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TURÁN TYPE INEQUALITIES FOR p-POLYGAMMA FUNCTIONS

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The aim of this paper is to establish new Turán-type inequalities involving the $p-$polygamma functions.

1. Introduction

The inequalities of the type

$$f_n(x)f_{n+2}(x) - f_{n+1}^2(x) \leq 0$$

have many applications in pure mathematics as in other branches of science. They are named by Karlin and Szegő [5], Turán-type inequalities because the first of these type of inequalities was introduced by Turán [14]. More precisely, he used some results of Szegő [13] to prove the previous inequality for $x \in (-1,1)$, where f_n is the Legendre polynomial of degree n. This classical result has been extended in many directions, as ultraspherical polynomials, Laguerre and Hermite polynomials, or Bessel functions, and so forth.Many results of Turán-type have been established on the zeros of special functions.
Recently, W. T. Sulaiman [12] proved some Turán-type inequalities for some q-special functions as well as the polygamma functions, by using the following inequality:

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Lemma 1.1. *Let $a \in R_+ \cup \{\infty\}$ and let f and g be two nonnegative functions. Then*

$$\Big(\int_0^a g(x) f^{\frac{m+n}{2}} d_q x\Big)^2 \leq \Big(\int_0^a g(x) f^m d_q x\Big)\Big(\int_0^a g(x) f^n d_q x\Big) \tag{1}$$

Lets give some definitions for gamma and polygamma function.

The Euler gamma function $\Gamma(x)$ is defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt.$$

The digamma (or psi) function is defined for positive real numbers x as the logarithmic derivative of Euler's gamma function, that is $\psi(x) = \frac{d}{dx} \ln \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}$. The following integral and series representations are valid (see [2]):

$$\psi(x) = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-xt}}{1 - e^{-t}} dt = -\gamma - \frac{1}{x} + \sum_{n \geq 1} \frac{x}{n(n+x)}, \tag{2}$$

where $\gamma = 0.57721\cdots$ denotes Euler's constant.

Euler gave another equivalent definition for the $\Gamma(x)$ (see [9],[10])

$$\Gamma_p(x) = \frac{p! p^x}{x(x+1) \cdot \ldots \cdot (x+p)} = \frac{p^x}{x(1+\frac{x}{1}) \cdot \ldots \cdot (1+\frac{x}{p})}, \quad x > 0, \tag{3}$$

where p is positive integer, and

$$\Gamma(x) = \lim_{p \to \infty} \Gamma_p(x). \tag{4}$$

The p-analogue of the psi function, as the logarithmic derivative of the Γ_p function (see [9]), is

$$\psi_p(x) = \frac{d}{dx} \ln \Gamma_p(x) = \frac{\Gamma_p'(x)}{\Gamma_p(x)}. \tag{5}$$

The following representations are valid:

$$\Gamma_p(x) = \int_0^p \Big(1 - \frac{t}{p}\Big)^p t^{x-1} dt, \tag{6}$$

$$\psi_p(x) = \ln p - \int_0^\infty \frac{e^{-xt}(1 - e^{-(p+1)t})}{1 - e^{-t}} dt, \tag{7}$$

and

$$\psi_p^{(m)}(x) = (-1)^{m+1} \cdot \int_0^\infty \frac{t^m \cdot e^{-xt}}{1 - e^{-t}} (1 - e^{-(p+1)t}) dt. \tag{8}$$

The p-zeta function is defined as (see [10])

$$\zeta_p(s) = \frac{1}{\Gamma_p(s)} \int_0^p \frac{t^{s-1}}{\left(1+\frac{t}{p}\right)^p - 1} dt$$

2. Main Result

Theorem 2.1. *For $n = 1,2,3,\ldots$, let $\psi_{p,n} = \psi_p^{(n)}$ the n-th derivative of the function ψ_p. Then*

$$\psi_{p,\frac{m}{s}+\frac{n}{l}}\left(\frac{x}{s}+\frac{y}{t}\right) \leq \psi_{p,m}^{\frac{1}{s}}(x)\psi_{p,n}^{\frac{1}{l}}(y), \tag{9}$$

where $\frac{m+n}{2}$ is an integer, $s > 1, \frac{1}{s}+\frac{1}{l} = 1$.

Proof. Let m and n be two integers of the same parity. From (8), it follows that:

$$\begin{aligned}
&\psi_{p,\frac{m}{s}+\frac{n}{l}}\left(\frac{x}{s}+\frac{y}{l}\right)\\
&= (-1)^{\frac{m}{s}+\frac{n}{l}+1}\int_0^\infty \frac{t^{\frac{m}{s}+\frac{n}{l}}e^{-\left(\frac{x}{s}+\frac{y}{l}\right)t}}{1-e^{-t}}(1-e^{-(p+1)t})dt\\
&= (-1)^{\frac{m+1}{s}}(-1)^{\frac{n+1}{l}}\int_0^\infty \frac{t^{\frac{m}{s}}e^{-\left(\frac{x}{s}\right)t}}{\left(1-e^{-t}\right)^{\frac{1}{s}}}(1-e^{-(p+1)t})^{\frac{1}{s}}\frac{t^{\frac{n}{l}}e^{-\left(\frac{y}{l}\right)t}}{\left(1-e^{-t}\right)^{\frac{1}{l}}}(1-e^{-(p+1)t})^{\frac{1}{l}}dt\\
&\leq \left((-1)^{m+1}\int_0^\infty \frac{t^m e^{-xt}}{\left(1-e^{-t}\right)}(1-e^{-(p+1)t})dt\right)^{\frac{1}{s}}\\
&\quad\times\left((-1)^{n+1}\int_0^\infty \frac{t^n e^{-yt}}{\left(1-e^{-t}\right)}(1-e^{-(p+1)t})dt\right)^{\frac{1}{l}}\\
&= \psi_{p,m}^{\frac{1}{s}}(x)\psi_{p,n}^{\frac{1}{l}}(y)
\end{aligned}$$

