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ON SOME SYMMETRIC q-SPECIAL FUNCTIONS

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In this paper, we define the q-analogue of gamma and Bessel function, symmetric under interchange of q and q^{-1}, and present some of its main properties.

1. Introduction

The study of q-analysis appeared in the literature a long time ago. In particular, a q-analogue of some special functions like a q-exponential, q-Gamma, q-Beta and q-Bessel functions have been studied intensively for $0 < q < 1$.

Recently, G.Dattoli and A.Torre [2] introduced a q-Bessel functions of integer index which are symmetric under the interchange of q and q^{-1}. The authors use a generating function obtained owing a product of symmetric q-exponential functions [9, 10].

In the present paper, we introduce a symmetric q- Gamma and q-Beta functions and we extend the symmetric q-Bessel function of real index.

The paper is organized as follows: In section 2, we present some preliminaries and notations that will be useful in the sequel. In section 3, we introduce a q-analogue of the Gamma and Beta functions, symmetric under the interchange q and q^{-1}. A q-analogue of Bohr-Mollerup theorem is proved. In section 4, we define and study a symmetric q-Bessel functions of real order. Finally, in section 5, we introduce and study a q-analogue of the normalized Bessel function

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j_ν symmetric under the interchange $q \leftrightarrow q^{-1}$. We establish a q-Bessel operator and we provide a q-integral representation of Mehler and Sonine type and as applications, we present a q- transmutation operator.

2. Notations and Preliminaries

Throughout this paper, we will fix $q > 0, q \neq 1$. We recall some usual notions and notations used in the q-theory (see [4] and [2]).
For $a \in \mathbb{C}$, the q-shifted factorials are defined by

$$(a;q)_0 = 1, \qquad (a;q)_n = \prod_{i=0}^{n-1}(1-aq^i), \quad n \in N^*. \tag{1}$$

$$[x]_q = \frac{1-q^x}{1-q}, \quad x \in \mathbb{C}. \tag{2}$$

We also denote

$$\widetilde{[x]}_q = \frac{q^x - q^{-x}}{q-q^{-1}}, \quad x \in \mathbb{C}, \tag{3}$$

and

$$\widetilde{[n]}_q! = \prod_{k=1}^{n} \widetilde{[k]}_q, \quad n \in \mathbb{N}^*, \ \widetilde{[0]}_q! = 1. \tag{4}$$

One can see that

$$\widetilde{[x]}_q = \widetilde{[x]}_{\frac{1}{q}}$$
$$\widetilde{[x+y]}_q = q^y\widetilde{[x]}_q + q^{-x}\widetilde{[y]}_q = q^x\widetilde{[y]}_q + q^{-y}\widetilde{[x]}_q$$
$$\widetilde{[0]}_q = 0 \qquad \widetilde{[1]}_q = 1.$$

We have

$$\widetilde{[x]}_q = q^{-(x-1)}[x]_{q^2}. \tag{5}$$

The symmetric q-derivative $\widetilde{D}_q f$ of a function f is given by

$$(\widetilde{D}_q f)(x) = \frac{f(qx) - f(q^{-1}x)}{(q-q^{-1})x}, \text{ if } x \neq 0, \tag{6}$$

$(\widetilde{D}_q f)(0) = f'(0)$ provided $f'(0)$ exists.
The following properties hold

$$\widetilde{D}_q(f(x)+g(x)) = \widetilde{D}_q f(x) + \widetilde{D}_q g(x) \tag{7}$$

$$\begin{aligned}\widetilde{D}_q(f(x)g(x)) &= g(q^{-1}x)\widetilde{D}_q f(x) + f(qx)\widetilde{D}_q g(x)\\ &= g(qx)\widetilde{D}_q f(x) + f(q^{-1}x)\widetilde{D}_q g(x)\end{aligned} \tag{8}$$

$$\widetilde{D}_q z^n = \widetilde{[n]}_q z^{n-1}. \tag{9}$$

We have the following relation

$$\widetilde{D}_q f(x) = D_{q^2} f(q^{-1}x) \tag{10}$$

where

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}. \tag{11}$$

Finally, we consider the sets:
$\widetilde{\mathbb{R}}_q = \{\pm q^n : n \in \mathbb{Z}\} \cup \{0\}$, $\widetilde{\mathbb{R}}_{q,+} = \{q^n : n \in \mathbb{Z}\} \cup \{0\}$.

