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ON DEGENERATE ELLIPTIC EQUATIONS IN MORREY SPACES

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1. Introduction.

Regularity of generalized solutions to degenerate elliptic PDE's has received a very strong impulse in the direction of finding the minimal assumptions under which regularity results hold true. Definitely, the point of view is that Lebesgue classes are not the right ones where to put lower order terms to ensure regularity. It is now clear that the right classes are Morrey spaces and some potential spaces (see [5]). The aim of this note is to show that regularity is possible for equations of the following kind

$$Lu - X_j^*(b_j u) = X_j^*(f_j) \tag{1}$$

where the lower order terms are assumed to belong to the degenerate Morrey class $L^{2,\lambda}(\Omega, \mathrm{X})$, with $2 < \lambda < Q$. The paper is a continuation of [1] where the case of operator in leading part has been studied. We prove extra integrability and local hölder continuity results. The proof is based on the study of a *non convolution integral operator* obtained from the representation formula for generalized solution by means of the generalized gradient of the Green function $G^x(y)$ of L. We study the boundedness properties of this operator in Theorem 5.3 using an idea of Hedberg. Despite

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of the lack of pointwise estimates for XG, we find regularity of generalized solutions handling the case of discontinuous coefficients. Our results extend properly those in [3].

2. Preliminaries

In this section we recall some basic definitions.

2.1. Let X $= (X_1, X_2, \ldots, X_m)$ a system of C^∞ vector fields in $\mathbb{R}^n$. We say that $X_1, X_2, \ldots, X_m$ satisfy Hörmander's condition in a bounded domain Ω if

$$rank\ Lie\{X_1, X_2, \ldots, X_m\} = n$$

at every point of Ω.

A piecewise C^1 curve $\gamma : [0, T] \to \mathbb{R}^n$ is called X–sub-unit, if

$$\langle \gamma'(t), \xi \rangle^2 \le \sum_{j=1}^{m} \langle X_j(\gamma(t)), \xi \rangle^2 \quad \forall\, \xi \in \mathbb{R}^n, \quad \text{a.e. } t \in [0, T]. \tag{2}$$

The X–sub-unit length of γ is by definition $l_S(\gamma) = T$. Given $x, y \in \mathbb{R}^n$, we denote by $\Phi(x, y)$ the collection of all X–sub-unit curves connecting x to y. As it is well known, $\Phi(x, y)$ is not empty by Chow theorem ([2]). Setting

$$\rho(x, y) = \inf\{l_S(\gamma) \ : \ \gamma \in \Phi(x, y)\} \tag{3}$$

we define a distance, usually called the Carnot–Caratheodory distance generated by the system X.

We denote by $B(x, r) = \{y \in \mathbb{R}^n \ : \ \rho(x, y) < r\}$ the metric ball centered at x of radius r and whenever x is not relevant we write B_r. The C-C balls satisfy the doubling property with respect to the Lebesgue measure. We set $Q = \log_2 C$ for the homogenous dimension of Ω.

Now we list the function spaces we will need in the sequel.

Let $1 \le p < \infty$. We say that $u \in L^p(\Omega)$ if

$$\|u\|_p^p \equiv \int_\Omega |u|^p \mathrm{d}x < \infty$$

and $u \in L^\infty(\Omega)$ if u is a bounded measurable function in Ω.

Definition 2.1. (Sobolev spaces). *Let $1 \le p < +\infty$. We say that u belongs to $W^{1,p}(\Omega, \mathrm{X})$ if u and $X_j u$, belong to $L^p(\Omega)$, $j = 1, 2, \ldots, m$ and we* *set*

$$\|u\|_{W^{1,p}(\Omega,\mathrm{X})} \equiv \|u\|_{L^p(\Omega)} + \sum_{j=1}^{m} \|X_j u\|_{L^p(\Omega)}. \tag{4}$$

We denote by $W_0^{1,p}(\Omega, \mathrm{X})$ *the completion of* $C_0^\infty(\Omega)$ *with respect to the above norm. As usual, when* $p = 2$ *we set* $H^1(\Omega, \mathrm{X})$ *and* $H_0^1(\Omega, \mathrm{X})$ *the spaces* $W^{1,2}(\Omega, \mathrm{X})$ *and* $W_0^{1,2}(\Omega, \mathrm{X})$ *respectively.*

We would like to point out that $X_j u$ denotes the distributional derivative of u defined by

$$< X_j u, \phi >= \int_\Omega u X_j^* \phi \mathrm{d}x, \quad \forall\, \phi \in C_0^\infty(\Omega)$$

where $X_j^* = -\sum_{i=1}^{n} \partial_i (c_{ij}\ \cdot)$ is the formal adjoint of $X_j = \sum_{i=1}^{n} c_{ij}\partial_i$.

