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ENTROPY SOLUTIONS TO NONLINEAR ELLIPTIC ANISOTROPIC PROBLEM WITH ROBIN BOUNDARY CONDITION

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We study a Robin boundary value problem of a nonlinear elliptic anisotropic type. We prove the existence and uniqueness of entropy solutions for L^1-data, via the existence and uniqueness of a weak solution.

1. Introduction

Let Ω be an open bounded domain of $\mathbb{R}^N$ $(N \geq 3)$ with a smooth boundary. Our aim is to prove the existence and uniqueness of an entropy solution to the following nonlinear anisotropic elliptic problem

$$\begin{cases} -\sum_{i=1}^{N} \dfrac{\partial}{\partial x_i} a_i(x, \dfrac{\partial}{\partial x_i} u) + |u|^{p_M(x)-2} u = f \text{ in } \Omega \\ \sum_{i=1}^{N} a_i(x, \dfrac{\partial}{\partial x_i} u)\nu_i = -|u|^{r(x)-2} u \qquad \text{on } \partial\Omega, \end{cases} \tag{1}$$

where the right-hand side $f \in L^1(\Omega)$ and $\nu_i(x)$ are the outer unit normals to the boundary at $x \in \partial\Omega$.

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We assume that for $i = 1,...,N$, the function $a_i : \Omega \times \mathbb{R} \to \mathbb{R}$ is Carathéodory and satisfies the following conditions: $a_i(x,\xi)$ is the continuous derivative with respect to ξ of the mapping $A_i : \Omega \times \mathbb{R} \to \mathbb{R}$, $A_i = A_i(x,\xi)$, i.e. $a_i(x,\xi) = \frac{\partial}{\partial \xi} A_i(x,\xi)$ such that the following equality and inequalities hold true.

$$A_i(x,0) = 0, \tag{2}$$

for almost every $x \in \Omega$.
There exists a positive constant C_1 such that

$$|a_i(x,\xi)| \le C_1(j_i(x) + |\xi|^{p_i(x)-1}), \tag{3}$$

for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}$, where j_i is a nonnegative function in $L^{p_i'(\cdot)}(\Omega)$, with $1/p_i(x) + 1/p_i'(x) = 1$.
There exists a positive constant C_2 such that

$$(a_i(x,\xi) - a_i(x,\eta)).(\xi - \eta) \ge \begin{cases} C_2|\xi-\eta|^{p_i(x)} & \text{if } |\xi-\eta| \ge 1 \\ C_2|\xi-\eta|^{p_i^-} & \text{if } |\xi-\eta| < 1, \end{cases} \tag{4}$$

for almost every $x \in \Omega$ and for every $\xi, \eta \in \mathbb{R}$, with $\xi \neq \eta$
and

$$|\xi|^{p_i(x)} \le a_i(x,\xi).\xi \le p_i(x) A_i(x,\xi), \tag{5}$$

for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}$.
We also assume that the variable exponents $p_i(.) : \overline{\Omega} \to [2,N)$ are continuous functions for all $i = 1,...,N$ such that

$$\frac{\bar{p}(N-1)}{N(\bar{p}-1)} < p_i^- < \frac{\bar{p}(N-1)}{N-\bar{p}}, \ \sum_{i=1}^{N} \frac{1}{p_i^-} > 1 \text{ and } \frac{p_i^+ - p_i^- - 1}{p_i^-} < \frac{\bar{p}-N}{\bar{p}(N-1)}, \tag{6}$$

where $\frac{1}{\bar{p}} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{p_i^-}$, $p_i^- := ess \inf_{x\in\Omega} p_i(x)$ and $p_i^+ := ess \sup_{x\in\Omega} p_i(x)$.
We put for all $x \in \Omega$, $p_M(x) = \max\{p_1(x),...,p_N(x)\}$ and for all $x \in \partial\Omega$,

$$p^{\partial}(x) = \begin{cases} \frac{(N-1)p(x)}{N-p(x)} & \text{if } \ p(x) < N \\ +\infty & \text{if } \ p(x) \ge N. \end{cases}$$

We introduce the numbers

$$q = \frac{N(\bar{p}-1)}{N-1}, \ q^* = \frac{Nq}{N-q} = \frac{N(\bar{p}-1)}{N-\bar{p}}. \tag{7}$$

We make the following additional assumption

$$r \in C(\overline{\Omega}) \text{ with } 1 < r^- \leq r^+ < \min_{x\in\partial\Omega}\{p_1^\partial(x), \cdots, p_N^\partial(x)\} \quad \forall x \in \partial\Omega \tag{8}$$

A prototype example which is covered by our assumptions is the following anisotropic equation:
Set $A_i(x,\xi) = (1/p_i(x))\,|\xi|^{p_i(x)}$, $a_i(x,\xi) = |\xi|^{p_i(x)-2}\,\xi$ where $p_i(x) \geq 2$. Then we get the following equation:

$$-\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial}{\partial x_i}u\right|^{p_i(x)-2}\frac{\partial}{\partial x_i}u\right) + |u|^{p_M(x)-2} = f$$

which is the anisotropic $p(.)$-Laplacian equation.
Equations involving variable exponent growth conditions have been extensively studied in the last decade. The large number of papers studying problems involving variable exponent growth conditions is motivated by the fact that this type of equations can serve as models in the theory of electrorheological fluids (see, e.g. [19]), image processing (see, e.g. [10]), the theory of elasticity (see, e.g. [24]), biology (see, e.g. [13]), the study of dielectric breakdown, electrical resistivity, and polycrystal plasticity (see, e.g. [6, 7]) or in the study of some models for growth of heterogeneous sandpiles (see, [5]).
Problems like (1) were studied by several authors with homogeneous Dirichlet boundary conditions in constant and variable exponent setting (see [2, 14–18, 22]). The Neumann or Robin type boundary conditions in the variable exponents setting are new and interesting problems and were for the first time studied by Boureanu and Radulescu (see [9]). The main difficulty for the study of problem in [9] was how to define an admissible space of solutions and to get useful properties of it. Another difficulty is that the famous Poincaré inequality does not apply. Boureanu and Radulescu [9] defined the appropriate space and a useful trace property which permit them to use minimization method and the famous Mountain Pass Theorem (see [21]) to prove the existence of weak solutions to the following problem

$$\begin{cases} -\sum_{i=1}^{N} \dfrac{\partial}{\partial x_i} a_i(x, \dfrac{\partial}{\partial x_i}u) + b(u)|u|^{p_M(x)-2}u = f(x,u) \text{ in } \Omega, \\ \\ u \geq 0 \text{ in } \Omega, \\ \\ \sum_{i=1}^{N} a_i(x, \dfrac{\partial}{\partial x_i}u)\nu_i = g(x,u) \qquad\qquad \text{on } \partial\Omega. \end{cases} \tag{9}$$

Note that in [9], if f and g satisfies monotonicity conditions, then (9) has a unique weak positive solution.

In this paper, we consider L^1-data f instead of the Carathéodory function $f(x,u)$ considered in [9]. When the data f is L^1, the suitable notion of solution is the entropy solution introduced for the first time by Bénilan *et al* (see [3]). We adapt this notion of entropy solution in our work. A first study in this direction was done by the authors of this paper for a Neumann homogeneous boundary condition (see [8]). When the Neumann condition is homogeneous, we don't need to give a sense of the values of the solution on the boundary. In this paper, we consider a Robin type boundary condition. Therefore, the values of the solution u at the boundary must be precised and the notion of solution considered must include the boundary condition. In this paper, we combine the notions of entropy solutions due to Bénilan *et al* [3] and Andreu *et al* [1]. The remaining part of the paper is as follows: in Section 2, we give some mathematical preliminaries, we prove the existence and uniqueness of a weak solution in Section 3 and in Section 4, we study the existence of a unique entropy solution of (1).

2. Preliminaries

In this section, we define Lebesgue, Sobolev and anisotropic spaces with a variable exponent and give some of their properties. Roughly speaking, anistropic Lebesgue and Sobolev spaces are functional spaces of Lebesgue's and Sobolev's type in which different space directions have different roles.