3. Symmetric q-Gamma and q-Beta functions

The q-Gamma function [4] is defined for positive real numbers x and $q \neq 1$ by

$$\Gamma_q(x) = \frac{(q,q)_\infty}{(q^x,q)_\infty}(1-q)^{1-x} \qquad for \qquad 0 < q < 1, \tag{12}$$

$$and \qquad \Gamma_q(x) = \frac{(q^{-1},q^{-1})_\infty}{(q^{-x},q^{-1})_\infty}(q-1)^{1-x} q^{\frac{x(x-1)}{2}} \qquad for \qquad q > 1. \tag{13}$$

Note that

$$\Gamma_{\frac{1}{q}}(x) = q^{-\frac{(x-1)(x-2)}{2}} \Gamma_q(x), \quad 0 < q < 1, \tag{14}$$

and when $q \to 1$ we obtain classical Euler's Gamma function.
The q-Gamma function satisfies the following equation

$$\Gamma_q(x+1) = [x]_q \Gamma_q(x), \qquad \Gamma_q(1) = 1. \tag{15}$$

A q-analogue of Legendre's duplication formula is given by (see [4])

$$\Gamma_q(2x)\Gamma_{q^2}(\frac{1}{2}) = (1+q)^{2x-1}\Gamma_{q^2}(x)\Gamma_{q^2}(x+\frac{1}{2}). \tag{16}$$

In [1], Askey proved a q-analogue of the Bohr-Mollerup theorem.

Theorem 3.1. *For $0 < q < 1$. The only one function $f \in C^2((0,\infty))$ satisfying the conditions:*
(i) $f(1) = 1$
(ii) $f(x+1) = [x]_q f(x)$
(iii) $\frac{d^2}{dx^2} Log f(x) \geq 0, \qquad x > 0$
is the q-Gamma function Γ_q.

In [11], Moak showed the following result.

Theorem 3.2. *For $q > 1$. The only one function $f \in C^2((0,\infty))$ satisfying the conditions:*
(i) $f(1) = 1$
(ii) $f(x+1) = [x]_q f(x)$
(iii) $\frac{d^2}{dx^2} Log f(x) \geq Log q, \qquad x > 0$
is the q-Gamma function Γ_q.

Now, we introduce the following definition

Definition 3.3. Let $q > 0, q \neq 1$. The symmetric q-Gamma function $\widetilde{\Gamma}_q$ is defined by

$$\widetilde{\Gamma}_q(x) = q^{-(x-1)(x-2)/2}\Gamma_{q^2}(x). \tag{17}$$

Proposition 3.4. *The q-Gamma function $\widetilde{\Gamma}_q$ has the following properties:*
(a) $\widetilde{\Gamma}_q(1) = 1$
(b) $\widetilde{\Gamma}_q(x+1) = \widetilde{[x]}_q \widetilde{\Gamma}_q(x)$
(c) $\widetilde{\Gamma}_q$ is symmetric under the interchange $q \leftrightarrow q^{-1}$.

Proof. (a) it is clear that $\widetilde{\Gamma}_q(1) = 1$.
(b) We have

$$\begin{aligned}
\widetilde{\Gamma}_q(x+1) &= q^{-x(x-1)/2}\Gamma_{q^2}(x+1) \\
&= q^{-x(x-1)/2}\frac{1-q^{2x}}{1-q^2}\Gamma_{q^2}(x) \\
&= \widetilde{[x]}_q \widetilde{\Gamma}_q(x).
\end{aligned}$$

(c) We assume that $0 < q < 1$, we have

$$\widetilde{\Gamma}_{q^{-1}}(x) = q^{(x-1)(x-2)/2}\Gamma_{q^{-2}}(x). \tag{18}$$

