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SUBORDINATION PRESERVING PROPERTIES ASSOCIATED WITH A CLASS OF OPERATORS

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The purpose of this paper is to find some subordination preserving properties of analytic functions associated with a class of operators with complex parameters. Due to the compositional structure of the involved operator, we take its advantage in deducing results which involve more familiar operators, thereby, exhibiting the usefulness of the main results.

1. Introduction

Let $\mathcal{H}(\mathbb{U})$ denotes a linear space of all analytic functions defined in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. For $a \in \mathbb{C}$, $n \in \mathbb{N}$, let

$$\mathcal{H}[a,n] = \{f \in \mathcal{H}(\mathbb{U}) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots\}.$$

We denote the special class of $\mathcal{H}[0,1]$ by $\mathcal{A}$ whose members are of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, z \in \mathbb{U}. \tag{1}$$

Let $\mathcal{K}$ denotes a subclass of $\mathcal{A}$ whose members are convex (univalent) in $\mathbb{U}$, satisfying

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, z \in \mathbb{U}.$$

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For two functions $f, g \in \mathcal{H}(\mathbb{U})$, we say f is subordinate to g in $\mathbb{U}$ and write $f(z) \prec g(z), z \in \mathbb{U}$, if there exists a Schwarz function ω, analytic in $\mathbb{U}$ with $\omega(0) = 0$, and $|\omega(z)| < 1, z \in \mathbb{U}$ such that $f(z) = g(\omega(z)), z \in \mathbb{U}$. Furthermore, if the function g is univalent in $\mathbb{U}$, then we have following equivalence:

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}). \tag{2}$$

Let for $m \in \mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$ and for $\mu > -1,\ k > 0$, a linear operator $\mathfrak{I}^m_{k,\mu} : \mathcal{A} \to \mathcal{A}$ be defined by

$$\begin{aligned} \mathfrak{I}^m_{k,\mu} f(z) &= f(z),\ \ m = 0, \qquad (3)\\ &= \frac{\mu+1}{k} z^{1-\frac{\mu+1}{k}} \int_0^z t^{\frac{\mu+1}{k}-2} \mathfrak{I}^{m+1}_{k,\mu} f(z)dt,\ \ m = -1, -2, ...,\\ &= \frac{k}{\mu+1} z^{2-\frac{\mu+1}{k}} \frac{d}{dz} \left(z^{\frac{\mu+1}{k}-1} \mathfrak{I}^{m-1}_{k,\mu} f(z) \right),\ \ m = 1, 2, ... \end{aligned}$$

Let for $A > 0,\ a, c \in \mathbb{C}$, be such that $\Re(c-a) \geq 0$, an Erdélyi-Kober type ([11], Ch. 5) integral operator $\widetilde{I}^{a,c}_A : \mathcal{A} \to \mathcal{A}$ be defined for $\Re(c-a) > 0$ and $\Re(a) > -A$ by

$$\widetilde{I}^{a,c}_A f(z) = \frac{\Gamma(c+A)}{\Gamma(a+A)} \frac{1}{\Gamma(c-a)} \int_0^1 (1-t)^{c-a-1} t^{a-1} f(zt^A) dt, \tag{4}$$

and

$$\widetilde{I}^{a,a}_A f(z) = f(z). \tag{5}$$

By iterations of the linear operators (defined above), a class of operators $\mathfrak{I}^m_{k,\mu}(a,c,A) : \mathcal{A} \to \mathcal{A}$ is defined for the purpose of this paper by

$$\mathfrak{I}^m_{k,\mu}(a,c,A) f(z) = \mathfrak{I}^m_{k,\mu} \left(\widetilde{I}^{a,c}_A f(z) \right) = \widetilde{I}^{a,c}_A \left(\mathfrak{I}^m_{k,\mu} f(z) \right), \tag{6}$$

whose series expansion for $m \in \mathbb{Z},\ \mu > -1,\ k > 0,\ A > 0,\ \Re(c-a) \geq 0$, $\Re(a) > -A$ and for f of the form (1) is given by

$$\mathfrak{I}^m_{k,\mu}(a,c,A) f(z) = z + \frac{\Gamma(c+A)}{\Gamma(a+A)} \sum_{n=2}^{\infty} \left(1 + \frac{k(n-1)}{\mu+1} \right)^m \frac{\Gamma(a+nA)}{\Gamma(c+nA)} a_n z^n. \tag{7}$$

