_q(\mathbb{R}_q)$. So, by Proposition 3.3 (a), ${}^tV_{\alpha,q}$ is a topological automorphism of $\mathscr{S}_q(\mathbb{R}_q)$. Moreover, using part (b) of Proposition 3.3 together with (22), we obtain

$$\begin{aligned} F_q\big(\partial_q({}^tV_{\alpha,q}(f))\big) &= i\lambda F_q\big({}^tV_{\alpha,q}(f)\big) \\ &= i\lambda F_D^{\alpha,q}(f) = F_D^{\alpha,q}(\Lambda_{\alpha,q} f) = F_q\big({}^tV_{\alpha,q}(\Lambda_{\alpha,q} f)\big), \end{aligned}$$

for all $f \in \mathscr{S}_q(\mathbb{R}_q)$. Hence, (23) follows from the injectivity of F_q. □

Theorem 4.12. *Let $\alpha, \beta \in]-\frac{1}{2}, +\infty[$ such that $\beta > \alpha$.*

(*a*) *The dual q-Dunkl-Sonine transform ${}^t\mathcal{S}^q_{\alpha,\beta}$ is a topological automorphism of $\mathscr{S}_q(\mathbb{R}_q)$, and satisfies the following relations:*

$${}^t\mathcal{S}^q_{\alpha,\beta}(f) = ({}^tV_{\alpha,q})^{-1} \circ {}^tV_{\beta,q}(f), \qquad f \in \mathscr{S}_q(\mathbb{R}_q), \tag{24}$$

$$\Lambda_{\alpha,q}({}^t\mathcal{S}^q_{\alpha,\beta}(f)) = {}^t\mathcal{S}^q_{\alpha,\beta}(\Lambda_{\beta,q} f), \qquad f \in \mathscr{S}_q(\mathbb{R}_q). \tag{25}$$

(*b*) *The q-Dunkl-Sonine transform $\mathcal{S}^q_{\alpha,\beta}$ is a topological automorphism of $\mathscr{E}_q(\mathbb{R}_q)$ and satisfies the following relations*

$$\mathcal{S}^q_{\alpha,\beta}(f) = V_{\beta,q} \circ (V_{\alpha,q})^{-1}(f), \qquad f \in \mathscr{E}_q(\mathbb{R}_q), \tag{26}$$

$$\Lambda_{\beta,q}(\mathcal{S}^q_{\alpha,\beta}(f)) = \mathcal{S}^q_{\alpha,\beta}(\Lambda_{\alpha,q} f), \qquad f \in \mathscr{E}_q(\mathbb{R}_q). \tag{27}$$

Proof. For part (a), by Corollary 4.8, we have ${}^t\mathcal{S}^q_{\alpha,\beta} = (F_D^{\alpha,q})^{-1} \circ F_D^{\beta,q}$. Hence, part (a) of Proposition 3.3 infers that ${}^t\mathcal{S}^q_{\alpha,\beta}$ is a topological automorphism of $\mathscr{S}_q(\mathbb{R}_q)$. Moreover, (24) follows immediately from (22). The proof of (25) runs in a similar way as the proof of (23) using Corollary 4.8 and Proposition 3.3 (b).

For (b), we start by proving (26) which is equivalent to $\mathcal{S}^q_{\alpha,\beta} \circ V_{\alpha,q} = V_{\beta,q}$. It suffices, therefore, to prove that for all $f \in \mathscr{E}_q(\mathbb{R}_q)$ and $g \in \mathscr{D}_q(\mathbb{R}_q)$ we have

$$\int_{-\infty}^{+\infty} \mathcal{S}^q_{\alpha,\beta}(V_{\alpha,q} f)(x) g(x) |x|^{2\beta+1} d_q = \int_{-\infty}^{+\infty} V_{\beta,q} f(x) g(x) |x|^{2\beta+1} d_q x.$$

Let $f \in \mathscr{E}_q(\mathbb{R}_q)$ and $g \in \mathscr{D}_q(\mathbb{R}_q)$. To simplify the notations, set

$$K_{\alpha,q} = \frac{(1+q)^{\alpha+\frac{1}{2}} \Gamma_{q^2}(\alpha+1)}{\Gamma_{q^2}(\frac{1}{2})} \qquad \text{and} \qquad K^q_{\alpha,\beta} = \frac{(1+q)^{\beta-\alpha} \Gamma_{q^2}(\beta+1)}{\Gamma_{q^2}(\alpha+1)}.$$

By Remark 4.6, we can apply the duality relation (20) as follows

$$\int_{-\infty}^{+\infty} \mathcal{S}^q_{\alpha,\beta}(V_{\alpha,q} f)(x) g(x) |x|^{2\beta+1} d_q = K^q_{\alpha,\beta} \int_{-\infty}^{+\infty} V_{\alpha,q} f(x) {}^t\mathcal{S}^q_{\alpha,\beta}(g)(x) |x|^{2\alpha+1} d_q x.$$

Since, obviously, ${}^t\mathcal{S}^q_{\alpha,\beta}(g) \in \mathscr{D}_q(\mathbb{R}_q)$, we can apply the duality relation (7). We obtain

$$\int_{-\infty}^{+\infty} V_{\alpha,q} f(x) {}^t\mathcal{S}^q_{\alpha,\beta}(g)(x)|x|^{2\alpha+1} d_q x = K_{\alpha,q} \int_{-\infty}^{+\infty} f(x)({}^tV_{\alpha,q} \circ {}^t\mathcal{S}^q_{\alpha,\beta})(g)(x) d_q x.$$

