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SOME INEQUALITIES OF HERMITE-HADAMARD TYPE FOR GA-CONVEX FUNCTIONS WITH APPLICATIONS TO MEANS

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In the paper, the authors, by Hölder's integral inequality, establish some Hermite-Hadamard type integral inequalities for GA-convex functions and apply these inequalities to construct several inequalities for special means.

1. Introduction

It is general knowledge that if $f : I \subseteq \mathbb{R} = (-\infty,\infty) \to \mathbb{R}$ is a convex function and $a,b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \le \frac{f(a)+f(b)}{2}. \tag{1.1}$$

This inequality is well known in the literature as Hermite-Hadamard's inequality for convex functions.

The usual concept of convex functions has been generalized in diverse manners. One of them is the so-called s-convex functions.

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Definition 1.1 ([6]). Let $s \in (0,1]$. A function $f : I \subseteq \mathbb{R}_0 = [0,\infty) \to \mathbb{R}$ is said to be s-convex (in the second sense) if

$$f(\lambda x + (1-\lambda)y) \le \lambda^s f(x) + (1-\lambda)^s f(y) \tag{1.2}$$

holds for all $x, y \in I$ and $\lambda \in [0,1]$.

The following Hermite-Hadamard type inequalities for the usual convex functions and the s-convex functions were obtained in [5, 8, 9].

Theorem 1.2 ([5, Theorem 2.2]). *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping and $a, b \in I^\circ$ with $a < b$. If $|f'(x)|$ is convex on $[a,b]$, then*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)\,\mathrm{d}x \right| \le \frac{(b-a)}{8} \big(|f'(a)| + |f'(b)|\big). \tag{1.3}$$

Theorem 1.3 ([8, Theorems 2.3 and 2.4]). *Let $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ be differentiable on I° and $a, b \in I$ with $a < b$. If $|f'(x)|^p$ is s-convex on $[a,b]$ for some $s \in (0,1]$ and $p > 1$, then*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x)\,\mathrm{d}x \right| \le \frac{b-a}{16} \left(\frac{4}{p+1}\right)^{1/p} \big(|f'(a)| + |f'(b)|\big) \tag{1.4}$$

and

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x)\,\mathrm{d}x \right| \le \frac{b-a}{4} \left(\frac{4}{p+1}\right)^{1/p} \Big\{ \big[|f'(a)|^{p/(p-1)} + 3|f'(b)|^{p/(p-1)}\big]^{1-1/p} + \big[3|f'(a)|^{p/(p-1)} + |f'(b)|^{p/(p-1)}\big]^{1-1/p} \Big\}. \tag{1.5}$$

Theorem 1.4 ([9, Theorem 3]). *Let $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ be differentiable on I°, $a, b \in I$ with $a < b$, and $f' \in L[a,b]$. If $|f'(x)|^q$ is s-convex on $[a,b]$ for some $s \in (0,1]$ and $q > 1$, then*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)\,\mathrm{d}x \right| \le \frac{b-a}{2} \left[\frac{q-1}{2(2q-1)}\right]^{1-1/q} \left(\frac{1}{s+1}\right)^{1/q} \times \left\{ \left[|f'(a)|^q + \left| f'\left(\frac{a+b}{2}\right) \right|^q \right]^{1/q} + \left[|f'(b)|^q + \left| f'\left(\frac{a+b}{2}\right) \right|^q \right]^{1/q} \right\}. \tag{1.6}$$

In recent years, some Hermite-Hadamard type inequalities for other types of convex functions were established in, for example, [1, 2, 4, 7, 14, 16–23].

Definition 1.5 ([10, 11]). A function $f: I \subseteq \mathbb{R}_0 \to \mathbb{R}$ is said to be a GA-convex function on I if

$$f\left(x^{\lambda} y^{1-\lambda}\right) \le \lambda f(x) + (1-\lambda) f(y) \tag{1.7}$$

holds for all $x, y \in I$ and $\lambda \in [0,1]$, where $x^{\lambda} y^{1-\lambda}$ and $\lambda f(x) + (1-\lambda) f(y)$ are respectively called the weighted geometric mean of two positive numbers x and y and the weighted arithmetic mean of $f(x)$ and $f(y)$.