Let us consider a measurable function $p(.) : \Omega \to [1,\infty)$. We define the Lebesgue space with the variable exponent $L^{p(\cdot)}(\Omega)$ as the set of all measurable functions $u : \Omega \to \mathbb{R}$ for which the convex modular

$$\rho_{p(.)}(u) := \int_\Omega |u|^{p(x)}\, dx$$

is finite. If the exponent is bounded, i.e., if $p_+ < +\infty$, then the expression

$$|u|_{p(.)} := \inf\left\{\lambda > 0 : \rho_{p(.)}(u/\lambda) \le 1\right\}$$

defines a norm in $L^{p(\cdot)}(\Omega)$, called the Luxemburg norm. The space $(L^{p(\cdot)}(\Omega), |.|_{p(.)})$ is a separable Banach space. Moreover, if $p^- > 1$, then $L^{p(\cdot)}(\Omega)$ is uniformly convex, hence reflexive, and its dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. Finally, we have the Hölder type inequality (see [11]).

Proposition 2.1. ***(Generalized Hölder inequality)***
(i) *For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, we have*

$$\left|\int_\Omega u\, v\, dx\right| \le \left(\frac{1}{p^-} + \frac{1}{p'^-}\right) |u|_{p(.)}\, |v|_{p'(.)}\,.$$

(ii) *If $p_1, p_2 \in \mathcal{C}_+(\overline{\Omega})$, $p_1(x) \le p_2(x)$ for any $x \in \overline{\Omega}$, then $L^{p_2(.)}(\Omega) \hookrightarrow L^{p_1(.)}(\Omega)$ and the imbedding is continuous.*

Moreover, the application $\rho_{p(.)} : L^{p(.)}(\Omega) \to \mathbb{R}$ called the $p(.)-$modular of the $L^{p(.)}(\Omega)$ space, is very useful in handling these Lebesgue's spaces with variable exponent. Indeed, we have the following properties(see [11]). If $u \in L^{p(.)}(\Omega)$ and $p^+ < +\infty$ then

$$|u|_{p(.)} < 1 \implies |u|_{p(.)}^{p^+} \le \rho_{p(.)}(u) \le |u|_{p(.)}^{p^-}, \tag{10}$$

$$|u|_{p(.)} > 1 \implies |u|_{p(.)}^{p^-} \le \rho_{p(.)}(u) \le |u|_{p(.)}^{p^+}, \tag{11}$$

$$|u|_{p(.)} < 1 (= 1;\, > 1) \implies \rho_{p(.)}(u) < 1 (= 1;\, > 1), \tag{12}$$

$$|u|_{p(.)} \to 0 (\to \infty) \iff \rho_{p(.)}(u) \to 0 (\to \infty). \tag{13}$$

If, in addition, $(u_n)_{n\in\mathbb{N}} \subset L^{p(.)}(\Omega)$, then
$\lim\limits_{n\to+\infty} |u_n - u|_{p(.)} = 0 \;\Leftrightarrow\; \lim\limits_{n\to+\infty} \rho_{p(.)}(u_n - u) = 0 \;\Leftrightarrow\; (u_n)_n$ converges to u in measure and $\lim\limits_{n\to+\infty} \rho_{p(.)}(u_n) = \rho_{p(.)}(u)$.
Now, let us introduce the definition of the isotropic Sobolev space with variable exponent, $W^{1,p(.)}(\Omega)$.
We set

$$W^{1,p(.)}(\Omega) := \left\{ u \in L^{p(.)}(\Omega) : |\nabla u| \in L^{p(.)}(\Omega) \right\},$$

which is a Banach space equipped with the norm

$$\|u\|_{1,p(.)} := |u|_{p(.)} + |\nabla u|_{p(.)}.$$

Now, we present a natural generalization of the variable exponent Sobolev space $W^{1,p(.)}(\Omega)$ that will enable us to study the problem (1) with sufficient accuracy. The *anisotropic variable exponent Sobolev space* $W^{1,\overrightarrow{p}(.)}(\Omega)$ is defined as follow:

$$W^{1,\overrightarrow{p}(.)}(\Omega) = \left\{ u \in L^{p_M(.)}(\Omega); \frac{\partial}{\partial x_i} u \in L^{p_i(.)}(\Omega), \text{ for all } i \in \{1, ..., N\} \right\}.$$

Endowed with the norm

$$\|u\|_{\overrightarrow{p}(.)} := |u|_{p_M(.)} + \sum_{i=1}^{N} \left| \frac{\partial}{\partial x_i} u \right|_{p_i(.)},$$

the space $\left(W^{1,\overrightarrow{p}(.)}(\Omega), \|.\|_{\overrightarrow{p}(.)} \right)$ is a reflexive Banach space (see [12], Theorem 2.1 and Theorem 2.2).
Let us introduce the following notations:

$$\overrightarrow{P}_+ = (p_1^+, ..., p_N^+), \overrightarrow{P}_- = (p_1^-, ..., p_N^-),$$

$$P_+^+ = max\left\{p_1^+,...,p_N^+\right\}, P_-^+ = max\left\{p_1^-,...,p_N^-\right\}, P_-^- = min\left\{p_1^-,...,p_N^-\right\},$$

$$P_-^* = \frac{N}{\sum_{i=1}^N \frac{1}{p_i^-} - 1} \text{ and } \quad P_{-,\infty} = max\left\{P_-^+, P_-^*\right\}.$$

Throughout the paper we will mean by T_γ the truncation function at height $\gamma (\gamma > 0)$ defined as follow

$$T_\gamma(s) = \begin{cases} s & \text{if } |s| \leq \gamma, \\ \\ \gamma \, \text{sign(s)} & \text{if } |s| > \gamma. \end{cases}$$

We have the following result:

Theorem 2.2. *([12], Corollary 2.1) Let $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ be a bounded open set and for all $i \in \{1,...,N\}$, $p_i \in L^\infty(\Omega)$, $p_i(x) \geq 1$ a.e. in Ω. Then, for any $r \in L^\infty(\Omega)$ with $r(x) \geq 1$ a.e. in Ω such that*

$$ess \inf_{x\in\Omega}(p_M(x) - r(x)) > 0,$$

we have the compact embedding

$$W^{1,\overrightarrow{p}(.)}(\Omega) \hookrightarrow L^{r(.)}(\Omega).$$

We also need the following trace theorem due to [9].

Theorem 2.3. *Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded open set with a smooth boundary and let $\overrightarrow{p} \in (C_+(\bar{\Omega}))^N$, $r \in C(\bar{\Omega})$ satisfy the condition*

$$1 \leq r(x) < \min_{x\in\partial\Omega}\{p_1^\partial(x), \cdots, p_N^\partial(x)\}, \quad \forall x \in \partial\Omega. \tag{14}$$

Then, there is a compact boundary trace embedding

$$W^{1,\overrightarrow{p}(.)}(\Omega) \hookrightarrow L^{r(.)}(\partial\Omega).$$

Next, we define

$$\mathcal{T}^{1,\overrightarrow{p}(.)}(\Omega) := \{u : \Omega \to \mathbb{R} : T_k(u) \in W^{1,\overrightarrow{p}(.)}(\Omega), \forall k > 0\}.$$

Finally, in this paper, we will use the Marcinkiewicz spaces $\mathcal{M}^q(\Omega)(1 < q < \infty)$ with constant exponent. Note that the Marcinkiewicz spaces $\mathcal{M}^{q(.)}(\Omega)$ in the variable exponent setting was introduced for the first time by Sanchon and

Urbano (see [20]).
Marcinkiewicz spaces $\mathcal{M}^q(\Omega)(1 < q < \infty)$ contain the measurable functions $h : \Omega \to \mathbb{R}$ for which the distribution function

$$\lambda_h(\gamma) = |\{x \in \Omega : |h(x)| > \gamma\}|,\ \gamma \geq 0,$$

satisfies an estimate of the form

$$\lambda_h(\gamma) \leq C\gamma^{-q},\ \text{for some finite constant } C > 0.$$