Using (24) we get

$$\int_{-\infty}^{+\infty} V_{\alpha,q} f(x) {}^t\mathcal{S}^q_{\alpha,\beta}(g)(x)|x|^{2\alpha+1} d_q x = K_{\alpha,q} \int_{-\infty}^{+\infty} f(x)({}^tV_{\beta,q})(g)(x) d_q x.$$

Hence, using the duality relation (7) and the fact that $K^q_{\alpha,\beta} = \dfrac{K_{\beta,q}}{K_{\alpha,q}}$, we deduce that

$$\int_{-\infty}^{+\infty} \mathcal{S}^q_{\alpha,\beta}(V_{\alpha,q} f)(x) g(x)|x|^{2\beta+1} d_q = \int_{-\infty}^{+\infty} V_{\beta,q} f(x) g(x)|x|^{2\beta+1} d_q x.$$

This achieves the proof of (26) which, together with the fact that, as mentioned in Section 2, $V_{\alpha,q}$ is a topological isomorphism of $\mathscr{E}_q(\mathbb{R}_q)$, infers that $\mathcal{S}^q_{\alpha,\beta}$ is a topological automorphism of $\mathscr{E}_q(\mathbb{R}_q)$.

To prove (27), let $f \in \mathscr{E}_q(\mathbb{R}_q)$. From the factorization Relation (26) and the transmutation relation (8), we have

$$\Lambda_{\beta,q}(\mathcal{S}^q_{\alpha,\beta}(f)) = \Lambda_{\beta,q} V_{\beta,q}(V^{-1}_{\alpha,q}(f)) = V_{\beta,q}\left(\partial_q V^{-1}_{\alpha,q}(f)\right).$$

Since (8) implies that $\partial_q\big(V^{-1}_{\alpha,q} f\big) = V^{-1}_{\alpha,q}\big(\Lambda_{\alpha,q} f\big)$, it follows that

$$\Lambda_{\beta,q}(\mathcal{S}^q_{\alpha,\beta} f) = V_{\beta,q} \circ V^{-1}_{\alpha,q}\big(\Lambda_{\alpha,q}(f)\big) = \mathcal{S}^q_{\alpha,\beta} \Lambda_{\alpha,q}(f).$$

This completes the proof of the theorem. □

5. Inversion formulas for the q-Dunkl-Sonine transform and its dual using q-pseudo-differential operators

In this section, we give inversion formulas for the q-Dunkl sonine transform $\mathcal{S}^q_{\alpha,\beta}$ and its dual ${}^t\mathcal{S}^q_{\alpha,\beta}$ using q-pseudo-differential operators. Next, we give Plancherel formula for the dual q-Dunkl-sonine transform.

We begin by introducing the q-analogues of the Lizorkin spaces (see [22]) :

- $\Phi^q_\alpha(\mathbb{R}_q) = \left\{ f \in \mathscr{S}_q(\mathbb{R}_q) \colon \int_{-\infty}^{+\infty} f(x)|x|^{2\alpha+k+1} = 0, \quad k = 0, 1, \dots \right\}$;
- $\Psi_q(\mathbb{R}_q) = \left\{ f \in \mathscr{S}_q(\mathbb{R}_q) \colon \partial^k_q f(0) = 0, \quad k = 0, 1, \dots \right\}$.

Proposition 5.1. (see [2]) *For every $\alpha \geq -1/2$, the q-Dunkl transform $F^{\alpha,q}_D$ is an isomorphism from $\Phi^q_\alpha(\mathbb{R}_q)$ into $\Psi_q(\mathbb{R}_q)$.*

Lemma 5.2. (see [2, 3]) *For every $\lambda \in \mathbb{C}$, the multiplication operator $M_\lambda : f \mapsto |x|^\lambda f$ is a topological automorphism of $\Psi_q(\mathbb{R}_q)$, its inverse operator is $M_{-\lambda}$.*

Proposition 5.3. *If $f \in \Phi^q_\alpha(\mathbb{R}_q)$, then for all $\lambda \in \mathbb{R}_q$, we have*

$$F_D^{\beta,q}(\mathcal{S}^q_{\alpha,\beta} g f)(\lambda) = \frac{(1+q)^\beta \Gamma_{q^2}(\beta+1)}{(1+q)^\alpha \Gamma_{q^2}(\alpha+1)} \frac{1}{|\lambda|^{2(\beta-\alpha)}} F_D^{\alpha,q}(f)(\lambda). \tag{28}$$

Proof. By the inversion formula (11), we have

$$f(x) = \frac{(1+q)^{-\alpha}}{2\Gamma_{q^2}(\alpha+1)} \int_{-\infty}^{+\infty} F_D^{\alpha,q}(f)(\lambda) \psi_\lambda^{\alpha,q}(x) |\lambda|^{2\alpha+1} d_q\lambda, \qquad x \in \mathbb{R}_q.$$

Using Fubini's theorem and the fact that $\psi_\lambda^{\beta,q} = \mathcal{S}^q_{\alpha,\beta}(\psi_\lambda^{\alpha,q})$, we obtain

$$\begin{aligned} \mathcal{S}^q_{\alpha,\beta} f(x) =& \frac{(1+q)^{-\alpha}}{2\Gamma_{q^2}(\alpha+1)} \int_{-\infty}^{+\infty} F_D^{\alpha,q}(f)(\lambda) \psi_\lambda^{\beta,q}(x) |\lambda|^{2\alpha+1} d_q\lambda \\ =& \frac{(1+q)^{-\beta}}{2\Gamma_{q^2}(\beta+1)} \int_{-\infty}^{+\infty} h^q_{\alpha,\beta}(\lambda) \psi_\lambda^{\beta,q}(x) |\lambda|^{2\beta+1} d_q\lambda, \end{aligned}$$

for all $x \in \mathbb{R}_q$, where

$$h^q_{\alpha,\beta}(\lambda) = \frac{(1+q)^\beta \Gamma_{q^2}(\beta+1)}{(1+q)^\alpha \Gamma_{q^2}(\alpha+1)} \frac{1}{|\lambda|^{2(\beta-\alpha)}} F_D^{\alpha,q}(f)(\lambda), \qquad \lambda \in \mathbb{R}_q.$$

