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TWO NEW SHARP OSTROWSKI-GRÜSS TYPE INEQUALITIES

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The purpose of this paper is to use a variant of the Grüss inequality to derive two new sharp Ostrowski-Grüss type inequalities related to a perturbed trapezoidal type rule and a perturbed generalized interior point rule, respectively, which provide improvements of some previous results in the literatures.

1. Introduction

In 1935, G.Grüss proved the following integral inequality which gives an approximation for the integral of the product of two functions in terms of the product of the integrals of the two functions (see for example [11,p.296]).

Theorem 1.1. *Let $h, g : [a,b] \to \mathbb{R}$ be two integrable functions such that $\phi \leq h(t) \leq \Phi$ and $\gamma \leq g(t) \leq \Gamma$ for all $t \in [a,b]$, where ϕ, Φ, γ and Γ are real numbers. Then we have*

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$$
\begin{aligned}
|T(h,g)| := |\frac{1}{b-a}\int_a^b h(t)g(t)\,dt \\
-\frac{1}{b-a}\int_a^b h(t)\,dt\cdot\frac{1}{b-a}\int_a^b g(t)\,dt| \\
\le \frac{1}{4}(\Phi-\phi)(\Gamma-\gamma), \quad (1)
\end{aligned}
$$

and the inequality is sharp, in the sense that the constant $\frac{1}{4}$ can not be replaced by a smaller one.

It is clear that the constant $\frac{1}{4}$ is achieved for

$$h(t) = g(t) = sgn(t - \frac{a+b}{2}).$$

From then on, (1) is well known in the literature as the Grüss inequality.

A premature Grüss inequality originated from the work of Grüss (see also [11, p. 296]). It is embodied in the following theorem and was also considered and applied for the first time in the paper [10, Theorem 5] by M. Matić, J. Pečarić and N. Ujević in 2000.

Theorem 1.2. *Let $h,g : [a,b] \to \mathbb{R}$ be integrable functions such that hg is also integrable, and $\gamma \le g(t) \le \Gamma$ for all $t \in [a,b]$, where $\gamma,\Gamma \in \mathbb{R}$ are constants.Then*

$$|T(h,g)| \le \frac{1}{2}\sqrt{T(h,h)}\,(\Gamma-\gamma). \quad (2)$$

In 2002, almost at the same time, by using similar, somewhat complicated methods, X. L. Cheng and J. Sun in [6, Theorem 1.1] as well as M. Matić in [9, Theorem 3] have proved the following variant of the Grüss inequality respectively.

Theorem 1.3. *Let $h,g : [a,b] \to \mathbb{R}$ be two integrable functions such that $\gamma \le g(t) \le \Gamma$ for some constants γ, Γ for all $t \in [a,b]$. Then*

$$
\begin{aligned}
\left|\int_a^b h(t)g(t)\,dt - \frac{1}{b-a}\int_a^b h(t)\,dt\int_a^b g(t)\,dt\right| \\
\le \frac{1}{2}\left(\int_a^b |h(t) - \frac{1}{b-a}\int_a^b h(y)\,dy|\,dt\right)(\Gamma-\gamma). \quad (3)
\end{aligned}
$$

Moreover, Matić has proved that there exists a function g for which the equality in (3) is attained, Cerone and Dragomir have proved in [4, Theorem 3] that the constant $\frac{1}{2}$ in (3) cannot be replaced by a smaller one. In [8, p.122], the author has provided a simple proof of Theorem 1.3 with the sharpness of inequality (3) in the sense that we can choose the function g such that either

$$g(x)=\begin{cases}\Gamma, & \text{if } h(x)-\frac{1}{b-a}\int_a^b h(y)\,dy\geq 0,\\ \gamma, & \text{if } h(x)-\frac{1}{b-a}\int_a^b h(y)\,dy<0\end{cases}$$

or

$$g(x)=\begin{cases}\gamma, & \text{if } h(x)-\frac{1}{b-a}\int_a^b h(y)\,dy\geq 0,\\ \Gamma, & \text{if } h(x)-\frac{1}{b-a}\int_a^b h(y)\,dy<0\end{cases}$$

to attain the equality in (3).