Since $0 < q < 1$, so $q^{-1} > 1$ and from the relation (14) we have

$$\Gamma_{q^{-2}}(x) = q^{-(x-1)(x-2)}\Gamma_{q^2}(x). \tag{19}$$

Therefore

$$\widetilde{\Gamma}_{q^{-1}}(x) = q^{(x-1)(x-2)/2}q^{-(x-1)(x-2)}\Gamma_{q^2}(x) = q^{-(x-1)(x-2)/2}\Gamma_{q^2}(x) = \widetilde{\Gamma_q}(x). \tag{20}$$

□

Proposition 3.5. *A q-analogue of Legendre's duplication formula is given by*

$$\widetilde{\Gamma}_q(2x)\widetilde{\Gamma}_{q^2}(\frac{1}{2}) = (\widetilde{[2]}_q)^{2x-1}\widetilde{\Gamma}_{q^2}(x)\widetilde{\Gamma}_{q^2}(x+\frac{1}{2}). \tag{21}$$

Proof. From the relation (16), we have

$$\Gamma_{q^2}(2x)\Gamma_{q^4}(\frac{1}{2}) = (1+q^2)^{2x-1}\Gamma_{q^4}(x)\Gamma_{q^4}(x+\frac{1}{2}).$$

So, by using the relation (17) we obtain

$$q^{\frac{(2x-1)(2x-2)}{2}}\widetilde{\Gamma}_q(2x)q^{\frac{3}{4}}\widetilde{\Gamma}_{q^2}(\frac{1}{2}) \\ = (1+q^2)^{2x-1}q^{(x-1)(x-2)}\widetilde{\Gamma}_{q^2}(x)q^{(x-\frac{1}{2})(x-\frac{3}{2})}\widetilde{\Gamma}_{q^2}(x+\frac{1}{2}).$$

$$\begin{aligned}\widetilde{\Gamma}_q(2x)\widetilde{\Gamma}_{q^2}(\frac{1}{2}) &= (1+q^2)^{2x-1}q^{-2x+1}\widetilde{\Gamma}_{q^2}(x)\widetilde{\Gamma}_{q^2}(x+\frac{1}{2}) \\ &= (q+q^{-1})^{2x-1}\widetilde{\Gamma}_{q^2}(x)\widetilde{\Gamma}_{q^2}(x+\frac{1}{2}) \\ &= (\widetilde{[2]}_q)^{2x-1}\widetilde{\Gamma}_{q^2}(x)\widetilde{\Gamma}_{q^2}(x+\frac{1}{2}).\end{aligned}$$

The result is a consequence of the fact that $\widetilde{\Gamma}_q$ and $\widetilde{\Gamma}_{q^2}$ are symmetric under the interchange $q \leftrightarrow q^{-1}$. □

The following result is a q-analogue of the Bohr-Mollerup theorem for $q \neq 1$.

Theorem 3.6. *Let $q > 0, q \neq 1$. The only one function $f \in C^2((0,\infty))$ satisfying the conditions:*
(i) $f(1) = 1$
(ii) $f(x+1) = \widetilde{[x]}_q f(x)$
(iii) $\frac{d^2}{dx^2} Log f(x) \geq |Log q|, \qquad x > 0,$
is the symmetric q-Gamma function $\widetilde{\Gamma}_q$.

Proof. Firstly, it is easy to see that the function $\widetilde{\Gamma}_q$ satisfies the conditions (i), (ii) and we have

$$Log\widetilde{\Gamma}_q(x) = -\frac{(x-1)(x-2)}{2}Log(q) + Log\Gamma_{q^2}(x), \tag{22}$$

so

$$\frac{d^2}{dx^2}Log\widetilde{\Gamma}_q(x) = -Log q + \frac{d^2}{dx^2}Log(\Gamma_{q^2}(x)).$$

Now, by using theorem 3.1 and theorem 3.2 we deduce that in both cases the condition (iii) follows.