Since $f \in \Phi^q_\alpha(\mathbb{R}_q)$, it follows from Proposition 5.1 that $F_D^{\alpha,q}(f) \in \Psi_q(\mathbb{R}_q)$ and hence, Lemma 5.2 infers that $h^q_{\alpha,\beta} \in \Psi_q(\mathbb{R}_q)$, and the conclusion of the proposition follows from the above inversion formula. □

Definition 5.4. We define the operators $K_{1,q}$, $K_{2,q}$ and $K_{3,q}$ by

$$\begin{aligned} K_{1,q}(f) =& \frac{(1+q)^\alpha \Gamma_{q^2}(\alpha+1)}{(1+q)^\beta \Gamma_{q^2}(\beta+1)} (F_D^{\alpha,q})^{-1}(|\lambda|^{2(\beta-\alpha)} F_D^{\alpha,q}(f)), \qquad f \in \Phi^q_\alpha(\mathbb{R}_q); \\ K_{2,q}(f) =& \frac{(1+q)^\alpha \Gamma_{q^2}(\alpha+1)}{(1+q)^\beta \Gamma_{q^2}(\beta+1)} (F_D^{\beta,q})^{-1}(|\lambda|^{2(\beta-\alpha)} F_D^{\beta,q}(f)), \qquad f \in \Phi^q_\beta(\mathbb{R}_q); \\ K_{3,q}(f) =& (F_D^{\alpha,q})^{-1}(|\lambda|^{(\beta-\alpha)} F_D^{\alpha,q}(f)), \qquad f \in \Phi^q_\alpha(\mathbb{R}_q). \end{aligned}$$

Using the q-pseudo-differential operators $K_{1,q}$, $K_{2,q}$, we give in the following theorem, inversion formulas for the q-Dunkl Sonine operator $\mathcal{S}^q_{\alpha,\beta}$ and its dual ${}^t\mathcal{S}^q_{\alpha,\beta}$, and, using the q-pseudo-differential operator $K_{3,q}$, we give Plancherel formula for ${}^t\mathcal{S}^q_{\alpha,\beta}$.

Theorem 5.5.

(a) Inversion formulas: *For all $f \in \Phi^q_\alpha(\mathbb{R}_q)$ and $g \in \Phi^q_\beta(\mathbb{R}_q)$, we have the following inversion formulas:*

(i) $f = \left({}^t\mathcal{S}^q_{\alpha,\beta} K_{2,q} \mathcal{S}^q_{\alpha,\beta}\right)(f)$.

(ii) $f = \left(K_{1,q}{}^t\mathcal{S}^q_{\alpha,\beta} \mathcal{S}^q_{\alpha,\beta}\right)(f)$.

(*iii*) $g = \big(K_{2,q}\mathcal{S}^q_{\alpha,\beta}\,{}^t\mathcal{S}^q_{\alpha,\beta}\big)(g)$.

(*iv*) $g = \big(\mathcal{S}^q_{\alpha,\beta}K_{1,q}\,{}^t\mathcal{S}^q_{\alpha,\beta}\big)(g)$.

(*b*) Plancherel formula: *For all* $f \in \Phi^q_\beta(\mathbb{R}_q)$*, we have*

$$\int_{-\infty}^{+\infty} |f(t)|^2|t|^{2\beta+1}d_qt = \int_{-\infty}^{+\infty} |K_{3,q}({}^t\mathcal{S}^q_{\alpha,\beta}(f))(t)|^2|t|^{2\alpha+1}d_qt.$$

Proof. Part (a) Follows immediately from Proposition 5.3 together with the factorization relation (24).

For part (b), let $f \in \Phi^q_\beta(\mathbb{R}_q)$. Using the Plancherel formula (12) together with the factorization relation (24), we obtain

$$\begin{aligned}\int_{-\infty}^{+\infty} |f(t)|^2|t|^{2\beta+1}d_qt &= \int_{-\infty}^{+\infty} \big|F_D^{\beta,q}(f)(\lambda)\big|^2|\lambda|^{2\beta+1}d_q\lambda\\ &= \int_{-\infty}^{+\infty} \big||\lambda|^{(\beta-\alpha)}F_D^{\alpha,q}({}^t\mathcal{S}^q_{\alpha,\beta}(f))(\lambda)\big|^2|\lambda|^{2\alpha+1}d_q\lambda\\ &= \int_{-\infty}^{+\infty} \big|F_D^{\alpha,q}(K_{3,q}({}^t\mathcal{S}^q_{\alpha,\beta}(f))(\lambda)\big|^2|\lambda|^{2\alpha+1}d_q\lambda\\ &= \int_{-\infty}^{+\infty} \big|K_{3,q}({}^t\mathcal{S}^q_{\alpha,\beta}(f)(t)\big|^2|t|^{2\alpha+1}d_q\lambda.\end{aligned}$$

This achieves the proof. □

6. Inversions of the q-Dunkl-Sonine transform and its dual by the use of the q-Dunkl wavelets

We begin this section by summarizing some facts about q-Dunkl wavelets introduced and studied in [2]. Next, we study the effect of the q-Dunkl-Sonine transform $\mathcal{S}^q_{\alpha,\beta}$ and its dual ${}^tS_{\alpha,\beta}$ on q-Dunkl wavelets and as applications, we give the inversion formulas for these integral transforms using q-Dunkl wavelets.