The result stated in Theorem 1.3 is of particular interest and very useful in the case when $\int_a^b \left|h(t)-\frac{1}{b-a}\int_a^b h(y)\,dy\right|\,dt$ can be evaluated exactly.

In [4, (2.19)], we can see that Theorem 1.2 improves Theorem 1.1 and Theorem 1.3 improves Theorem 1.2.

From [2, Theorem 10] and [3, Theorem 13], we see that applying the premature Grüss inequality (2) has derived two Ostrowski-Grüss type inequalities related to a perturbed trapezoidal type rule and a perturbed generalized interior point rule as follows:

Theorem 1.4. *Let $f:[a,b]\to\mathbb{R}$ be a mapping such that the derivative $f^{(n-1)}$ $(n\geq 1)$ is absolutely continuous on $[a,b]$. Assume that there exist constants $\gamma,\Gamma\in\mathbb{R}$ such that $\gamma\leq f^{(n)}(t)\leq\Gamma$ a.e. on $[a,b]$. Then for all $x\in[a,b]$, the following inequality holds*

$$\begin{aligned}\Bigg|\int_a^b f(t)\,dt-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}&[(x-a)^{k+1}f^{(k)}(a)+(-1)^k(b-x)^{k+1}f^{(k)}(b)]\\ &-\frac{(x-a)^{n+1}+(-1)^n(b-x)^{n+1}}{(n+1)!}\left[\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}\right]\Bigg|\\ &\leq\frac{\Gamma-\gamma}{2(n+1)!\sqrt{2n+1}}\{n^2(b-a)[(x-a)^{2n+1}+(b-x)^{2n+1}]\\ &\quad+(2n+1)(x-a)(b-x)[(x-a)^n-(x-b)^n]^2\}^{\frac{1}{2}}.\end{aligned}\tag{4}$$

Theorem 1.5. *Let $f:[a,b]\to\mathbb{R}$ be a mapping such that the derivative $f^{(n-1)}$ $(n\geq 1)$ is absolutely continuous on $[a,b]$. Assume that there exist constants $\gamma,\Gamma\in\mathbb{R}$ such that $\gamma\leq f^{(n)}(t)\leq\Gamma$ a.e. on $[a,b]$. Then for all $x\in[a,b]$, the following inequality holds*

$$
\begin{aligned}
&|\int_a^b f(t)\,dt-\textstyle\sum_{k=0}^{n-1}\frac{1}{(k+1)!}[(b-x)^{k+1}+(-1)^k(x-a)^{k+1}]f^{(k)}(x)\\
&\quad-\tfrac{(b-x)^{n+1}+(-1)^n(x-a)^{n+1}}{(n+1)!}[\tfrac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}]|\\
&\le\tfrac{\Gamma-\gamma}{2(n+1)!\sqrt{2n+1}}\{n^2(b-a)[(x-a)^{2n+1}+(b-x)^{2n+1}]\\
&\quad+(2n+1)(x-a)(b-x)[(x-a)^n-(x-b)^n]^2\}^{\frac{1}{2}}.
\end{aligned}
\tag{5}
$$

The purpose of this paper is to provide, using the variant of the Grüss inequality, two new sharp Ostrowski-Grüss type inequalities related to a perturbed trapezoidal type rule and a perturbed generalized interior point rule, which give improvements of the above inequalities (4) and (5), respectively. We need the following two lemmas:

Lemma 1.6. [2, Theorem 7] *Let $f:[a,b]\to\mathbb{R}$ be a mapping such that the derivative $f^{(n-1)}(n\ge 1)$ is absolutely continuous on $[a,b]$. Then for all $x\in[a,b]$ we have the identity:*

$$
\int_a^b f(t)\,dt=\sum_{k=0}^{n-1}\frac{1}{(k+1)!}[(x-a)^{k+1}f^{(k)}(a)+(-1)^k(b-x)^{k+1}f^{(k)}(b)]
+\frac{1}{n!}\int_a^b(x-t)^n f^{(n)}(t)\,dt. \tag{6}
$$

