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PLANE SECTIONS OF SPACE CURVES IN POSITIVE CHARACTERISTIC

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It is known that if C is a curve of degree $d \geq 6$ in $\mathbb{P}^3$ over an algebraically closed field of characteristic 0 such that its plane section is contained in an irreducible conic, then C lies on a quadric surface. We show under which conditions this result holds also in positive characteristic.

1. Introduction.

In this article we consider some results regarding the plane sections of curves in $\mathbb{P}^3$. It is known (Laudal's lemma) that if C is a curve of degree d in $\mathbb{P}^3_k$, where k is a field of characteristic 0, such that its general hyperplane section is contained in an irreducible curve of degree s and $d > s^2 + 1$, then C is contained in a surface of degree s.

If char $k = p > 0$, in [1, Theorem 2.1] Hartshorne proved that this result holds also in the case $s = 1$, unless C is the divisor dL on a surface S. In this paper we analyze the case $s = 2$, we show that the same result does not hold if the curve is non reduced and we prove the following result:

Theorem 1.1. *Let $C \subset \mathbb{P}^3_k$ be a reduced curve of degree d, where k is an algebraically closed field of positive characteristic p. Suppose that its*

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generic plane section, $\Gamma = C \cap H$*, is contained in an irreducible conic* q*. Then* C *is contained in a quadric surface if one of the following conditions holds:*

(1) $d \geq 6$ *and* $p > 2$*;*

(2) $d \geq 6$*,* $p = 2$ *and* C *is connected;*

(3) $d > 6$ *and* $p = 2$*.*

In this paper $\mathbb{P}^3$ is a 3-dimensional projective space over an algebraically closed field k and $S = k[x_0, x_1, x_2, x_3]$ is its associated polynomial ring. Given a curve C in $\mathbb{P}^3$, that is a locally C.M. subscheme equidimensional of dimension 1, we denote by I its homogeneous ideal in S and by J the homogeneous ideal of its general plane section $\Gamma = C \cap H$ in the polynomial ring $R = k[x_1, x_2, x_3]$.

Now we need the following definition:

Definition 1.2. *Let* V_1 *and* V_2 *vector spaces of finite dimension over a field* k*. A family of linear maps* $\psi_t : V_1 \to V_2$*,* $t \in k$*, is called regular if, fixed two basis of* V_1 *and* V_2*, the associated matrix is* $(f_{ij}(t))$*, where* $f_{ij} : k \to k$ *are polynomial functions.*

2. Characteristic 0 case and linear case.

Suppose that $\mathrm{char} k = 0$. This result has been proved:

Theorem 2.1. (Strano, [5]). *Let* C *be a curve in* $\mathbb{P}^3$ *and let* Γ *be its general plane section. If there exists* $s \in \mathbb{N}$ *such that* $\mathrm{Tor}_1^R(J, k)_n = 0$ *for any* n*,* $0 \leq n \leq s + 2$*, then the canonic map:*

$$H^0\big(\mathbb{P}^3, \mathscr{I}_C(s)\big) \to H^0\big(\mathbb{P}^2, \mathscr{I}_\Gamma(s)\big)$$

is surjective.

We note that $\dim \mathrm{Tor}_1^R(J, k)_n$ is the number of syzygies of degree n in a minimal free resolution of J and that the result means that every curve of degree s containing Γ can be lifted to a surface of degree s containing C.

The following corollary is an easy consequence:

Corollary 2.2. *Let* $C \subset \mathbb{P}^3$ *be a curve such that its general plane section* Γ *is contained in an irreducible curve of* H *of degree* s *and suppose that* $\deg C > s^2 + 1$*. Then* C *is contained in a surface of degree* s*.*

Proof. To get the result we only need to prove that the are no syzygies in degree $\leq s + 2$.