Definition 6.1. Let $g \in L^2_{\alpha,q}(\mathbb{R}_q)$. We say that g is a q-wavelet associated with the q-Dunkl operator $\Lambda_{\alpha,q}$, if it satisfies the following admissibility condition

$$0 < C_{\alpha,q}(g) = \int_{-\infty}^{\infty} |F_D^{\alpha,q}(g)(a)|^2\frac{d_qa}{|a|} < \infty. \tag{29}$$

Proposition 6.2. *Let* $g \neq 0$ *be a function in* $L^2_{\alpha,q}(\mathbb{R}_q)$*. If g satisfies*

- *The function* $F_D^{\alpha,q}(g)$ *is continuous at* 0.
- *There exists* $\nu > 0$ *such that*

$$F_D^{\alpha,q}(g)(x) - F_D^{\alpha,q}(g)(0) = O(x^\nu) \text{ as } x \to 0, x \in \mathbb{R}_q.$$

Then, the admissibility condition (29) *is equivalent to* $F_D^{\alpha,q}(g)(0) = 0$.

For $g \in \mathscr{S}_q(\mathbb{R}_q)$, $a \in \mathbb{R}_q$ and $b \in \mathbb{R}_q \cup \{0\}$, we define the function $g_{a,b}^{\alpha,q}$ by

$$g_{a,b}^{\alpha,q}(x) = \sqrt{|a|} T_b^{\alpha,q}(g_a^{\alpha,q})(x), \quad x \in \mathbb{R}_q,$$

where $T_b^{\alpha;q}$ is the generalized q-Dunkl translation operator defined by (13) and

$$g_a^{\alpha,q}(x) = \frac{1}{|a|^{2\alpha+2}} g\left(\frac{x}{a}\right), \qquad x \in \mathbb{R}_q. \tag{30}$$

In the following proposition, we give some properties of function $g_a^{\alpha,q}$, the proof is straightforward.

Proposition 6.3. *If $g \in \mathscr{S}_q(\mathbb{R}_q)$ and $a \in \mathbb{R}_q$ then:*

(a) $g_a^{\alpha,q} \in \mathscr{S}_q(\mathbb{R}_q)$.

(b) $F_D^{\alpha,q}(g_a^{\alpha,q})(\lambda) = F_D^{\alpha,q}(g)(a\lambda)$ *and* $\|g_a^{\alpha,q}\|_{2,\alpha,q} = \|g\|_{2,\alpha,q}$.

(c) $K_{1,q}(g_a^{\alpha,q}) = \dfrac{1}{|a|^{2(\beta-\alpha)}} \left(K_{2,q}(g)\right)_a^{\alpha,q}$ *and*

$K_{2,q}(g_a^{\alpha,q}) = \dfrac{1}{|a|^{2(\beta-\alpha)}} \left(K_{2,q}(g)\right)_a^{\beta,q}$

where $K_{1,q}$ and $K_{2,q}$ are given by Definition 5.4

(d) ${}^t\mathcal{S}_{\alpha,\beta}^q(g_a^{\alpha,q}) = \left({}^t\mathcal{S}_{\alpha,\beta}^q(g)\right)_a^{\beta,q}$ *and* ${}^t\mathcal{S}_{\alpha,\beta}^q(g_a^{\beta,q}) = \left({}^t\mathcal{S}_{\alpha,\beta}^q(g)\right)_a^{\alpha,q}$

Proposition 6.4. *If g is a q-wavelet associated with the q-Dunkl operator $\Lambda_{\alpha,q}$ in $\mathscr{S}_q(\mathbb{R}_q)$, then for all $a \in \mathbb{R}_q$ and $b \in \mathbb{R}_q \cup \{0\}$, the function $g_{a,b}^{\alpha,q}$ is a q-wavelet associated with the q-Dunkl operator $\Lambda_{\alpha,q}$ in $\mathscr{S}_q(\mathbb{R}_q)$ and we have*

$$C_{\alpha,q}(g_{a,b}^{\alpha,q}) = |a| \int_{-\infty}^{\infty} \left|\psi_b^{\alpha,q}(\frac{x}{a})\right|^2 |F_D^{\alpha,q}(g)(x)|^2 \frac{d_q x}{|x|}.$$

Definition 6.5. Let g be a q-wavelet associated with the q-Dunkl operator $\Lambda_{\alpha,q}$ in $\mathscr{S}_q(\mathbb{R}_q)$. We define the continuous q-wavelet transform associated with the q-Dunkl operator $\Lambda_{\alpha,q}$ for $f \in \mathscr{S}_q(\mathbb{R}_q)$ by

$$\begin{aligned}\Psi_g^{\alpha,q}(f)(a,b) &\\ &= \frac{(1+q)^{-\alpha}}{2\Gamma_{q^2}(\alpha+1)} \int_{-\infty}^{\infty} f(x)\overline{g_{a,b}^{\alpha,q}(-x)}|x|^{2\alpha+1} d_q x, \qquad a \in \mathbb{R}_q, b \in \mathbb{R}_q \cup \{0\},\end{aligned} \tag{31}$$

which can be written as

$$\Psi_g^{\alpha,q}(f)(a,b) = \sqrt{|a|}\left(f *_{\alpha,q} \overline{g_a^{\alpha,q}}\right)(b), \qquad a \in \mathbb{R}_q, b \in \mathbb{R}_q \cup \{0\}. \tag{32}$$

Theorem 6.6. Inversion formula: *If $g \in \mathscr{S}_q(\mathbb{R}_q)$ is a q-Dunkl wavelet associated with the q-Dunkl operator $\Lambda_{\alpha,q}$, then for all $f \in \mathscr{S}_q(\mathbb{R}_q)$ we have*

$$f(x) = \frac{(1+q)^{-\alpha}}{2\Gamma_{q^2}(\alpha+1)C_{\alpha,q}(g)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Psi_{q,g}^{\alpha}(f)(a,b) g_{a,b}^{\alpha,q}(-x)|b|^{2\alpha+1} \frac{d_q a d_q b}{|a|^2},$$

for all $x \in \mathbb{R}_q$.