The space $\mathcal{M}^q(\Omega)$ is a Banach space under the norm

$$\|h\|^*_{\mathcal{M}^q(\Omega)} = \sup_{t>0} t^{1/q} \left(\frac{1}{t} \int_0^t h^*(s)ds \right),$$

where h^* denotes the nonincreasing rearrangement of h:

$$h^*(t) = \inf\{\gamma > 0 : \lambda_h(\gamma) \leq t\}.$$

We will use the following pseudo norm

$$\|h\|_{\mathcal{M}^q(\Omega)} = \inf\left\{C : \lambda_h(\gamma) \leq C\gamma^{-q},\ \forall\gamma > 0\right\},$$

which is equivalent to the norm $\|h\|^*_{\mathcal{M}^q(\Omega)}$ (see [2]).
We need the following Lemma (see [4], Lemma A.2)

Lemma 2.4. *Let $1 \leq q < p < +\infty$. Then for every measurable function u on Ω, we have:*
(i) $\frac{(p-1)^p}{p^{p+1}} ||u||^p_{\mathcal{M}^p(\Omega)} \leq \sup_{\lambda>0}\{\lambda^p meas[x \in \Omega :\ |u| > \lambda]\} \leq ||u||^p_{\mathcal{M}^p(\Omega)}$.
Moreover
(ii) $\int_K |u|^q dx \leq \frac{p}{p-q} \left(\frac{p}{q}\right)^{q/p} ||u||^q_{\mathcal{M}^p(\Omega)} (meas(K))^{p-q/p}$,
for every measurable subset $K \subset \Omega$.
In particular, $\mathcal{M}^p(\Omega) \subset L^q_{loc}(\Omega)$, with continuous injection and $u \in \mathcal{M}^p(\Omega)$ implies $|u|^q \in \mathcal{M}^{p/q}(\Omega)$.

The following result is due to Troisi (see [23]).

Theorem 2.5. *Let $p_1, p_2, ..., p_N \in [1, +\infty)$; $g \in W^{1,(p_1,p_2,...,p_N)}(\Omega)$, and let*

$$\begin{cases} q = \bar{p}^* & \text{if } \ \bar{p}^* < N \\ q \in [1, +\infty) & \text{if } \ \bar{p}^* \geq N. \end{cases} \tag{15}$$

Then, there exists a constant $C > 0$ depending on $N, p_1, p_2, ..., p_N$ if $\bar{p} < N$ and also on q and meas(Ω) if $\bar{p} \geq N$ such that

$$||g||_{L^q(\Omega)} \leq C \prod_{i=1}^{N} \left[||g||_{L^{p_M}(\Omega)} + \left\| \frac{\partial g}{\partial x_i} \right\|_{L^{p_i}(\Omega)} \right]^{\frac{1}{N}}, \tag{16}$$

where $p_M = \max\{p_1, p_2, ..., p_N\}$ and $\frac{1}{\bar{p}} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{p_i}$.

We need the following lemma (see [8], Lemma 2.5).

Lemma 2.6. *Let g be a nonnegative function in $W^{1,\vec{p}(.)}(\Omega)$. Assume $\bar{p} < N$, and there exists a constant $C > 0$ such that*

$$\int_\Omega |T_\gamma(g)|^{p_M^-} dx + \sum_{i=1}^{N} \int_{\{|g| \leq \gamma\}} \left| \frac{\partial g}{\partial x_i} \right|^{p_i^-} dx \leq C(\gamma + 1), \quad \forall \gamma > 0. \tag{17}$$

Then, there exists a constant D, depending on C, such that

$$\|g\|_{\mathcal{M}^{\tilde{p}}(\Omega)} \leq D,$$

where $\tilde{p} = \dfrac{N(\bar{p}-1)}{N-\bar{p}}$.

In the sequel, we denote $E = W^{1,\vec{p}(.)}(\Omega)$ and $\|.\|_E = \|.\|_{W^{1,\vec{p}(.)}(\Omega)}$.

3. Weak solutions

Firstly, we define the notion of weak solution to problem (1) where the data f belongs to $L^\infty(\Omega)$.

Definition 3.1. A measurable function $u : \Omega \to \mathbb{R}$ is a weak solution of (1) if u belongs to $W^{1,\vec{p}(.)}(\Omega)$ and if the following equality holds true.

$$\int_\Omega \sum_{i=1}^{N} a_i(x, \frac{\partial}{\partial x_i} u) \frac{\partial}{\partial x_i} \varphi \, dx + \int_\Omega |u|^{p_M(x)-2} u \varphi \, dx + \int_{\partial\Omega} |u|^{r(x)-2} u \varphi \, d\sigma = \int_\Omega f \varphi \, dx, \tag{18}$$

for every $\varphi \in W^{1,\vec{p}(.)}(\Omega)$.

We have the following result.

Theorem 3.2. *Assume that (2)-(6), (8) and (14) hold true, $f \in L^\infty(\Omega)$. Then, there exists a unique weak solution of (1).*

Proof. $*$ **Existence** With the techniques that have become standard by now, it is not difficult to prove that (see for example [8, 9]) the functional $J : E \to \mathbb{R}$ defined by

$$J(u) = \int_\Omega \sum_{i=1}^N A_i(x, \frac{\partial}{\partial x_i} u)\, dx + \int_\Omega \frac{1}{p_M(x)} |u|^{p_M(x)}\, dx + \int_{\partial\Omega} \frac{1}{r(x)} |u|^{r(x)} d\sigma \\ - \int_\Omega f u\, dx$$

is well-defined on E, is of class $C^1(E, \mathbb{R})$ and is weakly lower semi-continuous with the Gateaux derivative given by

$$\langle J'(u), v\rangle = \int_\Omega \sum_{i=1}^N a_i(x, \frac{\partial}{\partial x_i} u) \frac{\partial}{\partial x_i} v\, dx + \int_\Omega |u|^{p_M(x)-2} u v\, dx + \int_{\partial\Omega} |u|^{r(x)-2} u v\, d\sigma \\ - \int_\Omega f(x) v\, dx.$$

To end the proof of the existence part, we just have to prove that J is bounded from below and coercive.
Let's denote

$$\Lambda(u) = \int_\Omega \sum_{i=1}^N A_i(x, \frac{\partial}{\partial x_i} u)\, dx,$$

$$I(u) = \int_\Omega \sum_{i=1}^N A_i(x, \frac{\partial}{\partial x_i} u)\, dx + \int_\Omega \frac{1}{p_M(x)} |u|^{p_M(x)}\, dx - \int_\Omega f u\, dx$$

and

$$L(u) = \int_{\partial\Omega} \frac{1}{r(x)} |u|^{r(x)} d\sigma.$$

Let $u \in E$ be such that $||u||_E \to \infty$. Using (5) we deduce that

$$\Lambda(u) \geq \frac{1}{p_M^+} \sum_{i=1}^N \int_\Omega \left| \frac{\partial}{\partial x_i} u \right|^{p_i(x)} dx.$$

We make the following notations:

$$\mathcal{I}_1 = \left\{ i \in \{1, ..., N\} : \quad \left| \frac{\partial}{\partial x_i} u \right|_{L^{p_i(\cdot)}(\Omega)} \leq 1 \right\}$$

and

$$\mathcal{I}_2 = \left\{ i \in \{1, ..., N\} : \quad \left| \frac{\partial}{\partial x_i} u \right|_{L^{p_i(\cdot)}(\Omega)} > 1 \right\}.$$

We then have

$$\Lambda(u) \geq \frac{1}{p_M^+} \sum_{i\in\mathcal{I}_1} \int_\Omega \left| \frac{\partial}{\partial x_i} u \right|^{p_i(x)} dx + \frac{1}{p_M^+} \sum_{i\in\mathcal{I}_2} \int_\Omega \left| \frac{\partial}{\partial x_i} u \right|^{p_i(x)} dx.$$