On the other hand, let f be a function satisfying the three conditions $(i), (ii)$ and

(iii) and let $g(x) = q^{\frac{(x-1)(x-2)}{2}} f(x)$.
We have

$$g(1) = f(1) = 1,$$

$$g(x+1) = q^{\frac{x(x-1)}{2}} f(x+1) = q^{x-1}\widetilde{[x]}_q q^{\frac{(x-1)(x-2)}{2}} f(x) = [x]_{q^2} g(x),$$

and

$$\frac{d^2}{dx^2} Log(g(x)) = Logq + \frac{d^2}{dx^2} Log(f(x)) \geq Logq + |Logq|,$$

which implies, from theorem 3.1 and theorem 3.2 that $g = \Gamma_{q^2}$.
Thus

$$f(x) = q^{-\frac{(x-1)(x-2)}{2}} \Gamma_{q^2}(x) = \widetilde{\Gamma}_q(x),$$

and the theorem is proved. □

Definition 3.7. Let $q > 0, q \neq 1$. The symmetric q-Beta function is defined by the relation

$$\widetilde{B}_q(x,y) = \frac{\widetilde{\Gamma}_q(x)\widetilde{\Gamma}_q(y)}{\widetilde{\Gamma}_q(x+y)}, \qquad x > 0 \quad y > 0. \tag{23}$$

It is clear that the q-Beta function $\widetilde{B}_q$ is symmetric under the interchange $q \leftrightarrow q^{-1}$.

Proposition 3.8. *The function $\widetilde{B}_q$ satisfies the following formulae for $x, y > 0$:*
1) $\widetilde{B}_q(x,y) = \widetilde{B}_q(y,x)$.
2)$\widetilde{B}_q(x+1,y) = \frac{\widetilde{[x]}_q}{\widetilde{[y]}_q} \widetilde{B}_q(x,y+1)$.
3) $\widetilde{B}_q(x,y+1) = \frac{\widetilde{[y]}_q}{\widetilde{[x+y]}_q} \widetilde{B}_q(x,y)$.

Proof. This can be proved straightforwardly from (23) and proposition 3.4. □

4. Symmetric q-Bessel function

In literature there are many definitions of the q-analogue of the Bessel function(see [3], [8])

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{x}{2})^{\nu+2n}}{n!\Gamma(\nu+n+1)}. \tag{24}$$

In [2], G. Dattoli and A. Torre gave a q-analogue of Bessel functions for an integer order, symmetric under the interchange of q and q^{-1} defined by:

$$J_n(x,q) = \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{x}{2})^{n+2k}}{\widetilde{[n+k]_q!}\widetilde{[k]_q!}}. \tag{25}$$

In this section, and by using the symmetric q-Gamma function, we give a generalization of the q-symmetric Bessel functions.

Definition 4.1. The symmetric q-Bessel function is defined by:

$$\widetilde{J}_\nu(x,q) = \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{x}{\widetilde{[2]}_{\sqrt{q}}})^{\nu+2n}}{\widetilde{\Gamma}_q(\nu+n+1)\widetilde{[n]}_q!}. \tag{26}$$

Proposition 4.2. *$\widetilde{J}_\nu(x,q)$ is symmetric under the interchange $q \longleftrightarrow q^{-1}$.*

We deduce the relation between (25) and (26) given by:

$$\widetilde{J}_n(x,q) = J_n(\frac{2x}{\widetilde{[2]}_{\sqrt{q}}},q). \tag{27}$$

Proposition 4.3. *The symmetric q-Bessel function satisfies the following recurrence relations*

$$\frac{\widetilde{[2\nu]}_{\sqrt{q}}}{x}\widetilde{J}_\nu(x,q) = q^{-\frac{(\nu-1)}{2}}\widetilde{J}_{\nu-1}(x\sqrt{q},q) + q^{\frac{\nu+1}{2}}\widetilde{J}_{\nu+1}(x\sqrt{q},q). \tag{28}$$

$$\frac{\widetilde{[2\nu]}_{\sqrt{q}}}{x}\widetilde{J}_\nu(x,q) = q^{\frac{(\nu-1)}{2}}\widetilde{J}_{\nu-1}(\frac{x}{\sqrt{q}},q) + q^{-\frac{(\nu+1)}{2}}\widetilde{J}_{\nu+1}(\frac{x}{\sqrt{q}},q). \tag{29}$$