□

Remark 2.2. Let p tend to ∞, then we have

$$\psi_{\frac{m}{s}+\frac{n}{l}}\left(\frac{x}{s}+\frac{y}{t}\right) \leq \psi_m^{\frac{1}{s}}(x)\psi_n^{\frac{1}{l}}(y), \tag{10}$$

On putting $y = x$ then we obtain

$$\psi_{\frac{m}{s}+\frac{n}{l}}(x) \leq \psi_m^{\frac{1}{s}}(x)\psi_n^{\frac{1}{l}}(y), \tag{11}$$

Another type via Minkowski's inequality is the following.

Theorem 2.3. *For $n = 1,2,3,\ldots$, let $\psi_{(p,q),n} = \psi_p^{(n)}$ the n-th derivative of the function ψ_p. Then*

$$\Big(\psi_{p,m}(x)+\psi_{p,n}(y)\Big)^{\frac{1}{p}} \le \psi_{p,m}^{\frac{1}{p}}(x)+\psi_{p,n}^{\frac{1}{p}}(y), \tag{12}$$

where $\frac{m+n}{2}$ is an integer, $p \ge 1$.

Proof. Since,

$$(a+b)^p \ge a^p+b^p, \quad a,b\ge 0, \quad p\ge 1,$$

$$\begin{aligned}
\Big(\psi_{p,m}(x)+\psi_{p,n}(y)\Big)^{\frac{1}{p}} &= \\
&= \Bigg[\int_0^\infty \Bigg[(-1)^{m+1}\frac{t^m e^{-xt}}{1-e^{-t}}\Big(1-e^{-(p+1)t}\Big)dt \\
&+(-1)^{n+1}\frac{t^n e^{-xt}}{1-e^{-t}}\Big(1-e^{-(p+1)t}\Big)dt\Bigg]^{\frac{1}{p}} \\
&= \Bigg[\int_0^\infty \Bigg[\Bigg[(-1)^{\frac{m+1}{p}}\frac{t^{\frac{m}{p}} e^{-\frac{xt}{p}}}{\big(1-e^{-t}\big)^{\frac{1}{p}}}\Big(1-e^{-(p+1)t}\Big)^{\frac{1}{p}}\Bigg]^p+ \\
&+\Bigg[(-1)^{\frac{n+1}{p}}\frac{t^{\frac{n}{p}} e^{-\frac{xt}{p}}}{\big(1-e^{-t}\big)^{\frac{1}{p}}}\Big(1-e^{-(p+1)t}\Big)^{\frac{1}{p}}\Bigg]^p\Bigg]dt\Bigg]^{\frac{1}{p}} \\
&\le \Bigg[\int_0^\infty \Bigg[(-1)^{\frac{m+1}{p}}\frac{t^{\frac{m}{p}} e^{-\frac{xt}{p}}}{\big(1-e^{-t}\big)^{\frac{1}{p}}}\Big(1-e^{-(p+1)t}\Big)^{\frac{1}{p}}+ \\
&+(-1)^{\frac{n+1}{p}}\frac{t^{\frac{n}{p}} e^{-\frac{xt}{p}}}{\big(1-e^{-t}\big)^{\frac{1}{p}}}\Big(1-e^{-(p+1)t}\Big)^{\frac{1}{p}}\Bigg]^p dt\Bigg]^{\frac{1}{p}}
\end{aligned}$$

$$\leq (-1)^{\frac{m+1}{p}}\left[\int_0^\infty \left[\frac{t^{\frac{m}{p}}e^{-\frac{xt}{p}}}{\left(1-e^{-t}\right)^{\frac{1}{p}}}\left(1-e^{-(p+1)t}\right)^{\frac{1}{p}}\right]^p dt\right]^{\frac{1}{p}}$$

$$+(-1)^{\frac{n+1}{p}}\left[\int_0^\infty \left[\frac{t^{\frac{n}{p}}e^{-\frac{xt}{p}}}{\left(1-e^{-t}\right)^{\frac{1}{p}}}\left(1-e^{-(p+1)t}\right)^{\frac{1}{p}}\right]^p dt\right]^{\frac{1}{p}}$$

$$=(-1)^{\frac{m+1}{p}}\left[\int_0^\infty \frac{t^m e^{-xt}}{1-e^{-t}}\left(1-e^{-(p+1)t}\right)dt\right]^{\frac{1}{p}}$$

$$+(-1)^{\frac{n+1}{p}}\left[\int_0^\infty \frac{t^n e^{-xt}}{1-e^{-t}}\left(1-e^{-(p+1)t}\right)dt\right]^{\frac{1}{p}}$$

$$=\psi_{p,m}^{\frac{1}{p}}(x)+\psi_{p,n}^{\frac{1}{p}}(y)$$

□

Remark 2.4. Let p tend to ∞, then we have

$$\Big(\psi_m(x)+\psi_n(y)\Big)^{\frac{1}{p}} \leq \psi_m^{\frac{1}{p}}(x)+\psi_n^{\frac{1}{p}}(y), \tag{13}$$