Now we define some classes of potential that are very useful for us in the sequel. We state also some related embedding properties.

Definition 2.2. (Stummel-Kato classes). *Let* $u : \Omega \subseteq \mathbb{R}^N \to \mathbb{R}$ *and* $r > 0$. *If*

$$\eta(r) \equiv \sup_{x\in\Omega} \int_{\{y\in\Omega|\rho(x,y)<r\}} |u(y)| \frac{\rho^2(x,y)}{|B(x,\rho(x,y))|} \mathrm{d}y < \infty, \quad \forall\, r > 0$$

we say that $u \in \tilde{S}(\Omega, \mathrm{X})$.

If, in addition, $\eta(r) \to 0$ *we say that* $u \in S(\Omega, \mathrm{X})$.

In the case $X_j = \partial_j,\ j = 1, \ldots, n$ we get the usual Stummel-Kato classes.

In the sequel we will use some properties of the above defined classes.

Lemma 2.3. ([4]) *Let* $V \in S(\Omega, \mathrm{X})$ *and* $u \in C_0^\infty(\Omega)$. *Then there exists* C *independent on* u*, such that*

$$\int_{B_R} |V(x)||u(x)|^2 \mathrm{d}x \le C\eta(2R) \int_{B_R} |\mathrm{X}u(x)|^2 \mathrm{d}x.$$

Moreover,

$$\int_\Omega |V(x)||u(x)|^2 \mathrm{d}x \le \varepsilon \int_\Omega |\mathrm{X}u(x)|^2 \mathrm{d}x + k(\varepsilon) \int_\Omega |u(x)|^2 \mathrm{d}x,$$

when $\varepsilon > 0$ *and* $k(\varepsilon) \sim \dfrac{\varepsilon}{(\eta_V^{-1}(\varepsilon))^{Q+2}}$

An immediate consequence of the previous Lemma, is the following

Proposition 2.4. *Let* $\Omega \subset \mathbb{R}^n$ *and* $\mathrm{X} = (X_1, \ldots, X_m)$ *a system of* C^∞ *vector fields, satisfying Hörmander's condition. We have*

$$\tilde{S}(\Omega, \mathrm{X}) \subset (H_0^1(\Omega, \mathrm{X}))^*.$$

Proof. Let $f \in \tilde{S}(\Omega, \mathrm{X})$ and $\varphi \in C_0^\infty(\Omega)$. Let B_r be a metric ball containing the support of φ. By the previous Lemma, we immediately get

$$\begin{aligned} | < f, \varphi > | &\leq \left(\int_{B_r} |f|\varphi^2 \mathrm{d}x \right)^{\frac{1}{2}} \left(\int_{B_r} |f| \mathrm{d}x \right)^{\frac{1}{2}} \\ &\leq C\eta(2r) \left(\int_{B_r} |\mathrm{X}\varphi|^2 \mathrm{d}x \right)^{\frac{1}{2}} \|f\|_{L^1(\Omega)} \leq C\|\varphi\|_{H_0^1(\Omega,\mathrm{X})}. \end{aligned}$$

□

Definition 2.5. (Morrey classes). *Let* Ω *be a bounded domain in* $\mathbb{R}^n$, $1 \leq p < \infty$ *and* $\lambda > 0$. *We say that* $f \in L^p_{loc}(\Omega)$ *belongs to the Morrey class* $L^{p,\lambda}(\Omega, \mathrm{X})$ *if*

$$\|f\|_{p,\lambda} \equiv \sup_B \left(\frac{r^\lambda}{|B|} \int_B |f(y)|^p \mathrm{d}y \right)^{\frac{1}{p}} < \infty$$

where $B \equiv B(x, r)$, *the supremum is taken with class of balls centered at* $x \in \Omega$ *of radius* r. f *is understood to be zero outside* Ω.

Remark 2.6. If $\lambda = Q$ then $L^{p,\lambda}(\Omega, \mathrm{X}) \equiv L^p_{loc}(\Omega)$ and if $\lambda > Q$ then $L^{p,\lambda}(\Omega, \mathrm{X}) \equiv \{0\}$.