Proof. From the definition (26), we have

$$\widetilde{[\nu]}_q\widetilde{J}_\nu(x,q) = \sum_{n=0}^{\infty} \frac{(-1)^n(q^n\widetilde{[\nu+n]}_q - q^{\nu+n}\widetilde{[n]}_q)(\frac{x}{\widetilde{[2]}_{\sqrt{q}}})^{\nu+2n}}{\widetilde{\Gamma}_q(\nu+n+1)\widetilde{[n]}_q!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n q^n (\frac{x}{\widetilde{[2]}_{\sqrt{q}}})^{\nu+2n}}{\widetilde{\Gamma}_q(\nu+n)\widetilde{[n]}_q!} - \sum_{n=1}^{\infty} \frac{(-1)^n q^{\nu+n} (\frac{x}{\widetilde{[2]}_{\sqrt{q}}})^{\nu+2n}}{\widetilde{\Gamma}_q(\nu+n+1)\widetilde{[n-1]}_q!}$$

$$= \frac{x}{\widetilde{[2]}_{\sqrt{q}}} \sum_{n=0}^{\infty} \frac{(-1)^n q^n (\frac{x}{\widetilde{[2]}_{\sqrt{q}}})^{\nu-1+2n}}{\widetilde{\Gamma}_q(\nu+n)\widetilde{[n]}_q!} + \sum_{n=0}^{\infty} \frac{(-1)^n q^{\nu+n+1} (\frac{x}{\widetilde{[2]}_{\sqrt{q}}})^{\nu+2n+2}}{\widetilde{\Gamma}_q(\nu+n+2)\widetilde{[n]}_q!} =$$

$$\frac{x}{\widetilde{[2]}_{\sqrt{q}}} q^{-\frac{(\nu-1)}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{\sqrt{q}x}{\widetilde{[2]}_{\sqrt{q}}})^{\nu-1+2n}}{\widetilde{\Gamma}_q(\nu+n)\widetilde{[n]}_q!} + \frac{x}{\widetilde{[2]}_{\sqrt{q}}} q^{\frac{(\nu+1)}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{\sqrt{q}x}{\widetilde{[2]}_{\sqrt{q}}})^{\nu+1+2n}}{\widetilde{\Gamma}_q(\nu+n+2)\widetilde{[n]}_q!}.$$

Using the relation $\widetilde{[2]}_{\sqrt{q}}\widetilde{[\nu]}_q = \widetilde{[2\nu]}_{\sqrt{q}}$, we obtain

$$\frac{\widetilde{[2\nu]}_{\sqrt{q}}}{x}\widetilde{J}_\nu(x,q) = q^{-\frac{(\nu-1)}{2}}\widetilde{J}_{\nu-1}(\sqrt{q}x,q) + q^{\frac{(\nu+1)}{2}}\widetilde{J}_{\nu+1}(\sqrt{q}x,q).$$

The relation (29) can be obtained from (28) by changing q by q^{-1}. □

Proposition 4.4. *The symmetric q-Bessel function satisfies the following relation*

$$q^{-\frac{(\nu-1)}{2}}\widetilde{J}_{\nu-1}(x\sqrt{q},q) = \frac{x}{\widetilde{[2]}_{\sqrt{q}}\, q}(1-q^2)\widetilde{J}_\nu(x,q) + q^{\frac{\nu-1}{2}}\widetilde{J}_{\nu-1}(\frac{x}{\sqrt{q}},q). \tag{30}$$