Lemma 1.7. [3, Theorem 7] *Let $f:[a,b]\to\mathbb{R}$ be a mapping such that the derivative $f^{(n-1)}(n\ge 1)$ is absolutely continuous on $[a,b]$. Then for all $x\in[a,b]$ we have the identity:*

$$
\int_a^b f(t)\,dt=\sum_{k=0}^{n-1}\frac{1}{(k+1)!}[(b-x)^{k+1}+(-1)^k(x-a)^{k+1}]f^{(k)}(x)
+(-1)^n\int_a^b K_n(x,t)f^{(n)}(t)\,dt, \tag{7}
$$

where the kernel $K_n:[a,b]^2\to\mathbb{R}$ is given by

$$
K_n(x,t):=\begin{cases}\frac{(t-a)^n}{n!}, & \textit{if } t\in[a,x],\\ \frac{(t-b)^n}{n!}, & \textit{if } t\in(x,b],\end{cases} \tag{8}
$$

and $x\in[a,b]$.

2. The results

In what follows, we will use the notations

$$H_x = \frac{1}{b-a}\int_a^b (x-t)^n\,dt = \frac{(x-a)^{n+1}-(x-b)^{n+1}}{(n+1)(b-a)}$$

and

$$K_x = \frac{(x-a)(x-b)^{n+1}+(b-x)(x-a)^{n+1}}{b-a}.$$

Theorem 2.1. *Let $f:[a,b]\to\mathbb{R}$ be a mapping such that the derivative $f^{(n-1)}$ $(n\geq 1)$ is absolutely continuous on $[a,b]$. Assume that there exist constants $\gamma,\Gamma\in\mathbb{R}$ such that $\gamma\leq f^{(n)}(t)\leq\Gamma$ a.e. on $[a,b]$. Then for all $x\in[a,b]$, the following inequality holds:*

$$\left|\int_a^b f(t)\,dt-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}[(x-a)^{k+1}f^{(k)}(a)+(-1)^k(b-x)^{k+1}f^{(k)}(b)]\right.$$
$$\left.-\frac{H_x[f^{(n-1)}(b)-f^{(n-1)}(a)]}{n!}\right|\leq\frac{\Gamma-\gamma}{(n+1)!}G(a,b,x;n) \quad (9)$$

where

$$G(a,b,x;n)=\begin{cases} nH_x\sqrt[n]{H_x}+K_x, & n \text{ odd}\\ nH_x\sqrt[n]{H_x}-K_x, & n \text{ even and } x\in[a,\xi]\\ 2nH_x\sqrt[n]{H_x}, & n \text{ even and } x\in(\xi,\eta)\\ nH_x\sqrt[n]{H_x}+K_x, & n \text{ even and } x\in[\eta,b]\end{cases} \quad (10)$$

where ξ and η are the real roots of the equations

$$(\xi-a)^n-H_\xi=0 \quad (11)$$

and

$$(\eta-b)^n-H_\eta=0 \quad (12)$$

respectively and $a<\xi<\frac{a+b}{2}<\eta<b$. The inequality (9) with (10)-(12) is sharp.

Proof. For all $x\in[a,b]$, it is clear that by applying (6) and (3) we can derive

$$\begin{aligned}\Bigg|\int_a^b f(t)\,dt-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}&[(x-a)^{k+1}f^{(k)}(a)+(-1)^k(b-x)^{k+1}f^{(k)}(b)]\\&-\frac{H_x[f^{(n-1)}(b)-f^{(n-1)}(a)]}{n!}\Bigg|\\&=\frac{1}{n!}\left|\int_a^b(x-t)^n f^{(n)}(t)\,dt-H_x\int_a^b f^{(n)}(t)\,dt\right|\\&\le\frac{\Gamma-\gamma}{2n!}\int_a^b|(x-t)^n-H_x|\,dt. \quad (13)\end{aligned}$$