It is worth to compare the Morrey and the Lebesgue classes (see e.g. [7]).

Proposition 2.7. *Let* $q \geq p$ *and* $\frac{\mu}{q} \leq \frac{\lambda}{p}$. *Then*

$$L^{q,\mu}(\Omega, \mathrm{X}) \subseteq L^{p,\lambda}(\Omega, \mathrm{X}).$$

The following weak Morrey classes will be also useful in the sequel.

Definition 2.8. *We say that* $f \in L_w^{p,\lambda}(\Omega, \mathrm{X})$ *if there exists* $C > 0$, *independent on* r *and* x_0, *such that*

$$\sup_{t>0} t^p |\{x \in \Omega \cap B_r(x_0) : |f(x)| > t\}| \le C \frac{|B_r(x_0)|}{r^\lambda}.$$

The relation between Morrey and weak Morrey classes is as expected

Proposition 2.9. *Let* $1 \le q < p < \infty$ *and* $0 < \lambda < Q$, *then*

$$L_w^{p,\lambda}(\Omega, \mathrm{X}) \subseteq L^{q,\mu}(\Omega, \mathrm{X})$$

where $\mu = \dfrac{\lambda}{p} q$.

Finally we recall the Hardy-Littlewood maximal function.

Definition 2.10. *Let* f *a locally integrable function. The function*

$$Mf(x) = \sup_B \frac{1}{|B|} \int_B |f(y)| \mathrm{d}y,$$

is called the Hardy-Littlewood maximal function of f. *The supremum is taken over all metric balls* B *centered at* x.

2.2. Let $\Omega \subset \mathbb{R}^n$ and $\mathrm{X} = (X_1, X_2, \ldots, X_m)$ a system of C^∞ vector fields in $\mathbb{R}^n$ satisfying Hörmander's condition in Ω.

Let $a_{ij} \in L^\infty(\Omega)$, $a_{ij} = a_{ji}$ for $i, j = 1, 2, \ldots, m$. Assume that there exist $\Lambda, \ \lambda > 0$ such that

$$\lambda |\xi|^2 \le \sum_{i,j=1}^m a_{ij} \xi_i \xi_j \le \Lambda |\xi|^2$$

$\forall \xi \in \mathbb{R}^m$, a.e.$x \in \Omega$.

In the sequel we set

$$Lu = \sum_{i,j=1}^m X_i^*(a_{ij} X_j u)$$

and consider u as a generalized solution to the equation $Lu = f$, for suitable f.

Definition 2.11. (Weak solution). *Let* $f \in (H_0^1(\Omega, \mathrm{X}))^*$. *We say that* $u \in H^1(\Omega, \mathrm{X})$ *is a weak solution of* $Lu = f$ *if*

$$\int_\Omega a_{ij}(x) X_j u(x) X_i \varphi(x) \mathrm{d}x = \int_\Omega f(x) \varphi(x) \mathrm{d}x, \quad \forall \, \varphi \in C_0^\infty(\Omega).$$

A more general kind of solution, is the following (compare [6])

Definition 2.12. (Very weak solution) *For a measure μ of bounded variation on Ω, we say that $u \in L^1(\Omega)$ is a very weak solution of $Lu = \mu$ vanishing on $\partial\Omega$, if*

$$< L^* v, u >= \int_\Omega v \mathrm{d}\mu, \ \forall\, v \in H_0^1(\Omega, \mathrm{X}) \cap C_b^0(\Omega) | L^* v \in C_b^0(\Omega), \tag{5}$$

where $C_b^0(\Omega)$ is the set of all bounded continuous functions in Ω.

Remark 2.13. The very weak solution is a solution vanishing at the boundary. This means that it is not just a solution of the equation but it is a solution of the Dirichlet problem.

So when we say " u is a very weak solution of $Lu = \mu$" we omit the words "vanishing at the boundary".

The Dirichlet problem is well posed for any bounded variation measure. Namely we have (see [1])

Theorem 2.14. *Let Ω be a bounded domain in $\mathbb{R}^n$. Then there exists the very weak solution of $Lu = \mu$ and it is unique. Moreover if $1 < p < \dfrac{Q}{Q-1}$ then $u \in W_0^{1,p}(\Omega, \mathrm{X})$ and there exists $C = C(\Omega, \lambda, Q)$ such that:*

$$\|u\|_{W_0^{1,p}} \leq C\|\mu\|.$$

In general, if μ is not in $(H_0^1(\Omega, \mathrm{X}))^*$ then weak solutions do not exist. The concept of very weak solutions properly extends the previous one. Indeed (see [1])

Proposition 2.15. *Let $\mu \in (H_0^1(\Omega, \mathrm{X}))^*$ a bounded variation measure on Ω and $u \in H_0^1(\Omega, X)$ be the weak solution. Let w be the very weak solution of $Lw = \mu$. Then $u = w$.*

3. An integral operator.

In this section we introduce an integral operator that we use to represent the very weak solution to the problem (6).