In what follows, we also need some notions of means. For positive numbers $a > 0$ and $b > 0$ with $a \ne b$, the quantities

$$A(a,b) = \frac{a+b}{2}, \quad L(a,b) = \frac{b-a}{\ln b - \ln a}, \tag{1.8}$$

and

$$L_p(a,b) = \begin{cases} \left[\dfrac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right]^{1/p}, & p \ne -1, 0 \\ L(a,b), & p = -1 \\ \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{1/(b-a)}, & p = 0 \end{cases} \tag{1.9}$$

are called the arithmetic mean, the logarithmic mean, and the generalized logarithmic mean of order $p \in \mathbb{R}$ respectively. For more information on means, please refer to [3, 12, 13, 15] and a number of references therein.

The goal of this paper is to establish some new integral inequalities of Hermite-Hadamard type for GA-convex functions and to apply them to construct inequalities of special means.

2. A lemma

To reach our goal, we need the following lemma.

Lemma 2.1. *Let $f: I \subseteq \mathbb{R}_+ = (0, \infty) \to \mathbb{R}$ be differentiable on I° and $a, b \in I^\circ$ with $a < b$. If $f' \in L([a,b])$, then*

$$[bf(b) - af(a)] - \int_a^b f(x)\,\mathrm{d}x = (\ln b - \ln a) \int_0^1 b^{2t} a^{2(1-t)} f'\left(b^t a^{1-t}\right) \mathrm{d}t. \tag{2.1}$$

Proof. Integrating by part and changing variables of definite integral yield

$$\begin{aligned} \int_0^1 b^{2t} a^{2(1-t)} f'\left(b^t a^{1-t}\right) \mathrm{d}t &= \frac{1}{\ln b - \ln a} \int_0^1 b^t a^{(1-t)} f'\left(b^t a^{1-t}\right) \mathrm{d}\left(b^t a^{1-t}\right) \\ &= \frac{1}{\ln b - \ln a} \int_a^b x f'(x)\,\mathrm{d}x = \frac{bf(b) - af(a)}{\ln b - \ln a} - \frac{1}{\ln b - \ln a} \int_a^b f(x)\,\mathrm{d}x. \end{aligned}$$

Lemma 2.1 is thus proved. □

3. Some new integral inequalities of Hermite-Hadamard type

Now we set off to create some integral inequalities of Hermite-Hadamard type for GA-convex functions.

Theorem 3.1. *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I$ with $a < b$, and $f' \in L([a,b])$. If $|f'(x)|^q$ is GA-convex on $[a,b]$ for $q \ge 1$, then*

$$\left|[bf(b) - af(a)] - \int_a^b f(x)\,\mathrm{d}x\right| \le \frac{[(b-a)A(a,b)]^{1-1/q}}{2^{1/q}} \times \left\{ \left[L\left(a^2,b^2\right) - a^2\right]|f'(a)|^q + \left[b^2 - L\left(a^2,b^2\right)\right]|f'(b)|^q \right\}^{1/q}. \quad (3.1)$$

Proof. Since $|f'(x)|^q$ is GA-convex on $[a,b]$, from Lemma 2.1 and Hölder's inequality, we drive

$$\begin{aligned}
&\left|[bf(b) - af(a)] - \int_a^b f(x)\,\mathrm{d}x\right| \\
&\le a^2(\ln b - \ln a)\int_0^1 \left(\frac{b}{a}\right)^{2t} \left|f'\left(b^t a^{1-t}\right)\right| \mathrm{d}t \\
&\le a^2(\ln b - \ln a)\left[\int_0^1 \left(\frac{b}{a}\right)^{2t} \mathrm{d}t\right]^{1-1/q} \\
&\quad \times \left\{\int_0^1 \left(\frac{b}{a}\right)^{2t} [(1-t)|f'(a)|^q + t|f'(b)|^q]\,\mathrm{d}t\right\}^{1/q} \\
&= (\ln b - \ln a)\left[\frac{b^2-a^2}{2(\ln b - \ln a)}\right]^{1-1/q} \left[\frac{1}{2(\ln b - \ln a)}\right]^{1/q} \\
&\quad \times \left[\frac{b^2 - 2a^2(\ln b - \ln a) - a^2}{2(\ln b - \ln a)}|f'(a)|^q \right. \\
&\quad \left. + \frac{2b^2(\ln b - \ln a) - b^2 + a^2}{2(\ln b - \ln a)}|f'(b)|^q\right]^{1/q} \\
&= \frac{[(b-a)A(a,b)]^{1-1/q}}{2^{1/q}} \left\{ \left[L\left(a^2,b^2\right) - a^2\right]|f'(a)|^q \right. \\
&\quad \left. + \left[b^2 - L\left(a^2,b^2\right)\right]|f'(b)|^q \right\}^{1/q}.
\end{aligned}$$