For brevity, put

$$p(x,t)=(x-t)^n-H_x,\quad (x,t)\in[a,b]\times[a,b]. \tag{14}$$

Then

$$\frac{\partial p(x,t)}{\partial t}=-n(x-t)^{n-1}. \tag{15}$$

If n is odd, we see from (15) that for any fixed $x\in[a,b]$, $p(x,t)$ is a strictly decreasing continuous function for $t\in[a,b]$, and from (14) we have

$$p(x,a)=(x-a)^n\left[1-\frac{x-a}{(n+1)(b-a)}\right]+\frac{(x-b)^{n+1}}{(n+1)(b-a)}>0,$$

$$p(x,b)=(x-b)^n\left[1-\frac{b-x}{(n+1)(b-a)}\right]-\frac{(x-a)^{n+1}}{(n+1)(b-a)}<0.$$

So, by the intermediate value theorem we can conclude that for any fixed $x\in[a,b]$, $p(x,t)$ has unique zero $t_1=x-\sqrt[n]{H_x}$ in (a,b). Thus the last integral in (13) is equal to

$$\int_a^b|p(x,t)|\,dt=\int_a^{t_1}p(x,t)\,dt-\int_{t_1}^b p(x,t)\,dt=\frac{2}{n+1}(nH_x\sqrt[n]{H_x}+K_x). \tag{16}$$

If n is even, we see from (15) that for any fixed $x\in[a,b]$, $p(x,t)$ is strictly decreasing for $t\in(a,x)$ and strictly increasing for $t\in(x,b)$. From (14), we have

$$p(x,a)=(x-a)^n-\frac{(x-a)^{n+1}+(b-x)^{n+1}}{(n+1)(b-a)},$$

$$p(x,b) = (x-b)^n - \frac{(x-a)^{n+1} + (b-x)^{n+1}}{(n+1)(b-a)},$$

and then

$$\frac{\partial p(x,a)}{\partial x} = (x-a)^{n-1}[n - \frac{x-a}{b-a}] + \frac{(b-x)^n}{b-a} > 0,$$

$$\frac{\partial p(x,b)}{\partial x} = (x-b)^{n-1}[n - \frac{b-x}{b-a}] - \frac{(x-a)^n}{b-a} < 0.$$

Thus we see that $p(x,a)$ is strictly increasing for $x \in [a,b]$ and $p(x,b)$ is strictly decreasing for $x \in [a,b]$, and

$$p(a,a) = p(b,b) = -\frac{(b-a)^n}{n+1} < 0,$$

$$p(\frac{a+b}{2},a) = p(\frac{a+b}{2},b) = \frac{n}{n+1}(\frac{b-a}{2})^n > 0.$$

So, by the intermediate value theorem we can conclude that there exist unique $\xi \in (a, \frac{a+b}{2})$ and unique $\eta \in (\frac{a+b}{2}, b)$ such that $p(\xi,a) = 0$ and $p(\eta,b) = 0$, i.e. ξ and η are the real roots of equations (11) and (12) respectively and $a < \xi < \frac{a+b}{2} < \eta < b$. If $x \in [a,\xi]$, then $p(x,a) \le 0$ and $p(x,b) > 0$. If $x \in (\xi,\eta)$, then $p(x,a) > 0$ and $p(x,b) > 0$. If $x \in [\eta,b]$, then $p(x,a) > 0$ and $p(x,b) \le 0$. Therefore, there are three possible cases to be determined.

(i) In case $x \in [a,\xi]$, $p(x,t) \le 0$ for $t \in [a,x]$ and $p(x,t)$ has a zero $t_2 = x + \sqrt[n]{H_x}$ in (x,b). We have

$$\int_a^b |p(x,t)|\,dt = -\int_a^{t_2} p(x,t)\,dt + \int_{t_2}^b p(x,t)\,dt = \frac{2}{n+1}(nH_x\sqrt[n]{H_x} - K_x). \quad (17)$$

(ii) In case $x \in (\xi,\eta)$, $p(x,t)$ has a zero $t_1 = x - \sqrt[n]{H_x}$ in $(a,x)$and a zero $t_2 = x + \sqrt[n]{H_x}$in (x,b). We have

$$\int_a^b |p(x,t)|\,dt = \int_a^{t_1} p(x,t)\,dt - \int_{t_1}^{t_2} p(x,t)\,dt + \int_{t_2}^b p(x,t)\,dt$$
$$= \frac{4n}{n+1}H_x\sqrt[n]{H_x}. \quad (18)$$