We note that this new class of operators $\mathfrak{I}^m_{k,\mu}(a,c,A)$ was, in fact, introduced recently in [18] by the authors in different perspective and its relationships with some known operators are exhibited therein. We may point out here that some of the special cases of the operator defined by (7) can be found in [3], [5], [8],

[10], [12], [19], [20] etc. In view of (3), (5) and (6), it is easy to notice the following relationships:

$$\Im_{k,\mu}^{m}(a,a,A)=\Im_{k,\mu}^{m},\ \Im_{k,\mu}^{0}(a,c,A)=\widetilde{I}_{A}^{a,c}. \tag{8}$$

From (7), we get for $f\in\mathcal{A}$:

$$\begin{aligned}&\Im_{k,\mu}^{m+1}(a,c,A)f(z)\\&\quad=\left(1-\frac{k}{\mu+1}\right)\Im_{k,\mu}^{m}(a,c,A)f(z)+\frac{k}{\mu+1}\,z\left(\Im_{k,\mu}^{m}(a,c,A)f(z)\right)'\end{aligned} \tag{9}$$

and

$$\Im_{k,\mu}^{m}(a+1,c,A)f(z)=\frac{a}{a+A}\Im_{k,\mu}^{m}(a,c,A)f(z)+\frac{A}{a+A}\,z\left(\Im_{k,\mu}^{m}(a,c,A)f(z)\right)'. \tag{10}$$

Following definitions are due to Miller and Mocanu.

Definition 1.1. ([14], Definition 2.2b, p.21) Denote by $\mathcal{Q}$ the class of functions f that are analytic and injective on $\overline{\mathbb{U}}\setminus E(f)$, where

$$E(f)=\left\{\zeta\in\partial\mathbb{U}:\lim_{z\to\zeta}f(z)=\infty\right\},$$

and are such that $f'(\zeta)\neq 0$ for $\zeta\in\partial\mathbb{U}\setminus E(f)$.

Definition 1.2. ([14], p. 16) Let $\psi:\mathbb{C}^2\to\mathbb{C}$ and let h be univalent in $\mathbb{U}$. If p is analytic in $\mathbb{U}$ and satisfies the following differential subordination

$$\psi\left(p(z),zp'(z)\right)\prec h(z), \tag{11}$$

then p is called a solution of the differential subordination (11). A univalent function q is called a dominant of the solutions of the differential subordination (11) or, more simply, a dominant if $p(z)\prec q(z)$ for all p satisfying (11). A dominant $\widetilde{q}$ that satisfies $\widetilde{q}(z)\prec q(z)$ for all dominants q of (11) is said to be the best dominant of (11).

A function $L(z,t):\mathbb{U}\times[0,\infty)\to\mathbb{C}$ is called a Löwner (subordination) chain if $L(.,t)$ is analytic and univalent in $\mathbb{U}$ for all $t\geq 0$, and $L(z,s)\prec L(z,t),\ 0\leq s\leq t$.

Recently, based on certain linear operators, some subordination preserving results have been obtained in [1], [2], [4], [6], [7], [13], [21] and [22] etc. In this paper, we obtain some subordination preserving properties associated with the new class of operators $\Im_{k,\mu}^{m}(a,c,A)$ involving complex parameters. The class $\Im_{k,\mu}^{m}(a,c,A)$ which is expressed as the composition of the operators (3) and (4) is evidently of a dual nature. Some results associated with the operators $\Im_{k,\mu}^{m}$ and $\widetilde{I}_{A}^{a,c}$ are also mentioned.

2. Preliminary results

We use following lemmas in proving our results.

Lemma 2.1. *([16], Theorem 1, p. 300) Let* $\beta,\gamma \in \mathbb{C}$ *with* $\beta \neq 0$, *and let* $h \in \mathcal{H}(\mathbb{U})$ *with* $h(0) = c$. *If* $\Re(\beta h(z) + \gamma) > 0$ *for* $z \in \mathbb{U}$, *then the differential equation*

$$q(z) + \frac{zq^{'}(z)}{\beta q(z) + \gamma} = h(z),\ q(0) = c,$$

has an analytic solution in $\mathbb{U}$, *that satisfy* $\Re(\beta q(z) + \gamma) > 0, z \in \mathbb{U}$.