The proof of Theorem 3.1 is thus complete. □

Corollary 3.2. *Under conditions of Theorem 3.1, if $q = 1$, then*

$$\left|[bf(b) - af(a)] - \int_a^b f(x)\,\mathrm{d}x\right| \le \frac{1}{2}\left\{ \left[L\left(a^2,b^2\right) - a^2\right]|f'(a)| + \left[b^2 - L\left(a^2,b^2\right)\right]|f'(b)| \right\}.$$

Theorem 3.3. *Let $f: I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be differentiable on I°, $a, b \in I$ with $a < b$, and $f' \in L([a,b])$. If $|f'(x)|^q$ is GA-convex for $q > 1$ on $[a,b]$, then*

$$\left|[bf(b)-af(a)]-\int_a^b f(x)\,\mathrm{d}x\right| \le (\ln b-\ln a) \times \left[L\left(a^{2q/(q-1)}, b^{2q/(q-1)}\right)\right]^{1-1/q}\left[A(|f'(a)|^q, |f'(b)|^q)\right]^{1/q}. \quad (3.2)$$

Proof. Since $|f'(x)|^q$ is a GA-convex function on $[a,b]$, from Lemma 2.1 and Hölder's inequality, we have

$$\begin{aligned}
&\left|[bf(b)-af(a)]-\int_a^b f(x)\,\mathrm{d}x\right| \\
&\le a^2(\ln b-\ln a)\int_0^1\left(\frac{b}{a}\right)^{2t}\left|f'\left(b^t a^{1-t}\right)\right|\mathrm{d}t \\
&\le a^2(\ln b-\ln a)\left[\int_0^1\left(\frac{b}{a}\right)^{2q/(q-1)t}\mathrm{d}t\right]^{1-1/q}\left[\int_0^1\left|f'\left(b^t a^{1-t}\right)\right|^q\mathrm{d}t\right]^{1/q} \\
&\le (\ln b-\ln a)\left[\frac{b^{2q/(q-1)}-a^{2q/(q-1)}}{2q(\ln b-\ln a)/(q-1)}\right]^{1-1/q} \\
&\quad\times\left\{\int_0^1[(1-t)|f'(a)|^q+t|f'(b)|^q]\,\mathrm{d}t\right\}^{1/q} \\
&=(\ln b-\ln a)\left[L\left(a^{2q/(q-1)}, b^{2q/(q-1)}\right)\right]^{1-1/q}\left[A(|f'(a)|^q, |f'(b)|^q)\right]^{1/q}.
\end{aligned}$$

The proof of Theorem 3.3 is complete. □

Theorem 3.4. *Let $f: I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be differentiable on I°, $a, b \in I$ with $a < b$, and $f' \in L([a,b])$. If $|f'(x)|^q$ is GA-convex on $[a,b]$ for $q \ge 1$, then*

$$\left|[bf(b)-af(a)]-\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{(\ln b-\ln a)^{1-1/q}}{(2q)^{1/q}} \times\left\{\left[L\left(a^{2q}, b^{2q}\right)-a^{2q}\right]|f'(a)|^q+\left[b^{2q}-L\left(a^{2q}, b^{2q}\right)\right]|f'(b)|^q\right\}^{1/q}. \quad (3.3)$$