If $\Delta HF(\Gamma)$ is the first difference function of the Hilbert function of Γ, then we know that:

$$\Delta HF(\Gamma) = \{1, 2, \ldots, s, s, h_{s+1}, h_{s+2}, \ldots\}$$

where $h_{s+1} \leq s$, $h_{s+r} \leq h_{s+r-1}$ and $h_{s+r} = 0$ if $r >> 0$. Now, by a result of Maggioni-Ragusa (Corollary 2.10, [3]), we see that $\Delta HF(\Gamma)$ must be of decreasing type and so the result follows as in [5], Corollary 2. □

This is what we get if char$k = 0$. Then, if char$k = p > 0$, in what could be called the "linear case" there is the following result:

Theorem 2.3. (Hartshorne's restriction theorem, [1]]. *Let C be a nondegenerate curve in $\mathbb{P}^3_k$. Assume that $d = \deg C \geq 3$ and that its general plane section is contained in a line of H. Then* char$k = p > 0$*, the support of C is a single line L and for any $P \in L$ there exists a surface S containing C, which is smooth at P, so that in a neighborhood of P the scheme C is just the divisor dL on S.*

Example 2.4. In his article Hartshorne shows with an example that for any $d \geq 3$ and for any characteristic $p > 0$ there is a curve C which is equal to the divisor dL on a surface S which is smooth at every point of L such that C is not a plane curve, but every general hyperplane section of C is contained on a line. Given d, p, take $q = p^r \geq d$ and let S be a surface defined by the equation:

$$w^q x + w^{q+1-d} y^d + z^q y = 0.$$

This surface contains the line $L : x = y = 0$. The singular locus of S does not meet L and so S is nonsingular along L. Let C be the curve dL on S. Consider the affine piece $w = 1$ and cut S with a plane $H : z = a+bx+cy$. In this way we get the plane curve Y given by:

$$x + a^q y + (y^d + b^q x^q y + c^q y^{q+1}) = 0.$$

Since the linear term is not constant, C is not contained in a line and the absence of terms of degrees between 2 and $d-1$ means that Y has d-fold intersection with its tangent line. Thus the general plane sections of C are all contained in a line.

3. Main theorem.

In this paragraph we will prove Theorem 1.1, but first we must recall the following lemma, which is one of the main technic al tools for the proof of theorem.

Lemma 3.1. (Re, [4]). *Let V_1 and V_2 be k-vector spaces of finite dimension and let $\varphi_t : V_1 \to V_2$ be a regular family such that $\varphi_t = f(t)\psi + \psi_t$, where $f : k \to k$ is a nonzero polynomial function such that $f(0) = 0$, ψ is a fixed linear map and ψ_t a regular family. Suppose that:*

(1) Im $\psi_t \subseteq U_2$, *where U_2 is a subspace of V_2;*

(2) rank$\varphi_0 \geq$ rankφ_t *for a general $t \in k$ and* rank$\varphi_0 \geq$rankψ.

Then $\psi(\ker\varphi_0) \subseteq U_2$.

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Start by observing that from the exact sequence:

$$0 \to \mathscr{I}_C(-1) \to \mathscr{I}_C \to \mathscr{I}_{\Gamma|H} \to 0 \tag{1}$$

twisting by 2 we get the cohomology long exact sequence sequence:

$$H^0\big(\mathscr{I}_C(1)\big) \to H^0\big(\mathscr{I}_C(2)\big) \to H^0\big(\mathscr{I}_{\Gamma|H}(2)\big) \to$$

$$\to H^1\big(\mathscr{I}_C(1)\big) \overset{\varphi_H}{\to} H^1\big(\mathscr{I}_C(2)\big) \to H^1\big(\mathscr{I}_{\Gamma|H}(2)\big).$$

If we consider the map:

$$\varphi_H : H^1\big(\mathscr{I}_C(1)\big) \to H^1\big(\mathscr{I}_C(2)\big)$$

then it's sufficient to show that $\ker\varphi_H = 0$ to get the result.