Using (10), (11) and (12) we deduce that

$$\begin{aligned}
\Lambda(u) &\geq \frac{1}{p_M^+} \sum_{i\in\mathcal{I}_1} \left| \frac{\partial}{\partial x_i} u \right|_{p_i(.)}^{p_i^+} + \frac{1}{p_M^+} \sum_{i\in\mathcal{I}_2} \left| \frac{\partial}{\partial x_i} u \right|_{p_i(.)}^{p_i^-} \\
&\geq \frac{1}{p_M^+} \sum_{i\in\mathcal{I}_2} \left| \frac{\partial}{\partial x_i} u \right|_{p_i(.)}^{p_i^-} \\
&\geq \frac{1}{p_M^+} \sum_{i\in\mathcal{I}_2} \left| \frac{\partial}{\partial x_i} u \right|_{p_i(.)}^{p_m^-} \\
&\geq \frac{1}{p_M^+} \left(\sum_{i=1}^N \left| \frac{\partial}{\partial x_i} u \right|_{p_i(.)}^{p_m^-} - \sum_{i\in\mathcal{I}_1} \left| \frac{\partial}{\partial x_i} u \right|_{p_i(.)}^{p_m^-} \right) \\
&\geq \frac{1}{p_M^+} \left(\sum_{i=1}^N \left| \frac{\partial}{\partial x_i} u \right|_{p_i(.)}^{p_m^-} - N \right).
\end{aligned}$$

By the generalized mean inequality or the Jensen's inequality applied to the convex function $z:\ \mathbb{R}^+ \to \mathbb{R}^+,\ z(t) = t^{p_m^-},\ p_m^- > 1$ we get

$$\sum_{i=1}^N \left| \frac{\partial}{\partial x_i} u \right|_{p_i(.)}^{p_m^-} \geq \frac{1}{N^{p_m^- - 1}} \left(\sum_{i=1}^N \left| \frac{\partial}{\partial x_i} u \right|_{p_i(.)} \right)^{p_m^-},$$

thus,

$$\Lambda(u) \geq \frac{1}{p_M^+} \left[\frac{1}{N^{p_m^- - 1}} \left(\sum_{i=1}^N \left| \frac{\partial}{\partial x_i} u \right|_{p_i(.)} \right)^{p_m^-} - N \right]. \tag{19}$$

Case 1: $|u|_{p_M(.)} \geq 1$. We have

$$\begin{aligned}
I(u) &\geq \frac{1}{p_M^+}\left[\frac{1}{N^{p_m^- -1}}\left(\sum_{i=1}^{N}\left|\frac{\partial}{\partial x_i}u\right|_{p_i(.)}\right)^{p_m^-} - N\right] + \frac{1}{p_M^+}|u|_{p_M(.)}^{p_M^-} - ||f||_{L^\infty(\Omega)}||u||_{L^1(\Omega)} \\
&\geq \frac{1}{p_M^+}\left[\frac{1}{N^{p_m^- -1}}\left(\sum_{i=1}^{N}\left|\frac{\partial}{\partial x_i}u\right|_{p_i(.)}\right)^{p_m^-} + |u|_{p_M(.)}^{p_m^-}\right] - C||f||_{L^\infty(\Omega)}||u||_E - \frac{N}{p_M^+} \\
&\geq \frac{1}{2^{p_m^- -1}p_M^+}\min\left(1, \frac{1}{N^{p_m^- -1}}\right)\left[\sum_{i=1}^{N}\left|\frac{\partial}{\partial x_i}u\right|_{p_i(.)} + |u|_{p_M(.)}\right]^{p_m^-} \\
&\qquad - C||f||_{L^\infty(\Omega)}||u||_E - \frac{N}{p_M^+}.
\end{aligned}$$

Therefore

$$I(u) \geq \frac{1}{2^{p_m^- -1}p_M^+}\min\left(1, \frac{1}{N^{p_m^- -1}}\right)||u||_E^{p_m^-} - C||f||_{L^\infty(\Omega)}||u||_E - \frac{N}{p_M^+}. \tag{20}$$

Case 2: $|u|_{p_M(.)} < 1$. Then $|u|_{p_M(.)}^{p_m^-} - 1 < 0$ and we get

$$\begin{aligned}
I(u) &\geq \frac{1}{p_M^+}\left[\frac{1}{N^{p_m^- -1}}\left(\sum_{i=1}^{N}\left|\frac{\partial}{\partial x_i}u\right|_{p_i(.)}\right)^{p_m^-} + |u|_{p_M(.)}^{p_m^-} - N - 1\right] - ||f||_{L^\infty(\Omega)}||u||_{L^1(\Omega)} \\
&\geq \frac{1}{p_M^+}\left[\frac{1}{N^{p_m^- -1}}\left(\sum_{i=1}^{N}\left|\frac{\partial}{\partial x_i}u\right|_{p_i(.)}\right)^{p_m^-} + |u|_{p_M(.)}^{p_m^-}\right] - C||f||_{L^\infty(\Omega)}||u||_E - \frac{N+1}{p_M^+} \\
&\geq \frac{1}{2^{p_m^- -1}p_M^+}\min\left(1, \frac{1}{N^{p_m^- -1}}\right)\left[\sum_{i=1}^{N}\left|\frac{\partial}{\partial x_i}u\right|_{p_i(.)} + |u|_{p_M(.)}\right]^{p_m^-} \\
&\qquad - C||f||_{L^\infty(\Omega)}||u||_E - \frac{N+1}{p_M^+}.
\end{aligned}$$

So, we obtain

$$I(u) \geq \frac{1}{2^{p_m^- -1}p_M^+}\min\left(1, \frac{1}{N^{p_m^- -1}}\right)||u||_E^{p_m^-} - C||f||_{L^\infty(\Omega)}||u||_E - \frac{N+1}{p_M^+}. \tag{21}$$

As L is positive, we deduce from (20) and (21) that

$$J(u) \to +\infty \text{ as } \|u\|_E \to +\infty. \tag{22}$$

We also deduce that J is bounded from below.

$*$ **Uniqueness.** Let $u, v \in E$ be two weak solutions to problem (1). Choosing as a test function in (18), $\varphi = v-u$ for the weak solution u and $\varphi = u-v$ for the weak solution v, we get

$$\begin{aligned}&\int_\Omega \sum_{i=1}^N a_i(x,\frac{\partial}{\partial x_i}u)\frac{\partial}{\partial x_i}(v-u)\,dx+\int_\Omega |u|^{p_M(x)-2}u(v-u)\,dx\\&+\int_{\partial\Omega}|u|^{r(x)-2}u(v-u)\,d\sigma-\int_\Omega f(x)(v-u)\,dx=0\end{aligned}\tag{23}$$

and

$$\begin{aligned}&\int_\Omega \sum_{i=1}^N a_i(x,\frac{\partial}{\partial x_i}v)\frac{\partial}{\partial x_i}(u-v)\,dx+\int_\Omega |v|^{p_M(x)-2}v(u-v)\,dx\\&+\int_{\partial\Omega}|u|^{r(x)-2}v(u-v)\,d\sigma-\int_\Omega f(x)(u-v)\,dx=0.\end{aligned}\tag{24}$$

Summing up (23) and (24) we obtain

$$\begin{aligned}&\int_\Omega \sum_{i=1}^N \left(a_i(x,\frac{\partial}{\partial x_i}u)-a_i(x,\frac{\partial}{\partial x_i}v)\right)\frac{\partial}{\partial x_i}(u-v)\,dx\\&+\int_\Omega\left(|u|^{p_M(x)-2}u-|v|^{p_M(x)-2}v\right)(u-v)\,dx\\&+\int_{\partial\Omega}\left(|u|^{r(x)-2}u-|v|^{r(x)-2}v\right)(u-v)\,d\sigma=0.\end{aligned}$$

Thus, by the monotonicity of the functions $a_i(x,.)$, $t\mapsto |t|^{p_M(x)-2}t$ and $t\mapsto |t|^{r(x)-2}t$, we deduce that $u=v$ almost everywhere. □

4. Entropy solutions

In this section, we study the existence and uniqueness of an entropy solution to problem (1) when the right-hand side $f\in L^1(\Omega)$. We first recall some notations.