Theorem 2.5. *For every $x>0$ and integers $N\geq 1$, we have:*

1. *If n is odd, then* $\left(\exp\psi_p^{(n)}(x)\right)^2 \geq \exp\psi_p^{(n+1)}(x)\cdot\exp\psi_p^{(n-1)}(x)$
2. *If n is even, then* $\left(\exp\psi_p^{(n)}(x)\right)^2 \leq \exp\psi_p^{(n+1)}(x)\cdot\exp\psi_p^{(n-1)}(x)$

Proof. We use (8) to estimate the expression

$$\psi_p^{(n)}(x)-\frac{\psi_p^{(n+1)}(x)+\psi_p^{(n-1)}(x)}{2}=$$

$$=(-1)^{n+1}\Big(\int_0^\infty \frac{t^n e^{-xt}}{1-e^{-t}}(1-e^{-(p+1)t})dt$$

$$+\frac{1}{2}\int_0^\infty \frac{t^{n+1}e^{-xt}}{1-e^{-t}}(1-e^{-(p+1)t})dt$$

$$+\frac{1}{2}\int_0^\infty \frac{t^{n-1}e^{-xt}}{1-e^{-t}}(1-e^{-(p+1)t})dt\Big)$$

$$=(-1)^{n+1}\Big(\int_0^\infty \frac{t^{n-1}e^{-xt}}{1-e^{-t}}(t+1)^2(1-e^{-(p+1)t})dt\Big)$$

Now, the conclusion follows by exponentiating the inequality

$$\psi_p^{(n)}(x) \geq (\leq) \frac{\psi_p^{(n+1)}(x)+\psi_p^{(n-1)}(x)}{2}$$

as n is odd, respective even. □

Remark 2.6. Let p tend to ∞, then we obtain generalization of Theorem 3.3 from [11]

Theorem 2.7. *For $x, y > 1$, we have*

$$\zeta_p\Big(\frac{x-1}{s}+\frac{y+1}{t}\Big) \leq \frac{\Big(\Big(\Gamma_p(x)\Big)^{\frac{1}{s}}\Gamma_p(y+1)\Big)^{\frac{1}{t}}}{\Gamma_p(\frac{x-1}{s}+\frac{y+1}{t})}\zeta_p(x-1)\zeta_p(y+1) \tag{14}$$

Proof.

$$\zeta_p\Big(\frac{x-1}{s}+\frac{y+1}{t}\Big) = \frac{1}{\Gamma_p(\frac{x-1}{s}+\frac{y+1}{t})}\int_0^p \frac{t^{\frac{x-1}{s}+\frac{y+1}{t}-1}}{\Big(1+\frac{t}{p}\Big)^p-1}dt$$

$$= \frac{1}{\Gamma_p(\frac{x-1}{s}+\frac{y+1}{t})}\int_0^p \frac{t^{\frac{x-2}{s}}t^{\frac{y}{t}}}{\Big(\Big(1+\frac{t}{p}\Big)^p-1\Big)^{\frac{1}{p}}\Big(\Big(1+\frac{t}{p}\Big)^p-1\Big)^{\frac{1}{s}}}dt$$

$$\leq \frac{1}{\Gamma_p(\frac{x-1}{s}+\frac{y+1}{t})}\Big(\int_0^p \frac{t^{x-2}}{\Big(\Big(1+\frac{t}{p}\Big)^p-1\Big)}dt\Big)^{\frac{1}{s}}$$

$$\times\Big(\int_0^p \frac{t^y}{\Big(\Big(1+\frac{t}{p}\Big)^p-1\Big)^{\frac{1}{p}}}dt\Big)^{\frac{1}{t}}$$

$$= \frac{\Big(\Big(\Gamma_p(x)\Big)^{\frac{1}{s}}\Gamma_p(y+1)\Big)^{\frac{1}{t}}}{\Gamma_p(\frac{x-1}{s}+\frac{y+1}{t})}\zeta_p(x-1)\zeta_p(y+1)$$

□

Remark 2.8. Let p tend to ∞, then we have

$$\zeta(\frac{x-1}{s}+\frac{y+1}{t}) \leq \frac{\Big(\big(\Gamma(x)\big)^{\frac{1}{s}}\Gamma(y+1)\Big)^{\frac{1}{t}}}{\Gamma(\frac{x-1}{s}+\frac{y+1}{t})}\zeta(x-1)\zeta(y+1) \tag{15}$$

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