We stress that the operator we are going to study is a not convolution operator.

Let Ω be a bounded domain of

$$\mathbb{R}^n,\ \Delta = \{(x, y) \in \Omega \times \Omega | x = y\}$$

and $K(x, y)$ a measurable real function locally integrable on $\Omega \times \Omega \setminus \Delta$ such that $(X_j K(x, y))(y) \in L^2_{loc}(\Omega \setminus \Delta)$. Let us suppose that for a.e. $x, y \in \Omega$, there exists $C \equiv C_K > 0$, such that. For $f \equiv (f_1, \dots, f_m)$ we define

$$Tf(x) = \int_\Omega (X_j K(x, y))(y) f_j(y) \mathrm{d}y.$$

It is clear that we need some assumptions on K and f in order that the integral is finite and Tf to be bounded as linear operator. What we need (and prove) in the sequel is that the operator T is bounded between some Morrey classes.

Lemma 3.1. *Let Ω be a bounded domain of $\mathbb{R}^n$, $\Delta = \{(x, y) \in \Omega \times \Omega | x = y\}$ and $K(x, y)$ a measurable real function locally integrable on $\Omega \times \Omega \setminus \Delta$ such that $(X_j K(x, y))(y) \in L^2_{loc}(\Omega \setminus \Delta)$. Let us suppose that for a.e. $x, y \in \Omega$, there exists $C \equiv C_K > 0$, such that*

$$i) \int_{\{y \in \Omega | R < \rho(x,y) < 2R\}} |(X_j K(x, y))(y)|^2 \mathrm{d}y \leq \frac{C}{R^2} \int_{\{y \in \Omega | R/2 < \rho(x,y) < 4R\}} |K(x, y)|^2 \mathrm{d}y,$$

$\forall\, R > 0\ j = 1, \dots, m;$

$$ii) \quad |K(x, y)| \leq C_K \frac{\rho(x, y)^2}{|B(x, \rho(x, y))|}, \quad \forall\, (x, y) \in \Omega \times \Omega \setminus \Delta.$$

Let $f \equiv (f_1, \dots, f_m)$ with $f_j \in L^{2,\lambda}(\Omega, \mathrm{X})$ where $2 < \lambda < Q$.
Then the integral operator

$$Tf(x) = \int_\Omega (X_j K(x, y))(y) f_j(y) \mathrm{d}y, \quad \forall\, f_j \in L^{2,\lambda}(\Omega, \mathrm{X})$$

is bounded from $L^{2,\lambda}(\Omega, \mathrm{X})$ to $L^{p_\lambda,\lambda}_w(\Omega, \mathrm{X})$, where

$$\frac{1}{p_\lambda} = \frac{1}{2} - \frac{1}{\lambda}.$$

Moreover, there exists $C \geq 0$ such that

$$\|Tf\|_{q,\lambda} \leq C \|f\|_{2,\lambda}, \quad \forall\, 1 \leq q < p_\lambda.$$

Proof. Let $\varepsilon > 0$ to choose later. We have

$$Tf(x) = \int_{\{y\in\Omega|\rho(x,y)<\varepsilon\}} (X_j K(x,y))(y) f_j(y)\mathrm{d}y + \int_{\{y\in\Omega|\rho(x,y)\geq\varepsilon\}} (X_j K(x,y))(y) f_j(y)\mathrm{d}y \equiv I + II.$$

Let us estimate I.

$$|I| \leq \sum_{k=0}^{\infty} \int_{\{y\in\Omega|\frac{\varepsilon}{2^{k+1}}\leq\rho(x,y)<\frac{\varepsilon}{2^k}\}} |(X_j K(x,y))(y)||f_j(y)|\mathrm{d}y$$

$$\leq \sum_{k=0}^{\infty} \left(\int_{\{y\in\Omega|\frac{\varepsilon}{2^{k+1}}\leq\rho(x,y)<\frac{\varepsilon}{2^k}\}} |(X_j K(x,y))(y)|^2\mathrm{d}y\right)^{\frac{1}{2}} \left(\int_{\{y\in\Omega|\frac{\varepsilon}{2^{k+1}}\leq\rho(x,y)<\frac{\varepsilon}{2^k}\}} |f_j(y)|^2\mathrm{d}y\right)^{\frac{1}{2}}.$$