Lemma 4.1. *Let $u\in\mathcal{T}^{1,\overrightarrow{p}(.)}(\Omega)$. Then there exists a unique measurable function $v_i:\Omega\to\mathbb{R}$ such that*

$$v_i\chi_{\{|u|<k\}}=\frac{\partial T_k(u)}{\partial x_i}\text{ for a.e. } x\in\Omega,\text{ for all } k>0\text{ and } i\in\{1,...,N\},$$

where χ_A denotes the characteristic function of a measurable set A. The functions v_i are denoted $\frac{\partial u}{\partial x_i}$. Moreover, if u belongs to $W^{1,\overrightarrow{p}(.)}(\Omega)$, then $v_i\in L^{p_i(.)}(\Omega)$ and coincides with the standard distributional gradient of u i.e. $v_i=\frac{\partial u}{\partial x_i}$.

Definition 4.2. We define the space $\mathcal{T}_{tr}^{1,\overrightarrow{p}(.)}(\Omega)$ as the set of functions $u \in \mathcal{T}^{1,\overrightarrow{p}(.)}(\Omega)$ such that there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset W^{1,\overrightarrow{p}(.)}(\Omega)$ satisfying:
(a) $u_n \to u$ a.e. in Ω,
(b) $\dfrac{\partial T_k(u_n)}{\partial x_i} \to \dfrac{\partial T_k(u)}{\partial x_i}$ in $L^1(\Omega)$, for all $k > 0$,
(c) there exists a measurable function v on $\partial\Omega$ such that $u_n \to v$ a.e. on $\partial\Omega$.

Definition 4.3. A measurable function u is an entropy solution of (1) if $u \in \mathcal{T}_{tr}^{1,\overrightarrow{p}(.)}(\Omega)$ and for every $k > 0$,

$$\sum_{i=1}^{N}\int_\Omega a_i(x,\frac{\partial u}{\partial x_i})\frac{\partial}{\partial x_i}T_k(u-\varphi)\,dx + \int_\Omega |u|^{p_M(x)-2}uT_k(u-\varphi)\,dx \\ + \int_{\partial\Omega}|u|^{r(x)-2}uT_k(u-\varphi)\,d\sigma \leq \int_\Omega f(x)T_k(u-\varphi)\,dx \tag{25}$$

for every $\varphi \in W^{1,\overrightarrow{p}(.)}(\Omega)\cap L^\infty(\Omega)$.

Our main result is the following:

Theorem 4.4. *Assume (2)-(6), (8) and $f \in L^1(\Omega)$. Then there exists a unique entropy solution u to problem (1).*

The proof of this theorem will be done in three steps.
Step 1: A priori estimates

Lemma 4.5. *Assume (2)-(6), (8) and $f \in L^1(\Omega)$. Let u be an entropy solution of (1). If there exists a positive constant M such that*

$$\sum_{i=1}^{N}\int_{\{|u|>t\}} t^{q_i(x)}dx \leq M, \text{ for all } t > 0, \tag{26}$$

then

$$\sum_{i=1}^{N}\int_{\left\{\left|\frac{\partial}{\partial x_i}u\right|^{\alpha_i(.)}>t\right\}} t^{q_i(x)}dx \leq \|f\|_1 + M \text{ for all } t > 0,$$

where $\alpha_i(.) = p_i(.)/(q_i(.)+1)$, for all $i = 1,...,N$.

Proof. Take $\varphi = 0$ in (25), we have

$$\sum_{i=1}^{N}\int_\Omega a_i(x,\frac{\partial}{\partial x_i}T_t(u)).\frac{\partial}{\partial x_i}T_t(u)dx + \int_\Omega |u|^{p_M(x)-2}uT_t(u)dx \\ + \int_{\partial\Omega}|u|^{r(x)-2}uT_t(u)d\sigma \leq \int_\Omega f(x)T_t(u)dx,$$

for all $t > 0$. Since the two last terms of the left-hand side are nonnegative, it follows that

$$\sum_{i=1}^{N} \int_{\Omega} a_i(x, \frac{\partial}{\partial x_i} T_t(u)).\frac{\partial}{\partial x_i} T_t(u) dx \leq \int_{\Omega} f(x) T_t(u) dx.$$

According to (5), we deduce that

$$\sum_{i=1}^{N} \int_{\Omega} \left| \frac{\partial}{\partial x_i} T_t(u) \right|^{p_i(x)} dx \leq t \, \|f\|_1 \, , \text{ for all } t > 0.$$

Therefore, defining $\psi := T_t(u)/t$, we have, for all $t > 0$,

$$\sum_{i=1}^{N} \int_{\Omega} t^{p_i(x)-1} \left| \frac{\partial}{\partial x_i} \psi \right|^{p_i(x)} dx = \sum_{i=1}^{N} \frac{1}{t} \int_{\Omega} \left| \frac{\partial}{\partial x_i} T_t(u) \right|^{p_i(x)} dx \leq \|f\|_1 \, .$$

From the above inequality, the definition of $\alpha_i(.)$ and (26), we have

$$\begin{aligned}
&\sum_{i=1}^{N} \int_{\left\{ \left| \frac{\partial}{\partial x_i} u \right|^{\alpha_i(.)} > t \right\}} t^{q_i(x)} dx \\
&\leq \sum_{i=1}^{N} \int_{\left\{ \left| \frac{\partial}{\partial x_i} u \right|^{\alpha_i(.)} > t \right\} \cap \{|u| \leq t\}} t^{q_i(x)} dx + \sum_{i=1}^{N} \int_{\{|u|>t\}} t^{q_i(x)} dx \\
&\leq \sum_{i=1}^{N} \int_{\left\{ \left| \frac{\partial}{\partial x_i} u \right|^{\alpha_i(.)} > t \right\} \cap \{\{|u| \leq t\}} t^{q_i(x)} \left(\frac{\left| \frac{\partial}{\partial x_i} u \right|^{\alpha_i(x)}}{t} \right)^{\frac{p_i(x)}{\alpha_i(x)}} dx + M \\
&\leq \sum_{i=1}^{N} \int_{\left\{ \left| \frac{\partial}{\partial x_i} u \right|^{\alpha_i(.)} > t; \, |u| \leq t \right\}} t^{q_i(x) - \frac{p_i(x)}{\alpha_i(x)}} \left| \frac{\partial}{\partial x_i} u \right|^{p_i(x)} dx + M \\
&\leq \sum_{i=1}^{N} \frac{1}{t} \int_{\left\{ \left| \frac{\partial}{\partial x_i} u \right|^{\alpha_i(.)} > t; \, |u| \leq t \right\}} \left| \frac{\partial}{\partial x_i} u \right|^{p_i(x)} dx + M \leq \|f\|_1 + M
\end{aligned}$$

for all $t > 0$. □

Lemma 4.6. *Assume (2)-(6), (8) and $f \in L^1(\Omega)$. Let u be an entropy solution of (1), then*

$$\frac{1}{h} \sum_{i=1}^{N} \int_{\{|u| \leq h\}} \left| \frac{\partial}{\partial x_i} T_h(u) \right|^{p_i(x)} dx \leq M \tag{27}$$

for every $h > 0$, with M a positive constant. Moreover, we have

$$\| |u|^{p_M(x)-2} u \|_{L^1(\Omega)} = \| |u|^{p_M(x)-1} \|_{L^1(\Omega)} \leq \|f\|_1 \tag{28}$$

$$\||u|^{r(x)-2}u\|_{L^1(\partial\Omega)} = \||u|^{r(x)-1}\|_{L^1(\partial\Omega)} \leq \|f\|_1 \tag{29}$$

and there exists a constant $D > 0$ such that

$$meas\{|u| > h\} \leq \frac{D}{h^{P_M^- - 1}} \quad \forall h > 1.$$

Proof. Taking $\varphi = 0$ in the entropy inequality (25) and using (5), we obtain

$$\sum_{i=1}^{N} \int_{\{|u|\leq h\}} \left|\frac{\partial}{\partial x_i} T_h(u)\right|^{p_i(x)} dx \leq h\|f\|_1 \leq Mh,$$

$$\int_{\Omega} |u|^{p_M(x)-2} u T_h(u) dx \leq h\|f\|_1$$

and

$$\int_{\partial\Omega} |u|^{r(x)-2} u T_h(u) d\sigma \leq h\|f\|_1,$$

for all $h > 0$.
Therefore

$$\int_{\{|u|>h\}} |u|^{p_M(x)-2} u T_h(u) dx \leq h\|f\|_1$$

and

$$\int_{\partial\Omega\cap\{|u|>h\}} |u|^{r(x)-2} u T_h(u) d\sigma \leq h\|f\|_1.$$