Note that for the continuous q-Dunkl wavelet transform, there are also Parseval and Plancherel formulas (see [2] for more details).

Lemma 6.7. (see [2])

(*a*) *If* $f \in \Phi^q_\alpha(\mathbb{R}_q)$ *and* $g \in \mathscr{S}_q(\mathbb{R}_q)$, *then* $f *_{\alpha,q} g \in \Phi^q_\alpha(\mathbb{R}_q)$ *and we have*

$$K_{1,q}\left(f *_{\alpha,q} g\right) = \left(K_{1,q} f\right) *_{\alpha,q} g. \tag{33}$$

(*b*) *If* $f \in \Phi^q_\beta(\mathbb{R}_q)$ *and* $g \in \mathscr{S}_q(\mathbb{R}_q)$, *then* $f *_{\beta,q} g \in \Phi^q_\beta(\mathbb{R}_q)$ *and we have*

$$K_{2,q}\left(f *_{\beta,q} g\right) = \left(K_{2,q} f\right) *_{\beta,q} g. \tag{34}$$

Proposition 6.8.

(*a*) *If* $f, g \in \mathscr{S}_q(\mathbb{R}_q)$, *then*

$${}^t\mathcal{S}^q_{\alpha,\beta}(f *_\beta g) = {}^t\mathcal{S}^q_{\alpha,\beta} f *_\alpha {}^t\mathcal{S}^q_{\alpha,\beta} g. \tag{35}$$

(*b*) *If* $f \in \mathscr{S}_q(\mathbb{R}_q)$ *and* $g \in \Phi^q_\alpha(\mathbb{R}_q)$, *then*

$$\mathcal{S}^q_{\alpha,\beta}\left({}^t\mathcal{S}^q_{\alpha,\beta} f *_{\alpha,q} g\right) = f *_{\beta,q} \mathcal{S}^q_{\alpha,\beta}(g). \tag{36}$$

Proof. Part (a) follows by applying $F_D^{\alpha,q}$ to both sides of the equation (35).

For part (b), by the inversion formula (ii) of Theorem 5.5, we have

$$\mathcal{S}^q_{\alpha,\beta}\left({}^t\mathcal{S}^q_{\alpha,\beta} f *_\alpha g\right) = \mathcal{S}^q_{\alpha,\beta}\left[{}^t\mathcal{S}^q_{\alpha,\beta} f *_\alpha (K_{1,q} {}^t\mathcal{S}^q_{\alpha,\beta}\mathcal{S}^q_{\alpha,\beta})(g)\right].$$

Using Lemma 6.7 (a), we get

$$\mathcal{S}^q_{\alpha,\beta}\left({}^t\mathcal{S}^q_{\alpha,\beta} f *_\alpha g\right) = \mathcal{S}^q_{\alpha,\beta} K_{1,q}\left({}^t\mathcal{S}^q_{\alpha,\beta} f *_\alpha {}^t\, \mathcal{S}^q_{\alpha,\beta}(\mathcal{S}^q_{\alpha,\beta} g)\right)$$

So, by part (a) we obtain

$$\mathcal{S}^q_{\alpha,\beta}\left({}^t\mathcal{S}^q_{\alpha,\beta} f *_\alpha g\right) = \mathcal{S}^q_{\alpha,\beta} K_{1,q} {}^t\mathcal{S}^q_{\alpha,\beta}\left(f *_\beta \mathcal{S}^q_{\alpha,\beta} g\right),$$

and (36) follows from the inversion formula (iv) of Theorem 5.5. □

Now, we are in a situation to discuss the effect of the q-Dunkl-Sonine transform $\mathcal{S}^q_{\alpha,\beta}$ and its dual ${}^tS_{\alpha,\beta}$ on the q-Dunkl wavelets.

Proposition 6.9. *If* g *is a* q*-wavelet associated with the* q*-Dunkl operator* $\Lambda_{\beta,q}$ *in* $\mathscr{S}_q(\mathbb{R}_q)$ *(respectively in* $\Phi^q_\beta(\mathbb{R}_q)$*), then* ${}^t\mathcal{S}^q_{\alpha,\beta} g$ *is a* q*-wavelet associated with the* q*-Dunkl operator* $\Lambda_{\alpha,q}$ *in* $\mathscr{S}_q(\mathbb{R}_q)$ *(respectively in* $\Phi^q_\alpha(\mathbb{R}_q)$*) and we have*