(iii) In case $x \in [\eta,b]$, $p(x,t)$ has a zero $t_1 = x - \sqrt[n]{H_x}$ in (a,x) and $p(x,t) \le 0$ for $t \in [x,b]$. We have

$$\int_a^b |p(x-t)|\,dt = \int_a^{t_1} p(x,t)\,dt - \int_{t_1}^b p(x,t)\,dt = \frac{2}{n+1}(nH_x\sqrt[n]{H_x}+K_x). \quad (19)$$

Consequently, the inequality (9) with (10)-(12) follows from (13) and (16)-(19). The function f for which the equality in (9) is attained is in fact such that $f^{(n)}$ is equal to Γ (or γ) when $p(x,t)\geq 0$ and equal to γ (or Γ) when $p(x,t)<0$. The proof is completed. □

Remark 2.2. It is not difficult to find that the sharp inequality (9) with (10)-(12) provides an improvement of the inequality (4).

Remark 2.3. If we take $x=\frac{a+b}{2}$, we can obtain the following sharp perturbed trapezoid inequality:

$$\begin{aligned}
&\Bigg|\int_a^b f(t)\,dt - \frac{b-a}{2}[f(a)+f(b)] \\
&-\sum_{k=1}^{n-1}\frac{1}{(k+1)!}(\frac{b-a}{2})^{k+1}[f^{(k)}(a)+(-1)^k f^{(k)}(b)] \\
&-\frac{1+(-1)^n}{(n+1)!}(\frac{b-a}{2})^{n+1}\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}\Bigg| \\
&\leq \frac{\Gamma-\gamma}{(n+1)!}[\frac{1-(-1)^n}{2}+\frac{n+(-1)^n n}{(n+1)\sqrt[n]{n+1}}](\frac{b-a}{2})^{n+1}.
\end{aligned}$$

Theorem 2.4. *Let $f:[a,b]\to\mathbb{R}$ be a mapping such that the derivative $f^{(n-1)}$ $(n\geq 1)$ is absolutely continuous on $[a,b]$. Assume that there exist constants $\gamma,\Gamma\in\mathbb{R}$ such that $\gamma\leq f^{(n)}(t)\leq\Gamma$ a.e. on $[a,b]$. Then for all $x\in[a,b]$, the following inequality holds:*

$$\begin{aligned}
&\Bigg|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1}\frac{1}{(k+1)!}[(b-x)^{k+1}+(-1)^k(x-a)^{k+1}]f^{(k)}(x) \\
&\qquad -\frac{H_x[f^{(n-1)}(b)-f^{(n-1)}(a)]}{n!}\Bigg| \leq \frac{\Gamma-\gamma}{(n+1)!}G(a,b,x;n) \quad (20)
\end{aligned}$$

where G(a,b,x;n) is as defined in (10)-(12). The inequality (20) with (10)-(12) is sharp.

Proof. For all $x \in [a,b]$, it is clear that by applying (7), (8) and (3) we can derive

$$\begin{aligned}
&\left|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1} \frac{1}{(k+1)!}[(b-x)^{k+1} + (-1)^k (x-a)^{k+1}] f^{(k)}(x) \right. \\
&\left. - \frac{H_x[f^{(n-1)}(b) - f^{(n-1)}(a)]}{n!}\right| \\
&= \left|\int_a^b K_n(x,t) f^{(n)}(t)\,dt - \frac{1}{b-a}\int_a^b K_n(x,t)\,dt \int_a^b f^{(n)}(t)\,dt\right| \\
&\leq \frac{\Gamma-\gamma}{2}\int_a^b \left|K_n(x,t) - \frac{1}{b-a}\int_a^b K_n(x,y)\,dy\right| dt \\
&= \frac{\Gamma-\gamma}{2n!}\left[\int_a^x |(t-a)^n - H_x|\,dt + \int_x^b |(t-b)^n - H_x|\,dt\right].
\end{aligned}$$

By substituting $t_1 = a+x-t$ and $t_2 = b+x-t$, we get

$$\begin{aligned}
&\int_a^x |(t-a)^n - H_x|\,dt + \int_x^b |(t-b)^n - H_x|\,dt \\
&= \int_a^x |(x-t_1)^n - H_x|\,dt_1 + \int_x^b |(x-t_2)^n - H_x|\,dt_2 \\
&= \int_a^b |(x-t)^n - H_x|\,dt,
\end{aligned}$$

and so the proof is reduced to the proof of Theorem 2.1. □

Remark 2.5. It is not difficult to find that the inequality (20) with (10)-(12) provides an improvement of the inequality (5).