Proof. From the definition (26), we obtain

$$\begin{aligned}
& q^{-\frac{(\nu-1)}{2}}\widetilde{J}_{\nu-1}(x\sqrt{q},q) - q^{\frac{\nu-1}{2}}\widetilde{J}_{\nu-1}(\frac{x}{\sqrt{q}},q) \\
= \;& q^{-\frac{(\nu-1)}{2}}\sum_{n=0}^{\infty}(-1)^n\frac{(\frac{x\sqrt{q}}{\widetilde{[2]}_{\sqrt{q}}})^{\nu-1+2n}}{\widetilde{\Gamma}_q(\nu+n)\widetilde{[n]}_q!} - q^{\frac{(\nu-1)}{2}}\sum_{n=0}^{\infty}(-1)^n\frac{(\frac{x}{\widetilde{[2]}_{\sqrt{q}}\sqrt{q}})^{\nu-1+2n}}{\widetilde{\Gamma}_q(\nu+n)\widetilde{[n]}_q!} \\
= \;& \sum_{n=0}^{\infty}(-1)^n q^n\frac{(\frac{x}{\widetilde{[2]}_{\sqrt{q}}})^{\nu-1+2n}}{\widetilde{\Gamma}_q(\nu+n)\widetilde{[n]}_q!} - \sum_{n=0}^{\infty}(-1)^n q^{-n}\frac{(\frac{x}{\widetilde{[2]}_{\sqrt{q}}})^{\nu-1+2n}}{\widetilde{\Gamma}_q(\nu+n)\widetilde{[n]}_q!} \\
= \;& (q-q^{-1})\sum_{n=0}^{\infty}(-1)^n\frac{\widetilde{[n]}_q(\frac{x}{\widetilde{[2]}_{\sqrt{q}}})^{\nu-1+2n}}{\widetilde{\Gamma}_q(\nu+n)\widetilde{[n]}_q!} \\
= \;& (q-q^{-1})\sum_{n=0}^{\infty}(-1)^{n+1}\frac{(\frac{x}{\widetilde{[2]}_{\sqrt{q}}})^{\nu+1+2n}}{\widetilde{\Gamma}_q(\nu+n+1)\widetilde{[n]}_q!} \\
= \;& \frac{x}{\widetilde{[2]}_{\sqrt{q}}q}(1-q^2)\widetilde{J}_\nu(x,q).
\end{aligned}$$

The proof is complete. □

4.1. Derivative and differential equation of the symmetric q-Bessel function

Proposition 4.5. *For* $\nu > 0$*, the following relations holds*

$$\widetilde{D}_q\left[x^\nu\widetilde{J}_\nu(x,q^2)\right] = x^\nu\widetilde{J}_{\nu-1}(x,q^2). \tag{31}$$

$$\widetilde{D}_q\left[x^{-\nu}\widetilde{J}_\nu(x,q^2)\right] = -x^{-\nu}\widetilde{J}_{\nu+1}(x,q^2). \tag{32}$$

Proof. We have

$$\begin{aligned}
x^{\nu}\widetilde{J}_{\nu}(x,q^2) &= \sum_{n=0}^{\infty}\frac{(-1)^n x^{2\nu+2n}(\frac{1}{\widetilde{[2]}_q})^{\nu+2n}}{\widetilde{\Gamma}_{q^2}(\nu+n+1)\widetilde{[n]}_{q^2}!}\\
\widetilde{D}_q[x^{\nu}\widetilde{J}_{\nu}(x,q^2)] &= \sum_{n=0}^{\infty}\frac{(-1)^n(\frac{1}{\widetilde{[2]}_q})^{\nu+2n}\widetilde{[2\nu+2n]}_q x^{2\nu+2n-1}}{\widetilde{\Gamma}_{q^2}(\nu+n+1)\widetilde{[n]}_{q^2}!}\\
&= \sum_{n=0}^{\infty}\frac{(-1)^n(\frac{1}{\widetilde{[2]}_q})^{\nu+2n}x^{2\nu+2n-1}}{\widetilde{\Gamma}_{q^2}(\nu+n)\widetilde{[n]}_{q^2}!}\;\frac{\widetilde{[2(\nu+n)]}_q}{\widetilde{[\nu+n]}_{q^2}}\\
&= \sum_{n=0}^{\infty}\frac{(-1)^n(\frac{1}{\widetilde{[2]}_q})^{\nu+2n}x^{2\nu+2n-1}}{\widetilde{\Gamma}_{q^2}(\nu+n)\widetilde{[n]}_{q^2}!}\;(q+q^{-1})\\
&= x^{\nu}\,\widetilde{J}_{\nu-1}(x,q^2).
\end{aligned}$$

It's the same for the relation (32). □

By induction, we obtain

Proposition 4.6. *For all $n\in\mathbb{N}$, we have*

$$(\frac{1}{x}\widetilde{D}_q)^n(x^{\nu}\widetilde{J}_{\nu}(x,q^2)) = x^{\nu-n}\widetilde{J}_{\nu-n}(x,q^2), \qquad \nu > n-1, \tag{33}$$

and

$$(-\frac{1}{x}\widetilde{D}_q)^n(x^{-\nu}\widetilde{J}_{\nu}(x,q^2)) = x^{-\nu-n}\widetilde{J}_{\nu+n}(x,q^2), \qquad \nu > 0. \tag{34}$$