Lemma 2.2. *([14], Theorem 2.3i, p. 35) Suppose that the function* $H : \mathbb{C}^2 \to \mathbb{C}$ *satisfies the condition*

$$\Re(H(is,t)) \leq 0,$$

for all $s,t \in \mathbb{R}$ *with* $t \leq -n\left(1+s^2\right)/2$, $n \in \mathbb{N}$. *If the function* $p(z) = 1 + p_n z^n + p_{n+1}z^{n+1} + \ldots$ *is analytic in* $\mathbb{U}$ *and*

$$\Re\left(H\left(p(z), zp^{'}(z)\right)\right) > 0, z \in \mathbb{U},$$

then $\Re(p(z)) > 0, z \in \mathbb{U}$.

Lemma 2.3. *([14], Lemma 2.2d, p. 24) Let* $q \in \mathcal{Q}$ *with* $q(0) = a$, *and let* $p(z) = a + a_n z^n + a_{n+1}z^{n+1} + \ldots$ *be analytic in* $\mathbb{U}$ *with* $p(z) \neq a, n \in \mathbb{N}$. *If p is not subordinate to q, then there exist the points* $z_0 = r_0 e^{i\theta} \in \mathbb{U}$ *and* $\zeta_0 \in \partial\mathbb{U} \setminus E(f)$ *such that* $p(\mathbb{U}_{r_0}) \subset q(\mathbb{U})$, $p(z_0) = q(\zeta_0)$, *and* $z_0 p^{'}(z_0) = m\zeta_0 q^{'}(\zeta_0), m \geq n$, *where* $\mathbb{U}_{r_0} = \{z \in \mathbb{C} : |z| < r_0\}$.

Lemma 2.4. *([15], Theorem 7, p. 822) Let* $q \in \mathcal{H}[a,1]$, *let* $\phi : \mathbb{C}^2 \to \mathbb{C}$, *and set* $\phi\left(q(z), zq^{'}(z)\right) \equiv h(z)$. *If* $L(z,t) = \phi\left(q(z), tzq^{'}(z)\right)$ *is a subordination chain and* $p \in \mathcal{H}[a,1] \cap \mathcal{Q}$, *then*

$$h(z) \prec \phi\left(p(z), zp^{'}(z)\right)$$

implies that

$$q(z) \prec p(z).$$

Furthermore, if the differential equation $\phi\left(q(z), zq^{'}(z)\right) = h(z)$ *has a univalent solution* $q \in \mathcal{Q}$, *then q is the best subordinant.*

Lemma 2.5. *([17], p. 159) Let* $L(z,t) = a_1(t)z + a_2(t)z^2 + \ldots$, *with* $a_1(t) \neq 0$ *for all* $t \geq 0$ *and* $\lim\limits_{t\to+\infty} |a_1(t)| = +\infty$. *Suppose that* $L(.,t)$ *is analytic in* $\mathbb{U}$ *for* *all $t \geq 0$, $L(z,.)$ is continuously differentiable on $[0,\infty)$ for all $z \in \mathbb{U}$. If $L(z,t)$ satisfies*

$$\Re\left(z\frac{\partial L/\partial z}{\partial L/\partial t}\right) > 0, z \in \mathbb{U}, t \geq 0$$

and

$$|L(z,t)| \leq K_0 \, |a_1(t)|\,, |z| < r_0 < 1, t \geq 0$$

for some positive constants K_0 and r_0, then $L(z,t)$ is a subordination chain.

3. Main Result

Theorem 3.1. *Let for $m \in \mathbb{Z}$, $\mu > -1$, $k > 0$, $A > 0$, $a, c \in \mathbb{C}$ satisfying $\Re(c-a) \geq 0$ and $\Re(a) > -A$, the operator $\Im_{k,\mu}^{m}(a,c,A)$ be defined by (6). Let for $0 \leq \lambda \leq 1$,*

$$\frac{(\mu+1)(a+A)}{(1-\lambda)k(a+A)+\lambda A(\mu+1)} =: \delta \tag{12}$$

be such that $\Re(\delta) \geq 1$ and for $g \in \mathcal{A}$,

$$\varphi(z) := (1-\lambda)\Im_{k,\mu}^{m+1}(a,c,A)g(z) + \lambda \Im_{k,\mu}^{m}(a+1,c,A)g(z),$$