$$C_{\alpha,q}\left({}^tS_{\alpha,\beta} g\right) = C_{\beta,q}(g).$$

Proof. Assume that g is a q-wavelet associated with the q-Dunkl operator $\Lambda_{\beta,q}$ in $\mathscr{S}_q(\mathbb{R}_q)$. Then Theorem 4.12 (a) implies that ${}^t\mathcal{S}^q_{\alpha,\beta}g \in \mathscr{S}_q(\mathbb{R}_q)$. Moreover, using the factorization relation (21) together with the Plancherel formula (12) we deduce that $C_{\alpha,q}({}^tS_{\alpha,\beta}g) = C_{\beta,q}(g)$. Hence, ${}^t\mathcal{S}^q_{\alpha,\beta}g$ is a q-wavelet associated with the operator $\Lambda_{\alpha,q}$ in $\mathscr{S}_q(\mathbb{R}_q)$. To complete the proof, it remains only to show that if $g \in \Phi^q_\beta(\mathbb{R}_q)$ then ${}^t\mathcal{S}^q_{\alpha,\beta}g \in \Phi^q_\alpha(\mathbb{R}_q)$. This follows from the fact that ${}^t\mathcal{S}^q_{\alpha,\beta} = (F^{\alpha,q}_D)^{-1} \circ F^{\beta,q}_D$ on $\mathscr{S}_q(\mathbb{R}_q)$ and Proposition 5.1.

□

Proposition 6.10. *Let g be a q-wavelet associated with the q-Dunkl operator $\Lambda_{\beta,q}$ in $\Phi^q_\beta(\mathbb{R}_q)$. If $f \in \Phi^q_\beta(\mathbb{R}_q)$, then*

$$\Psi^{\beta,q}_g(f)(a,b) = \mathcal{S}^q_{\alpha,\beta}\left[\Psi^{\alpha,q}_{{}^t\mathcal{S}^q_{\alpha,\beta}g}\big((\mathcal{S}^q_{\alpha,\beta})^{-1}f\big)(a,.)\right](b), \qquad a,b \in \mathbb{R}_q, \qquad (37)$$

and

$$\Psi^{\alpha,q}_{{}^t\mathcal{S}^q_{\alpha,\beta}g}(f)(a,b) = {}^t\mathcal{S}^q_{\alpha,\beta}\left[\Psi^{\beta,q}_g\big(({}^t\mathcal{S}^q_{\alpha,\beta})^{-1}(f)\big)(a,.)\right](b), \qquad a,b \in \mathbb{R}_q. \qquad (38)$$

Proof. It follows, from Definition 6.5 together with Proposition 6.8 (b), that

$$\begin{aligned}\Psi^{\beta,q}_g(f)(a,b) &= \sqrt{|a|}\big(\overline{g^{\beta,q}_a} *_\beta f\big)(b)\\ &= \sqrt{|a|}S_{\alpha,\beta}\big({}^t\mathcal{S}^q_{\alpha,\beta}(\overline{g^{\beta,q}_a}) *_\alpha (\mathcal{S}^q_{\alpha,\beta})^{-1}f\big)(b)\\ &= \mathcal{S}^q_{\alpha,\beta}\left(\sqrt{|a|}(\mathcal{S}^q_{\alpha,\beta})^{-1}f *_\alpha \overline{{}^t\mathcal{S}^q_{\alpha,\beta}(g^{\beta,q}_a)}\right)(b).\end{aligned}$$

Since ${}^t\mathcal{S}^q_{\alpha,\beta}(g^{\beta,q}_a) = \big({}^t\mathcal{S}^q_{\alpha,\beta}(g)\big)^{\alpha,q}_a$, as mentioned in Proposition 6.3 (d), and ${}^t\mathcal{S}^q_{\alpha,\beta}g$ is a q-Dunkl wavelet, by Proposition 6.9, we can write

$$\begin{aligned}\Psi^{\beta,q}_g(f)(a,b) &= \mathcal{S}^q_{\alpha,\beta}\left(\sqrt{|a|}(\mathcal{S}^q_{\alpha,\beta})^{-1}f *_\alpha \overline{\left({}^t\mathcal{S}^q_{\alpha,\beta}(g)\right)_{a,\beta}}\right)(b)\\ &= \mathcal{S}^q_{\alpha,\beta}\left[\Psi^{\alpha,q}_{{}^t\mathcal{S}^q_{\alpha,\beta}g}\big((\mathcal{S}^q_{\alpha,\beta})^{-1}f\big)(a,.)\right](b).\end{aligned}$$

Thus, we have proved (37). Similarly, (38) is derived using Proposition 6.8 (b) instead of Proposition 6.8 (a). □

Lemma 6.11.

(*a*) *If $g \in \Phi^q_\alpha(\mathbb{R}_q)$ is a q-wavelet associated with the q-Dunkl operator $\Lambda_{\alpha,q}$, then $K_{1,q}g$ is a q-wavelet associated with the q-Dunkl operator $\Lambda_{\alpha,q}$.*

(*b*) *If $g \in \Phi^q_\beta(\mathbb{R}_q)$ is a q-wavelet associated with the q-Dunkl operator $\Lambda_{\beta,q}$, then $K_{2,q}g$ is a q-wavelet associated with he q-Dunkl operator $\Lambda_{\beta,q}$.*

Proof. For part (a), by Proposition 5.1, we have $K_{1,q}g \in \Psi_q(\mathbb{R}_q)$ and by Definition 5.4, we have

$$F_D^{\alpha,q}(K_{1,q}g)(\lambda) = |\lambda|^{2(\beta-\alpha)} F_D^{\alpha,q}(g)(\lambda).$$

Since $F_D^{\alpha,q}(g)$ is continuous at the orgin, it follows that

$$F_D^{\alpha,q}(K_{1,q}g)(\lambda) = O(|\lambda|^{2(\beta-\alpha)}) \qquad (\lambda \to 0).$$

Hence, by Proposition 6.2, the function $K_{1,q}g$ is a q-wavelet associated with the q-Dunkl operator $\Lambda_{\alpha,q}$.