Consider $\alpha \in H^1\big(\mathscr{I}_C(1)\big)$, $\alpha \neq 0$, such that $\alpha H = 0$ and let H' be any plane different from H. We use Lemma 3.1 taking $\varphi_t = t\varphi_{H'} + \varphi_H$, that is setting $\psi = \varphi_{H'}$, $\psi_t = \varphi_H$, $f(t) = t$ and $U_2 = \mathrm{Im}\varphi_H$. Since $\alpha \in \mathrm{Ker}\varphi_H$, we get $\varphi_{H'}(\alpha) = \alpha H' \in \mathrm{Im}\varphi_H$ for any H'.

Take now four independent planes $H_0 = H$, H_1, H_2, H_3 and consider $L_i = H \cap H_i$, for $i = 1, 2, 3$. Let α_Γ be the image of α in $H^1\big(\mathscr{I}_{\Gamma|H}(1)\big)$. $\alpha H_i \in \mathrm{Im}\varphi_H$ implies that $\alpha_\Gamma L_i = 0$ in $H^1\big(\mathscr{I}_{\Gamma|H}(2)\big)$ for $i = 1, 2, 3$. This comes from the following commutative diagram:

$$\begin{array}{ccc} H^1(\mathscr{I}_C(1)) & \xrightarrow{H_i} & H^1(\mathscr{I}_C(2)) \\ \downarrow & & \downarrow \\ H^1(\mathscr{I}_{\Gamma|H}(1)) & \xrightarrow{L_i} & H^1(\mathscr{I}_{\Gamma|H}(2)) \end{array}$$

From the exact sequence:

$$0 \to H^0\big(\mathscr{I}_C(1)\big) \to H^0\big(\mathscr{O}_{\mathbb{P}^3}(1)\big) \to H^0\big(\mathscr{O}_C(1)\big) \to H^1\big(\mathscr{I}_C(1)\big) \to 0$$

it follows that there exists $\sigma \in H^0\big(\mathscr{O}_C(1)\big)$ with image in $H^1\big(\mathscr{I}_C(1)\big)$ equal to α. Let $H^1(\mathscr{I}_C(1)) \to H^1(\mathscr{I}_{\Gamma|H}(1))$ be induced by the restriction on the sequence (1) and let $\sigma_\Gamma \in H^0\big(\mathscr{O}_{\Gamma|H}(1)\big)$ be its restriction to Γ; the image of $\sigma_\Gamma L_i$ in $H^1\big(\mathscr{I}_{\Gamma|H}(2)\big)$ is $\alpha_\Gamma L_i = 0$. Now, from the exact sequence:

$$H^0\big(\mathscr{O}_H(2)\big) \to H^0\big(\mathscr{O}_{\Gamma|H}(2)\big) \to H^1\big(\mathscr{I}_{\Gamma|H}(2)\big) \to 0$$

we get that there exists $Q_i \in H^0\big(\mathscr{O}_H(2)\big)$ such that:

$$\sigma_\Gamma L_i = \overline{Q_i} \tag{2}$$

in $H^0\big(\mathscr{O}_{\Gamma|H}(2)\big)$.

Considered now H as a $\mathbb{P}^2$, if $R = k[x_1, x_2, x_3]$ is its associated polynomial ring, it's possible to suppose that $L_i = x_i$, $i = 1, 2, 3$. So we have found some homogeneous forms Q_i of degree 2 in R such that $\sigma_\Gamma x_i = \overline{Q_i}$ in $H^0(\mathscr{O}_{\Gamma|H}(2))$. From this it follows that $Q_i x_j = Q_j x_i \bmod H^0(\mathscr{I}_{\Gamma|H}(3))$, $i, j = 1, 2, 3$, that is:

$$Q_i x_j - Q_j x_i = f_{ij} \in H^0\big(\mathscr{I}_{\Gamma|H}(3)\big).$$

From:

$$x_k(Q_i x_j - Q_j x_i) - x_i(Q_k x_j - Q_j x_k) + x_j(Q_k x_i - Q_i x_k) = 0$$

we get a syzygy in degree 4 for $J = \sum_{n=0}^{\infty} H^0(\mathscr{I}_{\Gamma|H}(n))$. If $d \geq 7$, the plane conic q containing Γ is the only generator in degree ≤ 3 and so we have:

$$Q_i x_j - Q_j x_i = l_{ij} q \ \forall i, j$$

since $\deg(Q_i x_j - Q_j x_i) = 3$ for all i, j. In this way:

$$x_k l_{ij} - x_i l_{kj} + x_j l_{ki} = 0$$

$$\Rightarrow l_{ij} = a_i x_j - a_j x_i \ \forall i, j.$$

If $d = 6$ we have a complete intersection determined by q and by a form f of degree 3. So we can write:

$$Q_i x_j - Q_j x_i = l_{ij} q + b_{ij} f$$

for some $l_{ij} \in H^0(\mathscr{O}_{\mathbb{P}^2}(1))$ and $b_{ij} \in k$, for all i, j. So:

$$q(x_k l_{ij} - x_i l_{kj} + x_j l_{ki}) = -f(b_{ij} x_k - b_{kj} x_i + b_{ki} x_j).$$

Then $f | (x_k l_{ij} - x_i l_{kj} + x_j l_{ki})$, since $(f, q) = 1$, but:

$$\deg f = 3 > 2 = \deg(x_k l_{ij} - x_i l_{kj} + x_j l_{ki})$$

and so:

$$x_k l_{ij} - x_i l_{kj} + x_j l_{ki} = 0$$

and:

$$b_{ij} x_k - b_{kj} x_i + b_{ki} x_j = 0.$$

From this we get $b_{ij} = 0\ \forall i, j$ and as before:

$$l_{ij} = a_i x_j - a_j x_i\ \forall i, j.$$

In any case we get the following relation:

$$Q_i x_j - x_j Q_i = l_{ij} q$$

and so, by substitution:

$$x_i (Q_j - a_j q) = x_j (Q_i - a_i q)\ \forall i, j$$

$$\Rightarrow Q_i = Q x_i + a_i q\ \forall i = 1, 2, 3$$

where Q is a linear form in R.

Now, supposed x_i general, we see by (2) that $\sigma_\Gamma = \overline{Q}$ in $H^0(\mathcal{O}_{\Gamma|H}(1))$, that is σ_Γ is in the image of $H^0(\mathcal{O}_H(1)) \to H^0(\mathcal{O}_{\Gamma|H}(1))$. So the image of σ_Γ in $H^1(\mathcal{I}_{\Gamma|H}(1))$, that is α_Γ, is zero. From the exact sequence:

$$H^1\big(\mathcal{I}_C\big) \to H^1\big(\mathcal{I}_C(1)\big) \to H^1\big(\mathcal{I}_{\Gamma|H}(1)\big)$$

we get $\alpha \in \mathrm{Im}\varphi_H$. So $\exists\ \beta \in H^1(\mathcal{I}_C)$ such that $\alpha = \beta H$, with $\beta \in \ker\varphi_{H^2}$, with:

$$\varphi_{H^2} : H^1\big(\mathcal{I}_C\big) \to H^1\big(\mathcal{I}_C(2)\big).$$

Note that if C is connected $H^1(\mathcal{I}_C) = 0$ and so we immediately get the result.

Suppose now that C is not connected and apply Lemma 3.1, considering in this case the following map:

$$\varphi_t = \varphi_{H^2 + tH'^2} = t\varphi_{H'^2} + \varphi_{H^2} = t\psi + \psi_t$$

with:

$$\psi = \varphi_{H'^2},\ \ \psi_t = \varphi_{H^2} \text{ and } U_2 = \mathrm{Im}\varphi_{H^2}.$$

So $\beta \in \ker\varphi_{H^2} \Rightarrow \beta H'^2 \in \mathrm{Im}\varphi_{H^2}$ for every H'.

Consider now the planes H_1, H_2, H_3 and $H_1 + \lambda H_2 + \mu H_3$, for all λ and μ. One gets $\beta(H_1 + \lambda H_2 + \mu H_3)^2 \in \mathrm{Im}\varphi_{H^2}$ for all $\lambda, \mu \in k$.