satisfy

$$\Re\left(1+\frac{z\varphi''(z)}{\varphi'(z)}\right) > -\rho,\ z \in \mathbb{U}, \tag{13}$$

where $\rho = 0$ if $\Re(\delta) = 1$ and for $\Re(\delta) > 1$,

$$\rho \leq \begin{cases} \frac{\Re(\delta)-1}{2}, & 1 < \Re(\delta) \leq 2, \\ \frac{1}{2(\Re(\delta)-1)}, & \Re(\delta) > 2, \end{cases} \tag{14}$$

and

$$(Im(\delta))^2 \leq (\Re(\delta)-1-2\rho)\left(\frac{1}{2\rho}-\Re(\delta)+1\right), \tag{15}$$

the equality in (14) and (15) occur only when $Im(\delta) = 0$. If $f \in \mathcal{A}$ satisfies

$$(1-\lambda)\Im_{k,\mu}^{m+1}(a,c,A)f(z) + \lambda \Im_{k,\mu}^{m}(a+1,c,A)f(z) \prec \varphi(z), z \in \mathbb{U} \tag{16}$$

then

$$\Im_{k,\mu}^{m}(a,c,A)f(z) \prec \Im_{k,\mu}^{m}(a,c,A)g(z), z \in \mathbb{U}. \tag{17}$$

Moreover, the function $\Im_{k,\mu}^{m}(a,c,A)g(z)$ is the best dominant of (16).

Proof. Let

$$F(z) := \Im_{k,\mu}^{m}(a,c,A)\, f(z) \text{ and } G(z) := \Im_{k,\mu}^{m}(a,c,A)\, g(z). \tag{18}$$

By the hypothesis, we first show that the function G is convex (univalent). For, let

$$q(z) = 1 + \frac{zG^{''}(z)}{G^{'}(z)}, z \in \mathbb{U}. \tag{19}$$

Using (9) and (10) for $g \in \mathcal{A}$, we get

$$\varphi(z) = \left(1 - \frac{1}{\delta}\right) G(z) + \frac{zG^{'}(z)}{\delta}, \tag{20}$$

where δ is given by (12). On differentiating (20) and using (19), we obtain

$$\frac{\varphi^{'}(z)}{G^{'}(z)} = \left(1 - \frac{1}{\delta}\right) + \frac{1}{\delta} q(z),$$

which on differentiating further and using (19) yields

$$1 + \frac{z\varphi^{''}(z)}{\varphi^{'}(z)} = q(z) + \frac{zq^{'}(z)}{q(z) + \delta - 1} =: h(z). \tag{21}$$

From (13) and (14), we have

$$\Re\left(h(z) + \delta - 1\right) > 0, z \in \mathbb{U},$$

and by Lemma 2.1, we deduce that the differential equation (21) has a solution $q \in \mathcal{H}(\mathbb{U})$, with $q(0) = h(0) = 1$. Let

$$H(u,v) := u + \frac{v}{u + \delta - 1} + \rho, \tag{22}$$

where ρ is given by (14). From (13), (21) and (22), we obtain

$$\Re\left(H\left(q(z), zq^{'}(z)\right)\right) > 0, z \in \mathbb{U}.$$

For all $s \in \mathbb{R}$ and $t \leq -\left(1 + s^2\right)/2$, using (22), we get

$$\begin{aligned} \Re\left(H(is,t)\right) &= \Re\left(is + \frac{t}{is + \delta - 1} + \rho\right) \\ &= \frac{\left(\Re(\delta) - 1\right)t}{\left|is + \delta - 1\right|^2} + \rho. \end{aligned} \tag{23}$$