As $uT_h(u)\chi_{\{|u|>h\}} = h|u|\chi_{\{|u|>h\}}$, we get from the previous inequalities and Fatou's Lemma

$$\int_{\Omega} |u|^{p_M(x)-2}|u| dx \leq \|f\|_1$$

and

$$\int_{\partial\Omega} |u|^{r(x)-2}|u| d\sigma \leq \|f\|_1.$$

Since $|T_h(u)| \leq |u|$ we have

$$\int_{\Omega} |T_h(u)|^{p_M(x)-1} dx \leq \int_{\Omega} |u|^{p_M(x)-1} dx \leq \|f\|_1.$$

It follows that

$$\int_{\Omega} |T_h(u)|^{p_M^- -1}\, dx \leq D, \tag{30}$$

where D is a positive constant, indeed

$$\begin{aligned}
\int_{\Omega} |T_h(u)|^{p_M^- -1}\, dx &\leq \int_{\{|T_h(u)|\leq 1\}} |T_h(u)|^{p_M^- -1}\, dx + \int_{\{|T_h(u)|>1\}} |T_h(u)|^{p_M^- -1}\, dx \\
&\leq meas(\Omega) + \int_{\Omega} |T_h(u)|^{p_M(x)-1} dx \\
&\leq meas(\Omega) + \|f\|_1.
\end{aligned}$$

From (30) we have

$$\int_{\{|u|>h\}} |T_h(u)|^{p_M^- -1}\, dx \leq D.$$

Then

$$h^{p_M^- -1} meas\{|u|>h\} \leq D,$$

from which, we deduce that

$$meas\{|u|>h\} \leq \frac{D}{h^{p_M^- -1}}.$$

□

Lemma 4.7. *If u is an entropy solution of (1) then there exists a constant $C>0$ such that*

$$\int_{\Omega} |T_k(u)|^{p_M^-}\, dx + \sum_{i=1}^{N} \int_{\{|u|\leq k\}} \left|\frac{\partial}{\partial x_i} u\right|^{p_i^-} dx \leq C(k+1), \quad \forall k>0 \tag{31}$$

and

$$\int_{\partial\Omega} |T_k(u)|^{r^-}\, d\sigma \leq C(k+1), \quad \forall k>0. \tag{32}$$

Proof. Taking $\varphi = 0$ in the entropy inequality (25) and using (5) we get

$$\begin{aligned}
\int_{\Omega} |u|^{p_M(x)-2} u T_k(u) dx &+ \sum_{i=1}^{N} \int_{\{|u|\leq k\}} \left|\frac{\partial}{\partial x_i} T_k(u)\right|^{p_i(x)} dx \\
&+ \int_{\partial\Omega} |u|^{r(x)-2} u T_k(u) d\sigma \leq k\|f\|_1.
\end{aligned} \tag{33}$$

Note that all the integrals in (33) are nonnegative, therefore we obtain

$$\int_{\Omega} |u|^{p_M(x)-2} u T_k(u) dx + \sum_{i=1}^{N} \int_{\{|u|\leq k\}} \left|\frac{\partial}{\partial x_i} T_k(u)\right|^{p_i(x)} dx \leq k\|f\|_1 \tag{34}$$

and

$$\int_{\partial\Omega} |u|^{r(x)-2} u T_k(u) d\sigma \leq k \|f\|_1 . \tag{35}$$

Furthermore,

$$\begin{aligned}
\sum_{i=1}^{N} \int_{\{|u|\leq k\}} \left| \frac{\partial}{\partial x_i} T_k(u) \right|^{p_i^-} dx &= \sum_{i=1}^{N} \int_{\{|u|\leq k,\, |\frac{\partial}{\partial x_i} u| \leq 1\}} \left| \frac{\partial}{\partial x_i} T_k(u) \right|^{p_i^-} dx \\
&\quad + \sum_{i=1}^{N} \int_{\{|u|\leq k,\, |\frac{\partial}{\partial x_i} u| \geq 1\}} \left| \frac{\partial}{\partial x_i} T_k(u) \right|^{p_i^-} dx \\
&\leq N meas(\Omega) + \sum_{i=1}^{N} \int_{\{|u|\leq k,\, |\frac{\partial}{\partial x_i} u| \geq 1\}} \left| \frac{\partial}{\partial x_i} T_k(u) \right|^{p_i(x)} dx \\
&\leq N meas(\Omega) + \sum_{i=1}^{N} \int_{\{|u|\leq k\}} \left| \frac{\partial}{\partial x_i} T_k(u) \right|^{p_i(x)} dx
\end{aligned}$$

and

$$\begin{aligned}
\int_{\Omega} |T_k(u)|^{p_M^-} dx &\leq \int_{\{|T_k(u)|\leq 1\}} |T_k(u)|^{p_M^-} dx + \int_{\{|T_k(u)|>1\}} |T_k(u)|^{p_M^-} dx \\
&\leq meas(\Omega) + \int_{\Omega} |T_k(u)|^{p_M(x)} dx \\
&\leq meas(\Omega) + \int_{\Omega} |u|^{p_M(x)-2} u T_k(u) dx.
\end{aligned}$$

Therefore, we deduce according to (34) and the two last estimations that

$$\int_{\Omega} |T_k(u)|^{P_M^-} dx + \sum_{i=1}^{N} \int_{\{|u|\leq k\}} \left| \frac{\partial}{\partial x_i} u \right|^{p_i^-} dx \leq (N+1) meas(\Omega) + k \|f\|_1 , \quad \forall k > 0.$$

For the inequality (32), according to (35), we have the following

$$\begin{aligned}
\int_{\partial\Omega} |T_k(u)|^{r^-} d\sigma &\leq \int_{\partial\Omega \cap \{|T_k(u)|\leq 1\}} |T_k(u)|^{r^-} d\sigma + \int_{\partial\Omega \{|T_k(u)|>1\}} |T_k(u)|^{r^-} d\sigma \\
&\leq meas(\partial\Omega) + \int_{\partial\Omega} |T_k(u)|^{r(x)} d\sigma \\
&\leq meas(\partial\Omega) + \int_{\partial\Omega} |u|^{r(x)-2} u T_k(u) d\sigma \\
&\leq meas(\partial\Omega) + k \|f\|_1 .
\end{aligned}$$

□

Lemma 4.8. *If u is an entropy solution of (1) then*

$$\rho_{p_i'(.)}\left(\left|\frac{\partial}{\partial x_i}u\right|^{p_i(x)-1}\chi_F\right)\leq C,\quad \forall i=1,\cdots,N,$$

where $F=\{h<|u|\leq h+t\}$, $h>0$, $t>0$.