The proof of part (b) is similar to the proof of part (a). □

Proposition 6.12. *If g is a q-wavelet associated with the q-Dunkl operator $\Lambda_{\beta,q}$ in $\Phi_\beta^q(\mathbb{R}_q)$, then*

(a) *For all $f \in \Phi_\beta^q(\mathbb{R}_q)$ and all a, $b \in \mathbb{R}_q$, we have:*

$$\Psi_g^{\beta,q}(f)(a,b) = \frac{1}{|a|^{2(\beta-\alpha)}} \mathcal{S}_{\alpha,\beta}^q \left[\Psi^{\alpha,q}_{K_{q,1}({}^t\mathcal{S}^q_{\alpha,\beta}g)}({}^t\mathcal{S}^q_{\alpha,\beta}f)(a,.) \right](b).$$

(b) *For all $f \in \Phi_\alpha^q(\mathbb{R}_q)$ and all a, $b \in \mathbb{R}_q$ we have:*

$$\Psi^{\alpha,q}_{{}^t\mathcal{S}^q_{\alpha,\beta}g}(f)(a,b) = \frac{1}{|a|^{2(\beta-\alpha)}} {}^t\mathcal{S}^q_{\alpha,\beta} \left[\Psi^{\alpha,q}_{K_{q,2}g}(\mathcal{S}^q_{\alpha,\beta}(f))(a,.) \right](b).$$

Proof. We prove only part (a), the proof of part (b) is similar. Let $f \in \Phi_\alpha^q(\mathbb{R}_q)$. Since by the inversion formula (ii) of Theorem 5.5 we have

$$(\mathcal{S}^q_{\alpha,\beta})^{-1} f = K_{1,q}({}^t\mathcal{S}^q_{\alpha,\beta})(f),$$

it follows that

$$\Psi^{\alpha,q}_{{}^t\mathcal{S}^q_{\alpha,\beta}g}\left((\mathcal{S}^q_{\alpha,\beta})^{-1}f\right)(a,b) = \left((K_{1,q}{}^t\mathcal{S}^q_{\alpha,\beta})(f) *_{\alpha,q} \overline{({}^t\mathcal{S}^q_{\alpha,\beta}g)_a^{\alpha,q}}\right)(b).$$

Using Lemma 6.7 (a), we obtain

$$\Psi^{\alpha,q}_{{}^t\mathcal{S}^q_{\alpha,\beta}g}\left((\mathcal{S}^q_{\alpha,\beta})^{-1}f\right)(a,b) = \left({}^t\mathcal{S}^q_{\alpha,\beta}(f) *_{\alpha,q} K_{1,q}(\overline{\left({}^t\mathcal{S}^q_{\alpha,\beta}g\right)_a^{\alpha,q}})\right)(b).$$

By simple computation, we can see that

$$K_{1,q}(\overline{\left({}^t\mathcal{S}^q_{\alpha,\beta}g\right)_a^{\alpha,q}}) = \overline{K_{1,q}(\left({}^t\mathcal{S}^q_{\alpha,\beta}g\right)_a^{\alpha,q})}.$$

Using Proposition 6.3 (d), we obtain

$$\begin{aligned}\Psi^{\alpha,q}_{{}^t\mathcal{S}^q_{\alpha,\beta}g}\left((\mathcal{S}^q_{\alpha,\beta})^{-1}f\right)(a,b) &= \frac{1}{|a|^{2(\beta-\alpha)}}\left({}^t\mathcal{S}^q_{\alpha,\beta}(f) *_{\alpha,q} \overline{\left(K_{1,q}{}^t\mathcal{S}^q_{\alpha,\beta}g\right)_a^{\alpha,q}}\right)(b)\\ &= \frac{1}{|a|^{2(\beta-\alpha)}}\Psi^{\alpha,q}_{K_{1,q}({}^t\mathcal{S}^q_{\alpha,\beta}g)}\left({}^t\mathcal{S}^q_{\alpha,\beta}f\right)(a,b).\end{aligned}$$

Hence, (37) can be written as

$$\Psi_g^{\beta,q}(f)(a,b) = \frac{1}{|a|^{2(\beta-\alpha)}} \mathcal{S}_{\alpha,\beta}^q \left[\Psi^{\alpha,q}_{K_{q,1}({}^t\mathcal{S}^q_{\alpha,\beta} g)} ({}^t\mathcal{S}^q_{\alpha,\beta} f)(a,.) \right](b),$$

which achieves the proof. □

Now, using Theorem 6.6 together with Proposition 6.12, we get inversion formulas for the q-dunkl-Sonine transform and its dual. This is the purpose of the following theorem.

Theorem 6.13. *Let g be a q-wavelet associated with the q-Dunkl operator $\Lambda_{\beta,q}$ in $\Phi^q_\beta(\mathbb{R}_q)$ and set*

$$\widetilde{C}_{\beta,q}(g) = \frac{(1+q)^{-\beta}}{2\Gamma_{q^2}(\beta+1)C_{\beta,q}(g)}$$

(a) If $f \in \Phi^q_\alpha(\mathbb{R}_q)$, then for all $x \in \mathbb{R}_q$, we have

$$({}^t\mathcal{S}^q_{\alpha,\beta})^{-1}(f)(x) = \widetilde{C}_{\beta,q}(g)\times$$
$$\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \mathcal{S}^q_{\alpha,\beta} \left[\Psi^{\alpha,q}_{K_{q,1}({}^t\mathcal{S}^q_{\alpha,\beta} g)} (({}^t\mathcal{S}^q_{\alpha,\beta})(f))(a,.) \right](b) g^{\beta,q}_{a,b}(-x) \frac{|b|^{2\beta+1}}{|a|^{2(\beta-\alpha+1)}} d_q b \right) d_q a.$$