Applying $i)$ and $ii)$, we get

$$|I| \leq C\sum_k \frac{2^k}{\varepsilon}\left(\int_{\{y\in\Omega|\frac{\varepsilon}{2^{k+2}}\leq\rho(x,y)<\frac{\varepsilon}{2^{k-1}}\}} |K(x,y)|^2\mathrm{d}y\right)^{\frac{1}{2}} \cdot \left(\int_{\{y\in\Omega|\frac{\varepsilon}{2^{k+1}}\leq\rho(x,y)<\frac{\varepsilon}{2^k}\}} |f_j(y)|^2\mathrm{d}y\right)^{\frac{1}{2}}$$

$$\leq C\sum_k \frac{2^k}{\varepsilon}\left(\int_{\{y\in\Omega|\frac{\varepsilon}{2^{k+2}}\leq\rho(x,y)<\frac{\varepsilon}{2^{k-1}}\}} \frac{\rho^4(x,y)}{|B(x,\rho(x,y))|^2}\mathrm{d}y\right)^{\frac{1}{2}} \cdot |B(x,\frac{\varepsilon}{2^k})|(M|f_j|^2(x))^{\frac{1}{2}} \leq C(M|f_j|^2(x))^{\frac{1}{2}}\varepsilon.$$

Let us estimate now II.

$$|II| \leq \int_{\{y\in\Omega|2^k\varepsilon\leq\rho(x,y)<2^{k+1}\varepsilon\}} |(X_j K(x,y))(y)||f_j(y)|\mathrm{d}y$$

$$\leq C\sum_k \varepsilon^{\frac{2-Q}{2}} 2^{k\frac{2-Q}{2}} \frac{|B(x,2^k\varepsilon)|^{\frac{1}{2}}}{(2^k\varepsilon)^{\frac{\lambda}{2}}}\|f_j\|_{2,\lambda}$$

$$\leq C\varepsilon^{\frac{2-\lambda}{2}}\|f_j\|_{2,\lambda}\sum_k 2^{k\frac{2-\lambda}{2}} \leq C\varepsilon^{\frac{2-\lambda}{2}}\|f_j\|_{2,\lambda},$$

because $\lambda > 2$ and $f_j \in L^{2,\lambda}(\Omega, \mathrm{X})$.

Merging the two inequalities, we have

$$|Tf(x)| \leq C((M|f_j|^2)^{\frac{1}{2}}\varepsilon + \|f_j\|_{2,\lambda}\varepsilon^{\frac{2-\lambda}{2}})$$

and minimizing with respect to ε,

$$|Tf(x)| \leq C(M|f_j|^2)^{\frac{1}{p_\lambda}} \|f_j\|_{2,\lambda}^{1-\frac{2}{p_\lambda}}, \quad \text{a.e. } x \in \Omega,$$

where $\dfrac{1}{p_\lambda} = \dfrac{1}{2} - \dfrac{1}{\lambda}$.

Since M is a weak operator of type (1,1), for a.e. $x \in \Omega$ and $r > 0$, we get

$$|\{y \in \Omega \cap B(x,r) | |Tf(y)| > t\}| \leq C \frac{\|f_j\|_{2,\lambda}^{p_\lambda - 2}}{t^{p_\lambda}} \frac{|B(x,r)|}{r^\lambda} \|f_j\|_{2,\lambda}^2$$

and the result follows. □

4. A representation formula.

In this section we apply the result of the previous one to obtain a representation formula for the very weak solution to the Dirichlet problem

$$\begin{cases} Lu = X_j^* f_j, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{6}$$

assuming $f_j \in L^{2,\lambda}(\Omega, \mathrm{X})$ and $2 < \lambda < Q$.

Theorem 4.1. *Let $G^y(x)$ the Green function for L and Ω with pole at $y \in \Omega$. Then the function $u(x) = \int_\Omega X_j G^y(x) f_j(y) \mathrm{d}y$ is the unique solution of* (6).