Proof. Taking $\varphi=T_h(u)$ as a test function in the entropy inequality (25), we get

$$\sum_{i=1}^{N}\int_\Omega a_i(x,\frac{\partial}{\partial x_i}u).\frac{\partial}{\partial x_i}T_t(u-T_h(u))dx+\int_\Omega |u|^{p_M(x)-2}uT_t(u-T_h(u))dx$$
$$+\int_{\partial\Omega}|u|^{r(x)-2}uT_t(u-T_h(u))d\sigma\leq\int_\Omega f(x)T_t(u-T_h(u))dx.$$

It follows by using (5) that

$$\int_F\left|\frac{\partial}{\partial x_i}u\right|^{p_i(x)}dx\leq t\,\|f\|_1\,.$$

Therefore,

$$\rho_{p_i'(.)}\left(\left|\frac{\partial}{\partial x_i}u\right|^{p_i(x)-1}\chi_F\right)\leq C,\ \forall i=1,...,N.$$

□

Lemma 4.9. *If u is an entropy solution of (1) then*

$$\lim_{h\to+\infty}\int_\Omega |f|\chi_{\{|u|>h-t\}}=0,$$

where $h,t>0$.

Proof. By lemma 4.6, we deduce that

$$\lim_{h\to+\infty}|f|\chi_{\{|u|>h-t\}}=0$$

and as $f\in L^1(\Omega)$, it follows by using the Lebesgue dominated convergence theorem that

$$\lim_{h\to+\infty}\int_\Omega |f|\chi_{\{|u|>h-t\}}=0.$$

□

Step 2. Uniqueness of entropy solution. Let $t > 0$ and u, v two entropy solutions of (1). We write the entropy inequality (25) corresponding to the solution u, with $T_t v$ as a test function, and to the solution v, with $T_t u$ as test function. Upon addition and after the same calculus as in [18] by using above a priori estimates, we get

$$\sum_{i=1}^{N} \int_{\{|u-v|\leq t\}} \left(a_i(x, \frac{\partial}{\partial x_i} u) - a_i(x, \frac{\partial}{\partial x_i} u) \right) \cdot \frac{\partial}{\partial x_i}(u-v)\, dx = 0, \tag{36}$$

$$\int_{\Omega} \left(|u|^{p_M(x)-2} u - |v|^{p_M(x)-2} v \right) T_t(u-v)\, dx = 0 \tag{37}$$

and

$$\int_{\partial\Omega} \left(|u|^{r(x)-2} u - |v|^{r(x)-2} v \right) T_t(u-v)\, d\sigma = 0. \tag{38}$$

We deduce from (36), (37) and (38) that

$$u = v \text{ a.e on } \Omega.$$

Step 3. Existence of entropy solutions. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of bounded functions, strongly converging to $f \in L^1(\Omega)$ and such that

$$\|f_n\|_1 \leq \|f\|_1, \text{ for all } n \in \mathbb{N}. \tag{39}$$

We consider the problem

$$\begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, \frac{\partial}{\partial x_i} u_n) + |u_n|^{p_M(x)-2} u_n = f_n \text{ in } \Omega \\ \\ \sum_{i=1}^{N} a_i(x, \frac{\partial}{\partial x_i} u_n) \nu_i = -|u_n|^{r(x)-2} u_n \qquad \text{on } \partial\Omega. \end{cases} \tag{40}$$

It follows from Theorem 3.2 that problem (40) admits a unique weak solution $u_n \in W^{1,\vec{p}(.)}(\Omega)$ which satisfies

$$\sum_{i=1}^{N} \int_{\Omega} a_i(x, \frac{\partial}{\partial x_i} u_n) \frac{\partial}{\partial x_i} \varphi\, dx + \int_{\Omega} |u_n|^{p_M(x)-2} u_n \varphi\, dx + \int_{\partial\Omega} |u_n|^{r(x)-2} u_n \varphi\, d\sigma$$
$$= \int_{\Omega} f_n(x) \varphi\, dx, \tag{41}$$

for every $\varphi \in W^{1,\vec{p}(.)}(\Omega)$.
Our interest is to prove that these approximated solutions u_n tend, as n goes to infinity, to a measurable function u which is an entropy solution to the problem (1). For that purpose, we show some preliminary results.

Lemma 4.10. *If* u_n *is a weak solution of (41) then there exist two constants* $C_1, C_2 > 0$ *such that*
(i) $\|u_n\|_{\mathcal{M}^{\tilde{p}}(\Omega)} \le C_1$,
(ii) $\left\|\frac{\partial}{\partial x_i} u_n\right\|_{\mathcal{M}^{p_i^- q/\bar{p}}(\Omega)} \le C_2$, *for all* $i = 1, \ldots, N$.

Proof. (i) is a consequence of lemmas 2.6 and 4.7.
(ii) Let $\lambda_{|\frac{\partial}{\partial x_i} u_n|}(\alpha) = meas\{x \in \Omega : |\frac{\partial}{\partial x_i} u_n| > \alpha\}$ for all $i = 1, .., N$, and for any $\alpha > 0, \gamma > 1$, we have

$$\begin{aligned}
\lambda_{|\frac{\partial}{\partial x_i} u_n|}(\alpha) &\le meas\{x \in \Omega : |\frac{\partial}{\partial x_i} u_n| > \alpha, \, |u_n| > \gamma\} \\
&\quad + meas\{x \in \Omega : |\frac{\partial}{\partial x_i} u_n| > \alpha, \, |u_n| < \gamma\} \\
&\le \int_{\{|\frac{\partial}{\partial x_i} u_n| > \alpha, \, |u_n| < \gamma\}} \left(\frac{1}{\alpha} |\frac{\partial}{\partial x_i} u_n|\right)^{p_i^-} dx + \lambda_{|u_n|}(\gamma) \\
&\le \frac{1}{\alpha^{p_i^-}} \int_{\{|u_n| < \gamma\}} \left|\frac{\partial}{\partial x_i} u_n\right|^{p_i^-} dx + \lambda_{|u_n|}(\gamma).
\end{aligned}$$

Using Lemma 4.7 and (i) we get

$$\lambda_{|\frac{\partial}{\partial x_i} u_n|}(\alpha) \le C\left(\frac{\gamma}{\alpha^{p_i^-}} + \gamma^{-\tilde{p}}\right),$$

from which we deduce (ii). □

Lemma 4.11. *Assume that (2)-(6), (8) hold true and* u_n *is a weak solution of (40) then the following hold true.*
i) There exists $s > 1$ *such that* $u_n \to u$ *a.e. in* Ω *and moreover* $u_n \rightharpoonup u$ *in* $W^{1,s}(\Omega)$,
ii) $\frac{\partial}{\partial x_i} u_n$ *converges strongly to* $\frac{\partial}{\partial x_i} u$ *in* $L^s(\Omega)$, *for* $i = 1 \cdots N$. *Moreover,*
$a_i(x, \frac{\partial u_n}{\partial x_i}) \to a_i(x, \frac{\partial u}{\partial x_i})$ *strongly in* $L^1(\Omega)$, *for all* $i = 1, \ldots, N$.
iii) u_n *converges to some measurable function* v *a.e. in* $\partial\Omega$.