If $\Re\,(\delta)=1, \rho=0$, we get $\Re\,(H\,(is,t))=0$ and if $\Re\,(\delta)>1$,

$$\Re\,(H\,(is,t)) \leq -\frac{\Psi(s,\rho,\delta)}{2\,|is+\delta-1|^2} \tag{24}$$

where

$$\Psi(s,\rho,\delta)=(\Re\,(\delta)-1)\left(1+s^2\right)-2\rho\,|is+\delta-1|^2,$$

which on taking $\Re\,(\delta)-1=u$ and $\mathrm{Im}\,(\delta)=v$, we write

$$\Psi(s,\rho,\delta)=(u-2\rho)\,s^2-4\rho vs+u-2\rho\left(u^2+v^2\right).$$

If $v=0$, from (14), we get

$$\Psi(s,\rho,\delta)=(u-2\rho)\,s^2+u\,(1-2\rho u)\geq 0.$$

If $v\neq 0$, by hypothesis $u-2\rho>0$ for any $u>0$, and hence, we obtain

$$\begin{aligned}\Psi(s,\rho,\delta) &= (u-2\rho)\left(s-\frac{2\rho v}{u-2\rho}\right)^2-\frac{4\rho^2v^2}{u-2\rho}+u-2\rho\left(u^2+v^2\right)\\ &= (u-2\rho)\left(s-\frac{2\rho v}{u-2\rho}\right)^2+u\left[1-2\rho\left(u+\frac{v^2}{u-2\rho}\right)\right]\geq 0,\end{aligned}$$

from condition (15). Thus, $\Psi(s,\rho,\delta)\geq 0$ for all $s\in\mathbb{R}$. Hence, from (24) and (23), we have $\Re\,(H\,(is,t))\leq 0$ for all $s\in\mathbb{R}$ and $t\leq-\left(1+s^2\right)/2$. Thus, by using Lemma 2.2, we conclude that $\Re\,(q(z))>0$ for all $z\in\mathbb{U}$, which proves that the function G defined by (18) is convex (univalent) in $\mathbb{U}$.

We next prove that

$$F(z)\prec G(z), z\in\mathbb{U}, \tag{25}$$

if the subordination condition (16) holds. Without loss of generality, we can assume that G is analytic and univalent in $\overline{\mathbb{U}}$ and $G'(\zeta)\neq 0$ for $|\zeta|=1$. Otherwise, we replace F and G by $F_r(z)=F(rz)$ and $G_r(z)=G(rz)$, respectively, where $r\ (0<r<1)$. These functions satisfy the conditions of the theorem on $\overline{\mathbb{U}}$, and we need to prove that $F_r(z)\prec G_r(z)$ for all $r\ (0<r<1)$, which enables us to obtain (25) by letting $r\to 1^-$.

Let us define a function $L(z,t)$ by

$$L(z,t):=\left(1-\frac{1}{\delta}\right)G(z)+\frac{(1+t)\,zG'(z)}{\delta}, z\in\mathbb{U}, t\geq 0. \tag{26}$$

Then,

$$\left.\frac{\partial L(z,t)}{\partial z}\right|_{z=0}=G'(0)\left(1+\frac{t}{\delta}\right)=1+\frac{t}{\delta}\neq 0, t\geq 0,$$

and this shows that the function $L(z,t) = a_1(t)z + a_2(t)z^2 + \ldots$, with $a_1(t) = 1+\frac{t}{\delta} \neq 0$ for all $t \geq 0$ and $\lim_{t\to+\infty} |a_1(t)| = +\infty$.

From the definition (26) and using the assumption (12), for all $t \geq 0$, we get

$$\begin{aligned} \frac{|L(z,t)|}{|a_1(t)|} &\leq \frac{|\delta-1|}{|\delta+t|}|G(z)| + \frac{(1+t)}{|\delta+t|}\left|zG'(z)\right| \\ &\leq |G(z)| + \left|zG'(z)\right|. \end{aligned} \tag{27}$$

Since the function G is convex and normalized in the unit disk, i.e. $G \in \mathcal{K}$, we have the following growth and distortion sharp bounds (see [9]):

$$\begin{aligned} \frac{r}{1+r} &\leq |G(z)| \leq \frac{r}{1-r}, |z| \leq r < 1, \\ \frac{1}{(1+r)^2} &\leq \left|G'(z)\right| \leq \frac{1}{(1-r)^2}, |z| \leq r < 1. \end{aligned}$$

Using the the upper bounds from these inequalities in (27), we deduce that

$$\frac{|L(z,t)|}{|a_1(t)|} \leq \frac{r}{1-r} + \frac{r}{(1-r)^2} \leq \frac{r}{(1-r)^2}, |z| \leq r < 1, t \geq 0$$

and thus, the second assumption of Lemma 2.5 holds.