If $p > 2$, then from this we get that $\beta H_i H_j \in \mathrm{Im}\varphi_{H^2}$ for $i, j = 1, 2, 3$. Reasoning as done previously we find $\tau \in H^0(\mathcal{O}_{\Gamma|H})$ such that its image in

$H^1(\mathscr{I}_\Gamma)$ is $\beta_{\Gamma|H}$. As before, we consider $L_i = x_i$, for $i = 1, 2, 3$ on H and we find:

$$\tau x_i x_j = \overline{V_{ij}}\ \forall i, j$$

where $V_{ij} \in H^0(\mathscr{O}_H(2))$. Reasoning as before we find $a \in H^0(\mathscr{O}_H)$ such that:

$$\tau x_i x_j = \overline{a} x_i x_j\ \forall i, j \Rightarrow \tau = \overline{a}.$$

This means that its image in $H^1(\mathscr{I}_{\Gamma|H})$, that is to say β_Γ, must be zero, hence $\beta \in \mathrm{Im}\varphi_H$. From the following exact sequence:

$$0 = H^1\big(\mathscr{I}_C(-1)\big) \overset{\varphi_H}{\to} H^1\big(\mathscr{I}_C\big) \to H^1\big(\mathscr{I}_{\Gamma|H}\big)$$

we get $\beta = 0$. So $\alpha \in \mathrm{Ker}\varphi_H \subset H^1(\mathscr{I}_C(1))$ vanishes and the theorem is proved for $p > 2$ and $d \geq 6$.

Suppose now that $p = 2$. Repeating what we said before we can only say that $\beta {H_i}^2 \in \mathrm{Im}\varphi_{H^2}$ for $i = 1, 2, 3$.

We consider as before $\tau \in H^0(\mathscr{O}_{\Gamma|H})$ such that its image in $H^1(\mathscr{I}_{\Gamma|H})$ is β_Γ. Since $\beta {H_i}^2 \in \mathrm{Im}\varphi_{H^2}$, we can say that $\beta_\Gamma {x_i}^2 = 0$ in $H^1(\mathscr{I}_{\Gamma|H}(2))$. So the image of $\tau {x_i}^2$ in $H^1(\mathscr{I}_{\Gamma|H}(2))$ is zero. So from:

$$H^0\big(\mathscr{O}_H(2)\big) \to H^0\big(\mathscr{O}_{\Gamma|H}(2)\big) \to H^1\big(\mathscr{I}_{\Gamma|H}(2)\big) \to 0$$

we get that $\tau {x_i}^2 = \overline{V_i}$, for some $V_i \in H^0(\mathscr{O}_H(2))$. And so:

$${x_i}^2 V_j - {x_j}^2 V_i = f_{ij} \in H^0\big(\mathscr{I}_{\Gamma|H}(4)\big).$$

Note now that it's possible to choose on H some coordinates x_i, for $i = 1, 2, 3$, such that the irreducible conic q in $H^0(\mathscr{I}_{\Gamma|H}(2))$ has equation $q = x_i x_j + \sum_{r=1}^3 a_r {x_r}^2$, since q is irreducible and $p = 2$. Note also that J annihilates $H^1(\mathscr{I}_{\Gamma|H})$, since $H^1_*(\mathscr{I}_{\Gamma|H})$ is a quotient of $H^0_*(\mathscr{O}_{\Gamma|H})$. Then we get $\beta_\Gamma q = 0$ and so $\beta_\Gamma x_i x_j = \sum_{r=1}^3 a_r \beta_\Gamma {x_r}^2 = 0$. So we can say that:

$$\tau x_i x_j = \overline{U_{ij}}$$

in $H^0(\mathscr{O}_{\Gamma|H}(2))$, for some $U_{ij} \in H^0(\mathscr{O}_H(2))$. In this way we find that:

$$x_i U_{ij} - x_j V_i,\ x_j U_{ij} - x_i V_j \in H^0(\mathscr{I}_{\Gamma|H}(3)).$$