Remark 2.6. If we take $x = \frac{a+b}{2}$, we can obtain the following sharp perturbed midpoint inequality:

$$\begin{aligned}
&\left|\int_a^b f(t)\,dt - (b-a) f(\frac{a+b}{2}) - \sum_{k=1}^{n-1} \frac{1+(-1)^k}{(k+1)!}(\frac{b-a}{2})^{k+1} f^{(k)}(\frac{a+b}{2}) \right. \\
&\qquad \left. - \frac{1+(-1)^n}{(n+1)!}(\frac{b-a}{2})^{n+1} \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a}\right| \\
&\qquad \leq \frac{\Gamma-\gamma}{(n+1)!}\left[\frac{1-(-1)^n}{2} + \frac{n+(-1)^n n}{(n+1)\sqrt[n]{n+1}}\right](\frac{b-a}{2})^{n+1}.
\end{aligned}$$

Remark 2.7. Setting $x = a$ and $x = b$ in (9) and (20) yields the same left and right rectangle inequalities

$$\left| \int_a^b f(t)\,dt - (b-a)f(a) - \sum_{k=1}^{n-1} \frac{(b-a)^{k+1}}{(k+1)!} f^{(k)}(a) - \frac{(b-a)^{n+1}}{(n+1)!} \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a} \right| \le \frac{\Gamma - \gamma}{(n+1)!} \frac{n(b-a)^{n+1}}{(n+1)\sqrt[n]{n+1}}$$

and

$$\left| \int_a^b f(t)\,dt - (b-a)f(b) - \sum_{k=1}^{n-1} \frac{(-1)^k (b-a)^{k+1}}{(k+1)!} f^{(k)}(b) - \frac{(-1)^n (b-a)^{n+1}}{(n+1)!} \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a} \right| \le \frac{\Gamma - \gamma}{(n+1)!} \frac{n(b-a)^{n+1}}{(n+1)\sqrt[n]{n+1}}$$

respectively.

Remark 2.8. If we take $n = 2$ in (20), then we recapture the following sharp inequality

$$\left| f(x) - (x - \frac{a+b}{2}) f'(x) + [\frac{1}{24}(b-a)^2 + \frac{1}{2}(x - \frac{a+b}{2})^2] \frac{f'(b) - f'(a)}{b-a} - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \le (\Gamma_2 - \gamma_2) G(a,b,x)$$

where $G(a,b,x) =$

$$= \begin{cases} \frac{1}{3(b-a)}[(x-a)(\frac{a+b}{2} - x)(b-x) \\ +(\frac{1}{12}(b-a)^2 + (x - \frac{a+b}{2})^2)^{\frac{3}{2}}], & a \le x \le \frac{1}{3}(2a+b), \\ \frac{2}{3(b-a)}[\frac{1}{12}(b-a)^2 + (x - \frac{a+b}{2})^2]^{\frac{3}{2}}, & \frac{1}{3}(2a+b) < x < \frac{1}{3}(a+2b) \\ \frac{1}{3(b-a)}[(x-a)(x - \frac{a+b}{2})(b-x) \\ +(\frac{1}{12}(b-a)^2 + (x - \frac{a+b}{2})^2)^{\frac{3}{2}}], & \frac{1}{3}(a+2b) \le x \le b, \end{cases}$$

which has been proved in [5, Theorem 1.5] and [7, Theorem 2] in different ways.

Finally, it should be noticed that the bounds of inequality (9) in Theorem 2.1 and inequality (20) in Theorem 2.4 are the same which has verified and extended the results in [1].

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