Furthermore, from (26), we get

$$\Re\left(z\frac{\partial L(z,t)/\partial z}{\partial L(z,t)/\partial t}\right) = \Re(\delta) - 1 + (1+t)\Re\left(1+\frac{zG''(z)}{G'(z)}\right) > 0, z \in \mathbb{U}, t \geq 0,$$

and according to Lemma 2.5, the function $L(z,t)$ is a subordination chain. From the definition of the subordination chain combined with (2), we obtain

$$L(\zeta,t) \notin L(\mathbb{U},0) = \varphi(\mathbb{U}) \text{ whenever } \zeta \in \partial\mathbb{U}, t \geq 0.$$

Suppose that F is not subordinate to G, then by Lemma 2.3 there exists the points $z_0 \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U}$, and the number $t \geq 0$, such that

$$F(z_0) = G(\zeta_0) \text{ and } z_0F'(z_0) = (1+t)\zeta_0 G'(\zeta_0).$$

From these two relations, and by virtue of the subordination condition (16), we deduce that

$$\begin{aligned} L(\zeta_0,t) &= \left(1-\frac{1}{\delta}\right)G(\zeta_0) + \frac{(1+t)\ \zeta_0 G'(\zeta_0)}{\delta} \\ &= \left(1-\frac{1}{\delta}\right)F(z_0) + \frac{z_0F'(z_0)}{\delta} \\ &= (1-\lambda)\Im_{k,\mu}^{m+1}(a,c,A)f(z_0) + \lambda\Im_{k,\mu}^{m}(a+1,c,A)f(z_0) \in \varphi(\mathbb{U}), \end{aligned}$$

which contradicts the above observation that $L(\zeta,t) \notin \varphi(\mathbb{U})$. Therefore, the subordination condition (16) must imply the subordination given by (25). Considering $F(z) = G(z)$, we see that the function G is the best dominant, which completes the proof of the theorem. $\square$

In view of (8), taking $a = c$ and $m = 0$, respectively, in Theorem 3.1 and using the identities (9) and (10), we obtain following subordination properties.

Corollary 3.2. *Let $m \in \mathbb{Z}$, $\mu > -1$, $k > 0$, the operator $\Im_{k,\mu}^{m}$ be defined by (3), and for $0 \le \lambda \le 1$, $A > 0$, $a \in \mathbb{C}$ with $\Re(a) > -A$, $\Re(\delta) \ge 1$ be given by (12). Let for $g \in \mathcal{A}$:*

$$\psi_1(z) := \left(1 - \lambda + \lambda \frac{A(\mu+1)}{k(a+A)}\right) \Im_{k,\mu}^{m+1} g(z) + \lambda \left(1 - \frac{A(\mu+1)}{k(a+A)}\right) \Im_{k,\mu}^{m} g(z),$$

satisfy

$$\Re\left(1 + \frac{z\psi_1''(z)}{\psi_1'(z)}\right) > -\rho,\ z \in \mathbb{U},$$

where $\rho = 0$ if $\Re\ (\delta) = 1$, and for $\Re\ (\delta) > 1$, ρ is given by (14) with (15). If $f \in \mathcal{A}$ satisfies

$$\left(1 - \lambda + \lambda \frac{A(\mu+1)}{k(a+A)}\right) \Im_{k,\mu}^{m+1} f(z) + \lambda \left(1 - \frac{A(\mu+1)}{k(a+A)}\right) \Im_{k,\mu}^{m} f(z) \prec \psi_1(z), z \in \mathbb{U} \tag{28}$$

then

$$\Im_{k,\mu}^{m} f(z) \prec \Im_{k,\mu}^{m} g(z), z \in \mathbb{U}.$$

Moreover, the function $\Im_{k,\mu}^{m} g(z)$ is the best dominant of (28).