Proof. Since $|f'(x)|^q$ is a GA-convex function on $[a,b]$, from Lemma 2.1 and Hölder's inequality, we have

$$\begin{aligned}
&\left|[bf(b)-af(a)]-\int_a^b f(x)\,\mathrm{d}x\right| \\
&\le a^2(\ln b-\ln a)\int_0^1\left(\frac{b}{a}\right)^{2t}\left|f'\left(b^t a^{1-t}\right)\right|\mathrm{d}t \\
&\le a^2(\ln b-\ln a)\left(\int_0^1 1\,\mathrm{d}t\right)^{1-1/q}\left[\int_0^1\left(\frac{b}{a}\right)^{2qt}\left|f'\left(b^t a^{1-t}\right)\right|^q\mathrm{d}t\right]^{1/q}
\end{aligned}$$

$$
\begin{aligned}
&\le a^2(\ln b-\ln a)\left\{\int_0^1\left(\frac{b}{a}\right)^{2qt}[(1-t)|f'(a)|^q+t|f'(b)|^q]\,\mathrm{d}t\right\}^{1/q}\\
&=(\ln b-\ln a)\bigg[\frac{b^{2q}-a^{2q}(\ln b^{2q}-\ln a^{2q})-a^{2q}}{(\ln b^{2q}-\ln a^{2q})^2}|f'(a)|^q\\
&\quad+\frac{b^{2q}(\ln b^{2q}-\ln a^{2q})-b^{2q}+a^{2q}}{(\ln b^{2q}-\ln a^{2q})^2}|f'(b)|^q\bigg]^{1/q}\\
&\le\frac{(\ln b-\ln a)^{1-1/q}}{(2q)^{1/q}}\big\{\big[L\big(a^{2q},b^{2q}\big)-a^{2q}\big]|f'(a)|^q\\
&\quad+\big[b^{2q}-L\big(a^{2q},b^{2q}\big)\big]|f'(b)|^q\big\}^{1/q}.
\end{aligned}
$$

The proof of Theorem 3.4 is complete. □

Theorem 3.5. *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable function on I°, $a,b \in I$ with $a < b$, and $f' \in L([a,b])$. If $|f'(x)|^q$ is GA-convex on $[a,b]$ for $q > 1$ and $2q > p > 0$, then*

$$
\begin{aligned}
\left|[bf(b)-af(a)]-\int_a^b f(x)\,\mathrm{d}x\right| &\le \frac{(\ln b-\ln a)^{1-1/q}}{p^{1/q}}\\
&\quad\times\big[L\big(a^{(2q-p)/(q-1)},b^{(2q-p)/(q-1)}\big)\big]^{1-1/q}\\
&\quad\times\big\{[L(a^p,b^p)-a^p]|f'(a)|^q+[b^p-L(a^p,b^p)]|f'(b)|^q\big\}^{1/q}.
\end{aligned}
\tag{3.4}
$$

Proof. Since $|f'(x)|^q$ is GA-convex on $[a,b]$, from Lemma 2.1 and Hölder's inequality, we have

$$
\begin{aligned}
&\left|[bf(b)-af(a)]-\int_a^b f(x)\,\mathrm{d}x\right|\\
&\le a^2(\ln b-\ln a)\int_0^1\left(\frac{b}{a}\right)^{2t}\big|f'\big(b^ta^{1-t}\big)\big|\,\mathrm{d}t\\
&\le a^2(\ln b-\ln a)\left[\int_0^1\left(\frac{b}{a}\right)^{(2q-p)/(q-1)t}\mathrm{d}t\right]^{1-1/q}\\
&\quad\times\left[\int_0^1\left(\frac{b}{a}\right)^{pt}\big|f'\big(b^ta^{1-t}\big)\big|^q\,\mathrm{d}t\right]^{1/q}\\
&\le a^2(\ln b-\ln a)\left[\frac{(b/a)^{(2q-p)/(q-1)}-1}{(2q-p)(\ln b-\ln a)/(q-1)}\right]^{1-1/q}\\
&\quad\times\left\{\int_0^1\left(\frac{b}{a}\right)^{pt}[(1-t)|f'(a)|^q+t|f'(b)|^q]\,\mathrm{d}t\right\}^{1/q}\\
&=\frac{(\ln b-\ln a)^{1-1/q}}{p^{1/q}}\big[L\big(a^{(2q-p)/(q-1)},b^{(2q-p)/(q-1)}\big)\big]^{1-1/q}
\end{aligned}
$$

$$\times \left\{[L(a^p,b^p)-a^p]|f'(a)|^q+[b^p-L(a^p,b^p)]|f'(b)|^q\right\}^{1/q}.$$