If $d \geq 7$, then J has only the minimal generator q in degree ≤ 3 and so:

$$x_i U_{ij} - x_j V_i = l_1 q$$

$$x_j U_{ij} - x_i V_j = l_2 q$$

where $l_1, l_2 \in H^0(\mathscr{O}_H(1))$. So:

$$(3) \qquad {x_i}^2 V_j + {x_j}^2 V_i = (x_i l_2 + x_j l_1) q = m_{ij} q.$$

Considering now the ideal of six of the points of Γ, we find a complete intersection determined by q and by a cubic f. In this way we have:

$$x_k{}^2 V_i + x_i{}^2 V_k = m_{ki} q + l_{ki} f$$

$$x_k{}^2 V_j + x_j{}^2 V_k = m_{kj} q + l_{kj} f$$

and so from:

$$x_k{}^2 (x_i{}^2 V_j + x_j{}^2 V_i) + x_i{}^2 (x_k{}^2 V_j + x_j{}^2 V_k) + x_j{}^2 (x_k{}^2 V_i + x_i{}^2 V_k) = 0$$

we get:

$$(x_k{}^2 m_{ij} + x_i{}^2 m_{kj} + x_j{}^2 m_{ki}) q = (x_i{}^2 l_{kj} + x_j{}^2 l_{ki}) f.$$

Since q and f have no common factor, then $q \mid x_i{}^2 l_{kj} + x_j{}^2 l_{ki}$, but in this polynomial x_k appears in degree ≤ 1, while in q it appears in degree 2. So the only possibility is that $x_i{}^2 l_{kj} + x_j{}^2 l_{ki} = 0$, that is $l_{ki} = l_{kj} = 0$. So:

$$\begin{array}{c} x_k{}^2 V_i + x_i{}^2 V_k = m_{ki} q \\ x_k{}^2 V_j + x_j{}^2 V_k = m_{kj} q \end{array} \tag{4}$$

and:

$$x_k{}^2 m_{ij} + x_i{}^2 m_{kj} + x_j{}^2 m_{ki} = 0$$

$$\Rightarrow m_{rs} = b_r x_s{}^2 + b_s x_r{}^2 \ \forall r, s.$$

So by (3) and (4):

$$x_r{}^2 (V_s + b_s q) = x_s{}^2 (V_r + b_r q)$$

$$\Rightarrow V_s = b_s q + b x_s{}^2 \ \forall s$$

$$\Rightarrow \tau x_s{}^2 = \overline{b} x_s{}^2 \ \forall s$$

in $H^0(\mathscr{O}_{\Gamma|H}(2))$. Since this is true for all s and since $p = 2$, it must be $\tau l^2 = \overline{b} l^2 \ \forall l \in H^0(\mathscr{O}_H(1))$ and so $\tau = \overline{b}$. The conclusion follows as before and we get the result. $\square$

4. Example.

With the following example we find, for any $p > 0$, a nonreduced curve such that $\deg C_{red} = 1$, not lying on a quadric surface, whose general plane section is contained in an irreducible conic.

Consider the line $L : x = y = 0$ and, for any $d \geq 6$ and $q = p^r \geq d$, the divisor $C = dL$ on the following surface S containing L:

$$w^q x^2 + w^{q+2-d} y^d + z^{q+1} y = 0.$$

Consider the affine part $w = 1$. We get:

$$x^2 + y^d + z^{q+1}y = 0.$$

If Y is the intersection of S with the general plane $H : z = a + bx + cy$, then we get:

$$x^2 + a^q(a + bx + cy)y + y^d + (b^q x^q + c^q y^q)(a + bx + cy)y = 0.$$

Since there are no terms of degrees from 3 to $d - 1$, Y has d-uple intersection with the conic in the point given by $L \cap H$ and it follows that $C \cap H$ is contained in the irreducible conic given by $x^2 + a^q(a + bx + cy)y = 0$. However, since $x^2 + a^q(a + bx + cy)y + \lambda h$ is not constant for any $\lambda \in k(a, b, c)$, C doesn't lie on a quadric surface.

It's still unknown what happens in the case C reduced, non connected and $\deg C = 6$.

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