(b) If $f \in S_{\beta,q}(\mathbb{R}_q)$, then for all $x \in \mathbb{R}_q$, we have

$$S^{-1}_{\alpha,\beta}(f)(x) = \widetilde{C}_{\beta,q}(g)\times$$
$$\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} {}^tS_{\alpha,\beta} \left[\Psi^{\beta,q}_{K_{q,2}(g)}(f)(a,.) \right](b)\, {}^tS_{\alpha,\beta}(g^{\beta,q}_{a,b})(-x) \frac{|b|^{2\beta+1}}{|a|^{2(\beta-\alpha+1)}} d_q b \right) d_q a.$$

REFERENCES

[1] N. Bettaibi - R. H. Bettaieb, *q-Analogue of the Dunkl transform on the real line*, Tamsui Oxf. J. Math. Sci. 25 (2) (2009), 178–206.

[2] N. Bettaibi - R. H. Bettaieb - S. Bouaziz, *Wavelet transforms associated with the q-Dunkl operator*, Tamsui Oxf. J. Math. Sci. 26 (1) (2010), 77–101.

[3] F. Bouzeffour, *Inversion formulas for q-Riemann-Liouville and q-Weyl transforms*, J. Math. Analys. App. 336 (2007), 833–848.

[4] H. Chebli - A. Fitouhi - M. M. Hamza, *Expansion in series of Bessel functions and transmutations for perturbed Bessel operator*, J. Math. Anal. Appl. 181 (3) (1994), 789–802.

[5] J. Delsarte - J. Lions, *Transmutations d'opérateurs différentiels dans le domaine complexe*, Comment. Mat. Helv. 32 (1957), 113–128.

[6] C. F. Dunkl, *Differential-difference operators associated with reflections groups*, Trans. Amer. Math. Soc. 311 (1989), 167–183.

[7] C. F. Dunkl, *Integral kernels with reflection group invariance*, Canad. J. Math. 43 (1991), 1213–1227.

[8] C. F. Dunkl, *Hankel transforms associated to finite reflection groups*, Contemp. Math. 138 (1992), 123–138.

[9] A. Fitouhi - M. M. Hamza - F. Bouzeffour, *The $q-j_\alpha$ Bessel function*, J. Approximation Theory 115 (2002), 114–166.

[10] A. Fitouhi - N. Bettaibi - W. Binous, *Inversion formulas for the q-Riemann-Liouville and q-Weyl transforms using wavelets*, Fract. Calc. Appl. Anal. 10 (4) (2007), 327–342.

[11] G. Gasper - M. Rahman, *Basic Hypergeometric Series*, Encyclopedia of Mathematics and its application, Vol 35 Cambridge Univ. Press, Cambridge, UK, 1990.

[12] A. Jouini, *Dunkl wavelets and applications to inversion of the Dunkl intertwining operator and its dual*, Int. J. Math. Math. Sci. 6 (2004), 285–293.

[13] V. Kiryakova, *Generalized Fractional Calculus and Applications*, Longman, Harlow; John Wiley, N. Y. 1994.

[14] T. H. Koornwinder - R. F. Swarttouw, *On q-analogues of the Fourier and Hankel transforms*, Trans. Amer. Math. Soc. 333 (1992), 445–461.

[15] T. H. Koornwinder, *q-Special Functions, a Tutorial*, in Deformation theory and quantum groups with applications to mathematical physics, M. Gerstenhaber and J. Stasheff (eds), Contemp. Math. 134, Amer. Math. Soc., 1992.

[16] B. M. Levitan, *Transmutation Operators and the Inverse Spectral Problem*, Applications of hypergroups and related measure algebra (Joint Summer Reserche Conference Seattle 1993) American Mathematical Society, Providence, 1994.

[17] M. A. Mourou, *Inversion of the Dual Dunkl-Sonine Transform on $\mathbb{R}$ Using Dunkl Wavelets*, SIGMA 5 (2009), 071, arXiv: 0907.2341.

[18] M. A. Mourou, *Sonine-type integraral transform related to the Dunkl operator on the real line*, Integral Transform and Special Functions, 20 (11-12) (2009), 915–924.

[19] M. Rösler, *Bessel-type Signed Hypergroups on $\mathbb{R}$*, H. Heyer and A. Mukherjea (eds.): Probability measures on groups and related structures XI (Oberwolfach 1994), 292–304. World Sci. Publ., River Edge, NJ, 1995.

[20] R. L. Rubin, *A q^2-analogue operator for q^2-analogue Fourier Analysis*, J. Math. Analys. App. 212 (1997), 571–582.

[21] R. L. Rubin, *Duhamel solutions of non-homogenous q^2-analogue wave equations*, Proc. of Amer. Math. Soc. 135 (3) (2007), 777–785.

[22] F. Soltani, *Sonine Transform Associated to the Dunkl Kernel on the real line*, SIGMA 4 (2008), paper 092.

[23] K. Trimèche, *Generalized harmonic analysis and wavelet packets*, Gordon and Breach Science Publishers, Amsterdam, 2001.

[24] K. Trimèche, *The Dunkl intertwining operator on spaces of functions and distributions and integral representation of its dual*, Integral Transform. Spec. Funct. 12 (2001), 349–374.

[25] K. Trimèche, *Inversion of Lions transmutation operators using generalized wavelets*, Appl. Comput. Harmonic Anal. 4 (1997), 97–112.

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