Proof. We have to prove that the problem has a very weak solution. Thanks to Lemma 3.1, $u \in L^1(\Omega)$. Now let ϕ be a function in $H_0^1(\Omega, \mathrm{X}) \cap C_b^0(\Omega)$ such that $L^*\phi \in C_b^0(\Omega)$. We have

$$\int_\Omega \int_\Omega |L^*\phi(x)(X_j G^y(x))(y) f_j(y)| \mathrm{d}x \mathrm{d}y \leq \|L^*\phi\|_{L^\infty} \|u\|_{L^1} < \infty.$$

Hence, we have

$$\begin{aligned} \int_\Omega L^*\phi(x) \Big(\int_\Omega X_j G^y(x) f_j(y) \mathrm{d}y \Big) \mathrm{d}x &= \int_\Omega f_j(y) \Big(\int_\Omega L^*\phi X_j G^y(x) \mathrm{d}x \Big) \mathrm{d}y \\ &= \int_\Omega f_j(y) X_j \Big(\int_\Omega L^*\phi(x) G^y(x) \mathrm{d}x \Big) \mathrm{d}y \\ &= \int_\Omega f_j(y) X_j \phi \mathrm{d}y. \end{aligned}$$

In order to differentiate under the integral sign, let $\psi \in C_0^\infty(\Omega)$; since $\int_\Omega L^*\phi(x)G^y(x)\mathrm{d}x \in L^1(\Omega)$

$$
\begin{aligned}
< X_j \int_\Omega L^*\phi(x)G^y(x)\mathrm{d}x, \psi > &= < \int_\Omega L^*\phi(x)G^y(x)\mathrm{d}x, X_j^*\psi > \\
&= \int_\Omega \Big(\int_\Omega L^*\phi(x)G^y(x)\mathrm{d}x \Big) X_j^*\psi(y)\mathrm{d}y \\
&= \int_\Omega \int_\Omega L^*\phi(x)G^y(x)X_j^*\psi \mathrm{d}x\mathrm{d}y \\
&= \int_\Omega \int_\Omega L^*\phi(x)(X_jG^y(x))(y)\psi(y)\mathrm{d}x\mathrm{d}y \\
&= \int_\Omega \psi(y) \Big(\int_\Omega L^*\phi(x)(X_jG^y(x))(y)\psi(y)\mathrm{d}x \Big)\mathrm{d}y \\
&= < \int_\Omega L^*\phi(x)(X_jG^y(x))(y)\psi(y)\mathrm{d}x, \psi > .
\end{aligned}
$$

□

As a consequence of our previous result and Lemma 3.1 we get

Theorem 4.2. *A weak solution u to* (6) *belongs to the space $L_w^{p_\lambda,\lambda}(\Omega, \mathrm{X})$ and*

$$\|u\|_{L_w^{p_\lambda,\lambda}(\Omega,X)} \leq C\|f\|_{L^{2,\lambda}(\Omega,X)},$$

where C is independent on u and f.

5. Regularity.

Let $f_j \in L^2(\Omega)$ for $j = 1, \dots, m$. A function $u \in L^1(\Omega)$ is the very weak solution of the equation $Lu = X_j^* f_j$ if $\forall \varphi \in C_b^0(\Omega) \cap H_0^1(\Omega, \mathrm{X}) | L^*\varphi \in C_b^0(\Omega)$, results

$$< L^*\varphi, u > = \int_\Omega f_j X_j \varphi \mathrm{d}x.$$

We treat an operator with lower order terms. Namely, if $b_j \in L^{2,\lambda}(\Omega, \mathrm{X})$

we consider the Dirichlet problem

$$\begin{cases} Lu - X_j^*(b_j u) = X_j^* f_j \\ u = 0, \quad \text{on } \Omega \end{cases} \tag{7}$$

A function $u \in H_0^1(\Omega, \mathrm{X})$ is a weak solution of (7) if ,

$$\int_\Omega a_{ij} X_i u X_j \varphi \mathrm{d}x - \int_\Omega b_j u X_j \varphi \mathrm{d}x = \int_\Omega f_j X_j \varphi \mathrm{d}x, \quad \forall \varphi \in C_0^\infty(\Omega)$$

Thanks to Proposition 2.15 a weak solution of (7) is a very weak one.