Proof. The proof of (i) and (ii) can be found in [15] (see also [8]). We will then only prove (iii). From i), it follows that $T_k(u_n)$ converges weakly to $T_k(u)$ in $W^{1,s}(\Omega)$ and a.e. in Ω. Since for every $1 \le s \le +\infty$,

$$\tau : W^{1,s}(\Omega) \to L^s(\partial\Omega), u \mapsto \tau(u) = u|_{\partial\Omega}$$

is compact, we deduce that $T_k(u_n)$ converges strongly to $T_k(u)$ in $L^s(\partial\Omega)$ and so, up to a subsequence we can assume that $T_k(u_n)$ converges to $T_k(u)$ a.e. on

$\partial\Omega$. In other words, there exists $C \subset \partial\Omega$ such that $T_k(u_n)$ converges to $T_k(u)$ on $\partial\Omega \backslash C$ with $\sigma(C) = 0$ where σ is the area measure on $\partial\Omega$.
Furthermore, from Lemma 4.7, the following inequality holds true

$$\int_{\partial\Omega} |T_k(u_n)|^{r^-} \, d\sigma \leq Ck \tag{42}$$

for $k > 1$. By using Fatou's Lemma in (42), as n goes to infinity, we get

$$\int_{\partial\Omega} |T_k(u)|^{r^-} \, d\sigma \leq Ck. \tag{43}$$

For every $k > 1$, let $A_k := \{x \in \partial\Omega : |T_k(u)| < k\}$ and $C' = \partial\Omega \backslash \cup_{k>0} A_k$. We have

$$\sigma(C') = \frac{1}{k} \int_{C'} |T_k(u)| \, d\sigma \leq \frac{1}{k} \int_{\partial\Omega} |T_k(u)| \, d\sigma.$$

We now use the fact that the embedding $L^{r^-}(\partial\Omega) \hookrightarrow L^1(\partial\Omega)$ is continuous to get

$$\sigma(C') \leq \frac{C_1}{k} \|T_k(u)\|_{L^{r^-}(\partial\Omega)}.$$

Thus, from (43) we have

$$\sigma(C') \leq C_2 k^{\frac{1}{r^-}-1}.$$

We deduce that $\sigma(C') \to 0$ as k goes to infinity.
Let us define on $\partial\Omega$ the function v by

$$v(x) := T_k(u(x)) \text{ if } x \in A_k.$$

We take $x \in \partial\Omega \backslash (C \cup C')$; then there exists $k > 0$ such that $x \in A_k$ and we have

$$u_n(x) - v(x) = (u_n(x) - T_k(u_n(x))) + (T_k(u_n(x)) - T_k(u(x))).$$

Since $x \in A_k$, we have $|T_k(u(x))| < k$ and so $|T_k(u_n(x))| < k$, from which we deduce that $|u_n(x)| < k$. Therefore

$$u_n(x) - v(x) = (T_k(u_n(x)) - T_k(u(x))) \to 0, \text{ as } n \to \infty.$$

This means that u_n converges to v a.e. on $\partial\Omega$ □

Now, let $\varphi \in W^{1,\vec{p}(.)}(\Omega) \cap L^\infty(\Omega)$. For any $k > 0$, we choose $T_k(u_n - \varphi)$ as a test function in (41) to get

$$\sum_{i=1}^{N} \int_\Omega a_i(x, \frac{\partial}{\partial x_i} u_n) \frac{\partial}{\partial x_i} T_k(u_n - \varphi) \, dx + \int_\Omega |u_n|^{p_M(x)-2} u_n T_k(u_n - \varphi) \, dx$$
$$+ \int_{\partial\Omega} |u_n|^{r(x)-2} u_n T_k(u_n - \varphi) \, d\sigma = \int_\Omega f_n(x) T_k(u_n - \varphi) \, dx. \tag{44}$$

Now, by the same calculus as in [8], we pass to the limit in (44) to get

$$\sum_{i=1}^{N}\int_{\Omega} a_i(x, \frac{\partial}{\partial x_i}u)\frac{\partial}{\partial x_i}T_k(u-\varphi)\,dx + \int_{\Omega}|u|^{p_M(x)-2}uT_k(u-\varphi)\,dx$$
$$+\int_{\partial\Omega}|u|^{r(x)-2}uT_k(u-\varphi)\,d\sigma \leq \int_{\Omega} f(x)T_k(u-\varphi)\,dx.$$

REFERENCES

[1] F. Andreu - J. M. Mazón - S. Segura De Léon - J. Toledo, *Quasi-linear elliptic and parabolic equations in L^1 with nonlinear boundary conditions*, Adv. Math. Sci. Appl 7 (1) (1997), 183–213 .

[2] M. Bendahmane - K. H. Karlsen; *Anisotropic nonlinear elliptic systems with measure data and anisotropic harmonic maps into spheres*, Electron. J. Differential Equations 46 (2006), 30 pp.

[3] Ph. Bénilan L. Boccardo - T. Gallouët - R. Gariepy - M. Pierre - J. L. Vazquez, *An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations*, Ann. Sc. Norm. Super. Pisa, Cl. Sci. 22, (1995), 241–273.

[4] Ph. Bénilan - H. Brezis - M. G. Crandall, *A semilinear equation in $L^1(\mathbb{R})^N$*, Ann. Scuola. Norm. Sup. Pisa 2 (1975), 523–555 .

[5] M. Bocea- M. Mihăilescu - M. Pérez-Llanos - J. D. Rossi, *Models for growth of heterogeneous sandpiles via Mosco convergence*, Asymptotic Anal. 78 (1-2) (2012), 11–36.

[6] M. Bocea - M. Mihăilescu, *Γ-convergence of power-law functionals with variable exponents*, Nonlinear Anal. TMA 73 (1) (2010), 110–121.

[7] M. Bocea - M. Mihăilescu - C. Popovici, *On the asymptotic behavior of variable exponent power-law functionals and applications*, Ric. Mat. 59 (2) (2010), 207–238.

[8] B. K. Bonzi - S. Ouaro - F. D. Zongo, *Entropy solutions for nonlinear elliptic anisotropic homogeneous Neumann problem*, Int. J. Differ. Equ. Article 476781 (2013), 14 pp.

[9] M. M. Boureanu - V. D. Rădulescu, *Anisotropic Neumann problems in Sobolev spaces with variable exponent*, Nonlinear Anal. TMA 75 (12) (2012), 4471–4482.

[10] Y. Chen - S. Levine - M. Rao, *Variable exponent, linear growth functionals in image restoration*, SIAM J. Appl. Math. 66 (4) (2001), 1383–1406.

[11] X. Fan - D. Zhao, *On the $L^{p(.)}(\Omega)$ and $W^{1,p(.)}(\Omega)$*, J. Math. Anal. Appl. 263 (2001), 424–446.

[12] X. Fan, *Anisotropic variable exponent Sobolev spaces and $\vec{p}(.)$-Laplacian equations*, Complex Var. Elliptic Equ. 55 (2010), 1–20.

[13] G. Fragnelli, *Positive periodic solutions for a system of anisotropic parabolic equations*, J. Math. Anal. Appl. 367 (2010), 204–228.

[14] B. Koné - S. Ouaro - S. Traoré, *Weak solutions for anisotropic nonlinear elliptic equations with variable exponent*, Electron. J. Diff. Equ. 144 (2009), 1–11.

[15] B. Koné - S. Ouaro - S. Soma, *Weak solutions for anisotropic nonlinear elliptic problems with variable exponent and measure data*, Int. J. Evol. Equ. 5 (3) (2011), 327–350.

[16] M. Mihăilescu - P. Pucci - V. Rădulescu, *Eigenvalue problems for anisotropic quasilinear elliptic equations with variable exponent* J. Math. Anal. Appl. 340 (1) (2008), 687–698.

[17] M. Mihăilescu - G. Morosanu, *Existence and multiplicity of solutions for an anisotropic elliptic problem involving variable exponent growth conditions*, Appl. Anal. 89 (2) (2010), 257–271.

[18] S. Ouaro, *Well-posedness results for anisotropic nonlinear elliptic equations with variable exponent and L^1-data*, Cubo 12 (1) (2010), 133–148.

[19] M. Ruzicka, *Electrorheological fluids: modelling and mathematical theory*, Lecture Notes in Mathematics 1748, Springer-Verlag, Berlin, 2000.

[20] M. Sanchon - J. M. Urbano, *Entropy solutions for the $p(x)$-Laplace Equation*, Trans. Am. Math. Soc. 361 (12) (2009), 6387–6405.

[21] M. Struwe, *Variational Methods: Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems*, Springer, Heidelberg, 1996.

[22] D. Stancu-Dumitru; *Two nontrivial solutions for a class of anisotropic variable exponent problems*, Taiwanese J. Math. 16 (4) (2012), 1205–1219.

[23] M. Troisi, *Teoremi di inclusione per spazi di Sobolev non isotropi*, Ric. Mat. 18 (1969), 3–24.

[24] V. V. Zhikov, *Meyer-type estimates for solving the nonlinear Stokes system*, Differentsial'nye Uravneniya, 33 (1) (1997), 107–114.

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