The proof of Theorem 3.5 is complete. □

Corollary 3.6. *Under conditions of Theorem 3.5, when $p=q$, we have*

$$\begin{aligned}\left|[bf(b)-af(a)]-\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{(\ln b-\ln a)^{1-1/q}}{q^{1/q}} \\ \times \left[L\left(a^{q/(q-1)},b^{q/(q-1)}\right)\right]^{1-1/q} \\ \times \left\{[L(a^q,b^q)-a^q]|f'(a)|^q+[b^q-L(a^q,b^q)]|f'(b)|^q\right\}^{1/q}.\end{aligned}$$

4. Applications to special means

Finally we apply Hermite-Hadamard type inequalities obtained in the above section to construct several inequalities for special means.

Theorem 4.1. *For $b>a>0$, $s>0$, $q\ge 1$, and $sq\ne 1$, we have*

$$\begin{aligned}2[L_{s+1}(a,b)]^{s+1} \le (a+b)^{1-1/q} \\ \times \left\{(sq+2)[L_{sq+1}(a,b)]^{sq+1}-sqL\left(a^2,b^2\right)[L_{sq-1}(a,b)]^{sq-1}\right\}^{1/q}. \quad (4.1)\end{aligned}$$

Proof. Let

$$f(x)=\frac{x^{s+1}}{s+1} \quad (4.2)$$

for $x\in\mathbb{R}_+$ and $s>0$. Then $|f'(x)|^q=x^{sq}$ is a GA-convex function on $\mathbb{R}_+$ and both sides of the inequality (3.1) in Theorem 3.1 become

$$\left|[bf(b)-af(a)]-\int_a^b f(x)\,\mathrm{d}x\right| = \frac{b^{s+2}-a^{s+2}}{s+2}=(b-a)[L_{s+1}(a,b)]^{s+1} \quad (4.3)$$

and

$$\begin{aligned}&\frac{[(b-a)A(a,b)]^{1-1/q}}{2^{1/q}}\left\{\left[L\left(a^2,b^2\right)-a^2\right]|f'(a)|^q+\left[b^2-L\left(a^2,b^2\right)\right]|f'(b)|^q\right\}^{1/q} \\ &=\frac{(b-a)(a+b)^{1-1/q}}{2}\left[\frac{(sq+2)(b^{sq+2}-a^{sq+2})}{(sq+2)(b-a)}-L\left(a^2,b^2\right)\frac{sq(b^{sq}-a^{sq})}{sq(b-a)}\right]^{1/q} \\ &=\frac{(b-a)(a+b)^{1-1/q}}{2}\Big\{(sq+2)[L_{sq+1}(a,b)]^{sq+1} \\ &\quad -sqL\left(a^2,b^2\right)\left[L_{sq-1}(a,b)\right]^{sq-1}\Big\}^{1/q}.\end{aligned}$$

Combining the above two equalities leads to (4.1). The proof of Theorem 4.1 is complete. □

Corollary 4.2. *Under conditions of Theorem 4.1, when $q = 1$ and $s \neq 1$, we have*

$$L\big(a^2, b^2\big)[L_{s-1}(a,b)]^{s-1} \le [L_{s+1}(a,b)]^{s+1}. \tag{4.4}$$

Theorem 4.3. *For $b > a > 0$, $s > 0$, and $q > 1$, we have*

$$[L_{s+1}(a,b)]^{s+1} L(a,b) \le \big[L\big(a^{2q/(q-1)}, b^{2q/(q-1)}\big)\big]^{1-1/q} [A(a^{sq}, b^{sq})]^{1/q}. \tag{4.5}$$