Lemma 5.1. *Let* $0 < \mu < 2$, $f \in L^{2,\mu}(\Omega, \mathrm{X})$ *and* $u \in L^{2,\nu}(\Omega, \mathrm{X})$ *such that* $|\mathrm{X}u| \in L^{2,\nu+2}(\Omega, \mathrm{X})$, *with* $0 < \nu \le Q - 2$. *Then* $fu \in L^{2,\mu+\nu}(\Omega, \mathrm{X})$ *and*

$$\|fu\|_{2,\nu+\mu} \le C\|f\|_{2,\mu}(\|\mathrm{X}u\|_{2,\nu+2} + \|u\|_{2,\nu}).$$

Proof. Let $x \in \Omega$, $\varepsilon > 0$, $B_\varepsilon \equiv B_\varepsilon(x)$ and $u_\varepsilon \equiv \dfrac{1}{|B_\varepsilon|} \displaystyle\int_{B_\varepsilon \cap \Omega} u(y)dy$. We have

$$\begin{aligned} \int_{B_\varepsilon} f^2 u^2 \mathrm{d}y &\le \int_{B_\varepsilon} (|fu|)(|u - u_\varepsilon||f|)\mathrm{d}y + |u_\varepsilon| \int_{B_\varepsilon} f^2 |u| \mathrm{d}y \\ &\equiv I + II \end{aligned}$$

Let us separately estimate I and II.

$$\begin{aligned} I &\le \Big(\int_{B_\varepsilon} |f|^2 u^2 \mathrm{d}y\Big)^{1/2} \Big(\int_{B_\varepsilon} |f|^2 |u - u_\varepsilon|^2 \mathrm{d}y\Big)^{1/2} \\ &\le \Big(\int_{B_\varepsilon} |f|^2 u^2 \mathrm{d}y\Big)^{1/2} \Big[\eta_{f^2}^{1/2}(2\varepsilon)\Big(\int_{B_\varepsilon} |\mathrm{X}u|^2 \mathrm{d}y\Big)^{1/2}\Big] \\ &\le C\Big(\int_{B_\varepsilon} |f|^2 u^2 \mathrm{d}y\Big)^{1/2} \varepsilon^{-\frac{\mu+\nu}{2}} |B_\varepsilon|^{1/2} \|\mathrm{X}u\|_{2,\nu+2}. \end{aligned}$$

For the second integral, we have

$$\begin{aligned} II &= \Big(\fint_{B_\varepsilon} u \mathrm{d}y\Big)\Big(\int_{B_\varepsilon} |f|^2 |u| \mathrm{d}y\Big) \\ &\le \Big(\frac{1}{|B_\varepsilon|^{1/2}} \int_{B_\varepsilon} |u|^2 \mathrm{d}y\Big)^{1/2} \Big(\int_{B_\varepsilon} |f|^2 \mathrm{d}y\Big)^{1/2} \Big(\int_{B_\varepsilon} |f|^2 |u|^2 \mathrm{d}y\Big)^{1/2} \\ &\le \|u\|_{2,\nu} \|f\|_{2,\mu} \Big(\int_{B_\varepsilon} |f|^2 |u|^2 \mathrm{d}y\Big)^{1/2} \varepsilon^{-\frac{\nu+\mu}{2}} |B_\varepsilon|^{1/2}. \end{aligned}$$

Finally we get

$$\Big(\int_{B_\varepsilon} |f|^2|u|^2 \mathrm{d}y\Big)^{1/2} \leq C\|f\|_{2,\mu}\Big(\|u\|_{2,\nu} + \|\mathrm{X}u\|_{2,\nu+2}\Big)\Big(\varepsilon^{-(\nu+\mu)}|B_\varepsilon|\Big)^{1/2}$$

that means that $fu \in L^{2,\nu+\mu}$ and

$$\|fu\|_{2,\nu+\mu} \leq C\|f\|_{2,\mu}(\|\mathrm{X}u\|_{2,\nu+2} + \|u\|_{2,\nu}).$$

□

To prove regularity we need a Caccioppoli type inequality. Namely

Lemma 5.2. *Let $u \in H_0^1(\Omega, \mathrm{X})$ a weak solution of*

$$\begin{cases} Lu - X_j^*(b_j u) = X_j^* f_j \\ u = 0, \qquad \partial\Omega. \end{cases}$$

where $b_j \in L^{2,\mu}(\Omega, \mathrm{X})$, $f_j \in L^{2,\lambda}$, with $0 < \mu < 2 < \lambda < Q$.