Corollary 3.3. *Let for $A > 0$, $a, c \in \mathbb{C}$ satisfying $\Re(c-a) \ge 0$ and $\Re(a) > -A$, the operator $\widetilde{I}_A^{a,c}$ be defined by (4). Let for $0 \le \lambda \le 1$, $\mu > -1$, $k > 0$, $\Re\ (\delta) \ge 1$ be given by (12) and for $g \in \mathcal{A}$,*

$$\psi_2(z) := (1-\lambda)\left(1 - \frac{k(a+A)}{A(\mu+1)}\right) \widetilde{I}_A^{a,c} g(z) + \left((1-\lambda)\frac{k(a+A)}{A(\mu+1)} + \lambda\right) \widetilde{I}_A^{a+1,c} g(z),$$

satisfy

$$\Re\left(1 + \frac{z\psi_2''(z)}{\psi_2'(z)}\right) > -\rho,\ z \in \mathbb{U},$$

where $\rho = 0$ *if* $\Re\,(\delta) = 1$*, and for* $\Re\,(\delta) > 1$*,* ρ *is given by (14) with (15). If* $f \in \mathcal{A}$ *satisfies*

$$(1-\lambda)\left(1-\frac{k\,(a+A)}{A\,(\mu+1)}\right)\widetilde{I}_A^{a,c}f(z)+\left((1-\lambda)\,\frac{k\,(a+A)}{A\,(\mu+1)}+\lambda\right)\widetilde{I}_A^{a+1,c}f(z) \prec \psi_2(z) \quad z \in \mathbb{U} \quad (29)$$

then

$$\widetilde{I}_A^{a,c}f(z) \prec \widetilde{I}_A^{a,c}g(z), z \in \mathbb{U}.$$

Moreover, the function $\widetilde{I}_A^{a,c}g(z)$ *is the best dominant of (29).*

Following results are the direct consequences, if we put $\lambda = 0$ and 1, respectively, in Corollaries 3.2 and 3.3.

Corollary 3.4. *Let* $f, g \in \mathcal{A}$ *and* $\mu > -1$*,* $k > 0$ *be such that* $\frac{\mu+1}{k} \geq 1$*, for* $m \in \mathbb{Z}$*, the operator* $\Im_{k,\mu}^{m}$ *be defined by (3). Further, let*

$$\Re\left(1+\frac{z\chi''(z)}{\chi'(z)}\right) > -\xi, \qquad z \in \mathbb{U},\ \chi(z) := \Im_{k,\mu}^{m+1}g(z),$$

where $\xi = 0$ *if* $\frac{\mu+1}{k} = 1$*, and for* $\frac{\mu+1}{k} > 1$*,*

$$\xi = \begin{cases} \frac{\mu+1-k}{2k}, & 1 < \frac{\mu+1}{k} \leq 2, \\ \frac{k}{2(\mu+1-k)}, & \frac{\mu+1}{k} > 2. \end{cases}$$

Then $\Im_{k,\mu}^{m+1}f(z) \prec \Im_{k,\mu}^{m+1}g(z) \Rightarrow \Im_{k,\mu}^{m}f(z) \prec \Im_{k,\mu}^{m}g(z), z \in \mathbb{U}$*. Moreover, the function* $\Im_{k,\mu}^{m}g(z)$ *is the best dominant.*

Corollary 3.5. *Let* $f, g \in \mathcal{A}$ *and for* $A > 0$*,* $a, c \in \mathbb{C}$ *satisfying* $\Re\,(c-a) \geq 0$ *and* $\Re(a) \geq 0$*, the operator* $\widetilde{I}_A^{a,c}$ *be defined by (4). Further, let*

$$\Re\left(1+\frac{z\,\kappa''(z)}{\kappa'(z)}\right) > -\sigma, \quad z \in \mathbb{U},\ \kappa(z) := \widetilde{I}_A^{a+1,c}g(z),$$

where $\sigma = 0$ *if* $\Re(a) = 0$ *and for* $\Re(a) > 0$*,*

$$\sigma \quad \leq \quad \begin{cases} \frac{\Re(a)}{2A}, & \Re(a) \leq A, \\ \frac{A}{2\Re(a)}, & \Re(a) > A, \end{cases} \qquad (30)$$

$$(Im\,(a))^2 \quad \leq \quad (\Re(a)-2\sigma A)\left(\frac{A}{2\sigma}-\Re(a)\right), \qquad (31)$$

equality in (30) and (31) occur only if $Im\,(a) = 0$*. Then* $\widetilde{I}_A^{a+1,c}f(z) \prec \widetilde{I}_A^{a+1,c}g(z) \Rightarrow \widetilde{I}_A^{a,c}f(z) \prec \widetilde{I}_A^{a,c}g(z), z \in \mathbb{U}$*. Moreover, the function* $\widetilde{I}_A^{a,c}g(z)$ *is the best dominant.*

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