Proof. Applying the function (4.2) to the upper bound of the inequality (3.2) in Theorem 3.3 results in

$$\begin{aligned}(\ln b - \ln a)\big[L\big(a^{2q/(q-1)}, b^{2q/(q-1)}\big)\big]^{1-1/q} [A(|f'(a)|^q, |f'(b)|^q)]^{1/q} \\ = (\ln b - \ln a)\big[L\big(a^{2q/(q-1)}, b^{2q/(q-1)}\big)\big]^{1-1/q} [A(a^{sq}, b^{sq})]^{1/q}.\end{aligned}$$

Combining this with (4.3) and rearranging yield (4.5). The proof of Theorem 4.3 is complete. □

Theorem 4.4. *Let $b > a > 0$, $s > 0$, $q \ge 1$, and $sq \neq 1$. Then*

$$\begin{aligned}[L_{s+1}(a,b)]^{s+1} [L(a,b)]^{1-1/q} \le \frac{1}{(2q)^{1/q}} \big\{(s+2)q [L_{(s+2)q-1}(a,b)]^{(s+2)q-1} \\ - sqL\big(a^{2q}, b^{2q}\big)[L_{sq-1}(a,b)]^{sq-1}\big\}^{1/q}.\end{aligned} \tag{4.6}$$

Proof. The upper bound of the inequality (3.3) in Theorem 3.4 applied to the function (4.2) becomes

$$\begin{aligned}&\frac{(\ln b - \ln a)^{1-1/q}}{(2q)^{1/q}} \big\{\big[b^{2q} - L\big(a^{2q}, b^{2q}\big)\big] |f'(b)|^q \\ &+ \big[L\big(a^{2q}, b^{2q}\big) - a^{2q}\big] |f'(a)|^q\big\}^{1/q} \\ = &\frac{(\ln b - \ln a)^{1-1/q}}{(2q)^{1/q}} (b-a)^{1/q} \big\{(s+2)q [L_{(s+2)q-1}(a,b)]^{(s+2)q-1} \\ &- sqL\big(a^{2q}, b^{2q}\big)[L_{sq-1}(a,b)]^{sq-1}\big\}^{1/q}.\end{aligned}$$

Combining this with (4.3) and rearranging yield (4.6). The proof of Theorem 4.4 is complete. □

Theorem 4.5. *Let $0 < a < b$, $s > 0$, $q > 1$, $2q > p > 0$, and $sq \neq 1$. Then*

$$\begin{aligned}[L_{s+1}(a,b)]^{s+1} [L(a,b)]^{1-1/q} \le \frac{1}{p^{1/q}} \big[L\big(a^{(2q-p)/(q-1)}, b^{(2q-p)/(q-1)}\big)\big]^{1-1/q} \\ \times \big\{(p+sq)[L_{p+sq-1}(a,b)]^{p+sq-1} - sqL(a^p, b^p)[L_{sq-1}(a,b)]^{sq-1}\big\}^{1/q}.\end{aligned} \tag{4.7}$$

Proof. The upper bound of the inequality (3.4) in Theorem 3.5 applied to the function (4.2) is reduced to

$$\begin{aligned}
&\frac{(\ln b-\ln a)^{1-1/q}}{p^{1/q}}\big[L\big(a^{(2q-p)/(q-1)},b^{(2q-p)/(q-1)}\big)\big]^{1-1/q} \\
&\qquad\times\big\{[b^p-L(a^p,b^p)]|f'(b)|^q+[L(a^p,b^p)-a^p]|f'(a)|^q\big\}^{1/q} \\
&=\frac{(\ln b-\ln a)^{1-1/q}}{p^{1/q}}(b-a)^{1/q}\big[L\big(a^{(2q-p)/(q-1)},b^{(2q-p)/(q-1)}\big)\big]^{1-1/q} \\
&\qquad\times\big\{(p+sq)[L_{p+sq-1}(a,b)]^{p+sq-1}-sqL(a^p,b^p)[L_{sq-1}(a,b)]^{sq-1}\big\}^{1/q}.
\end{aligned}$$

Combining this with (4.3) and simplifying produce (4.7). The proof of Theorem 4.4 is complete. □

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