Then for all $\varepsilon > 0$ there exists a positive constant C (independent on ε and the ball) such that

$$\int_{B_\varepsilon} |\mathrm{X}u|^2 \mathrm{d}x \leq \frac{C}{\varepsilon^2}\int_{B_{2\varepsilon}} u^2 \mathrm{d}x + \int_{B_{2\varepsilon}} f^2 \mathrm{d}x + \int_{B_{2\varepsilon}} |b|^2 u^2 \mathrm{d}x.$$

The proof uses standard techniques and so we omit it. Now we prove our regularity result.

Theorem 5.3. *Let $u \in H_0^1(\Omega, \mathrm{X})$ the solution of* (7) *with* $2 < \lambda < Q$ *and* $f_j \in L^{2,\lambda}(\Omega, \mathrm{X})$. *Then $u \in L_w^{p_\lambda,\lambda}(\Omega, \mathrm{X})$, where $\dfrac{1}{p_\lambda} = \dfrac{1}{2} - \dfrac{1}{\lambda}$.*

Proof. We have $u \in L^{2,Q-2}(\Omega, \mathrm{X})$, in fact

$$\int_{B\varepsilon} |u|^2 \mathrm{d}x \leq C\Big(\int_{B_\varepsilon} |u|^{2^*} \mathrm{d}x\Big)^{\frac{2}{2^*}}|B_\varepsilon|\varepsilon^{Q\left(\frac{-2}{2^*}\right)} \leq C\|u\|_{2^*}^2|B_\varepsilon|\varepsilon^{2-Q}$$

where 2^* is the Sobolev exponent $1/2^* = 1/2 - 1/Q$. We also have $\mathrm{X}u \in L^{2,Q}(\Omega, X)$ so, applying Lemma 5.1 we get $b_j u \in L^{2,Q-2+\mu}(\Omega, \mathrm{X})$. If $\mu_0 \equiv Q + \mu - 2 \leq \lambda$ we get the assert by Theorem 4.2.

Let us suppose that $\mu_0 > \lambda$ so $b_j u$, $f_j \in L^{2,\mu_0}(\Omega, \mathrm{X})$ and by Theorem 4.2 we have that $u \in L_w^{p_{\mu_0},\mu_0}(\Omega, \mathrm{X})$ where $\dfrac{1}{p_{\mu_0}} = \dfrac{1}{2} - \dfrac{1}{\mu_0}$.

Applying Proposition 2.9 we get $u \in L^{2,\overline{\mu_0}}(\Omega, \mathrm{X})$, where $\overline{\mu_0} = \dfrac{2\mu_0}{p_{\mu_0}} = \mu_0 - 2$.

Thanks to Lemma 5.2, observing that $f \in L^{2,\mu_0}(\Omega, X)$, we get, for every ball

$$\int_{B_\varepsilon} |Xu|^2 dx \leq C \frac{|B_\varepsilon|}{\varepsilon^{\mu_0}} (\|u\|_{2,\mu_0-2} + \|f\|_{2,\mu_0} + \||b|u\|_{2,\mu_0}).$$

Hence we have

$$u \in L^{2,\mu_0-2}(\Omega, X), \quad b_j \in L^{2,\mu}(\Omega, X) \quad \text{and} \quad |Xu| \in L^{2,\mu_0}(\Omega, X)$$

and it is possible to apply Lemma 5.1 that gives $b_j u \in L^{2,\mu_1}(\Omega, X)$ where $\mu_1 \equiv \mu_0 - 2 + \mu$.

Let us compare λ and μ_1; if $\lambda \geq \mu_1$ we apply Theorem 4.2 to get the assertion. If $\lambda < \mu_1$ then $b_j u$, $f \in L^{2,\mu_1}(\Omega, X)$ and hence $u \in L_w^{p_{\mu_1},\mu_1}(\Omega, X)$, where $\dfrac{1}{p_{\mu_1}} = \dfrac{1}{2} - \dfrac{1}{\mu_1}$. Applying Proposition 2.9 we get $u \in L^{2,\overline{\mu_1}}(\Omega, X)$, where $\overline{\mu_1} \equiv 2\dfrac{\mu_1}{p_{\mu_1}} = \mu_1 - 2$.

Applying Lemma 5.2 we get $|Xu| \in L^{2,\mu_1}(\Omega, X)$ and thanks to Lemma 5.1, we finally have $b_j u \in L^{2,\mu_1+\mu-2}(\Omega, X)$ and we compare $\mu_1 + \mu - 2$ with λ.

Since at every step we decrease the index μ_i of a quantity dependent only on μ, the iteration is finite and so assertion is proved. □

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