Proposition 4.7. *The function $\widetilde{J}_{\nu}(x,q^2)$ is a solution of the equation*

$$q^{-1}\widetilde{D}_q^2 f(x) + \frac{1}{x}\widetilde{D}_q f(qx) + (1 - \frac{\widetilde{[\nu]}_q^2}{x^2})f(x) = 0. \tag{35}$$

Proof. By using the relations (31) and (32), we obtain

$$\frac{1}{x^{\nu+1}}\widetilde{D}_q(x^{2\nu+1}(\widetilde{D}_q(x^{-\nu}\widetilde{J}_{\nu}(x,q^2)))) = -\widetilde{J}_{\nu}(x,q^2). \tag{36}$$

On the other hand, from the relation (7), we have

$$\frac{1}{x^{\nu+1}}\widetilde{D}_q(x^{2\nu+1}\widetilde{D}_q(x^{-\nu}f(x))) = q^{-1}\widetilde{D}_q^2 f(x) + \frac{1}{x}\widetilde{D}_q f(qx) - \frac{\widetilde{[\nu]}_q^2}{x^2}f(x),$$

and the result follows. □

Proposition 4.8. *The function $\widetilde{J}_\nu(x,q)$ satisfies the following relations:*

$$\widetilde{[2]}_{\sqrt{q}}\widetilde{D}_q\widetilde{J}_\nu(x,q) = q^{-\frac{(\nu-1)}{2}}\widetilde{J}_{\nu-1}(\sqrt{q}x,q) - q^{-\frac{(\nu+1)}{2}}\widetilde{J}_{\nu+1}(\frac{x}{\sqrt{q}},q). \tag{37}$$

$$\widetilde{[2]}_{\sqrt{q}}\widetilde{D}_q\widetilde{J}_\nu(x,q) = q^{\frac{(\nu-1)}{2}}\widetilde{J}_{\nu-1}(\frac{1}{\sqrt{q}}x,q) - q^{\frac{(\nu+1)}{2}}\widetilde{J}_{\nu+1}(x\sqrt{q},q). \tag{38}$$

Proof. We have

$$\widetilde{D}_q\widetilde{J}_\nu(x,q) = \frac{1}{\widetilde{[2]}_{\sqrt{q}}}\sum_{n=0}^{\infty}\frac{(-1)^n\widetilde{[\nu+2n]}_q(\frac{x}{\widetilde{[2]}_{\sqrt{q}}})^{\nu+2n-1}}{\widetilde{\Gamma}_q(\nu+n+1)\widetilde{[n]}_q!}. \tag{39}$$

The relation $\widetilde{[\nu+2n]}_q = q^n\widetilde{[\nu+n]}_q + q^{-\nu-n}\widetilde{[n]}_q$, gives

$$\begin{aligned}
\widetilde{[2]}_{\sqrt{q}}\widetilde{D}_q\widetilde{J}_\nu(x,q) &= \sum_{n=0}^{\infty}\frac{(-1)^nq^n(\frac{x}{\widetilde{[2]}_{\sqrt{q}}})^{\nu-1+2n}}{\widetilde{\Gamma}_q(\nu-1+n+1)\widetilde{[n]}_q!} + \sum_{n=1}^{\infty}\frac{(-1)^nq^{-\nu-n}(\frac{x}{\widetilde{[2]}_{\sqrt{q}}})^{\nu-1+2n}}{\widetilde{\Gamma}_q(\nu+n+1)\widetilde{[n-1]}_q!}\\
&= q^{-\frac{\nu-1}{2}}\sum_{n=0}^{\infty}\frac{(-1)^n(\frac{\sqrt{q}x}{\widetilde{[2]}_{\sqrt{q}}})^{\nu-1+2n}}{\widetilde{\Gamma}_q(\nu-1+n+1)\widetilde{[n]}_q!} + \sum_{n=0}^{\infty}\frac{(-1)^{n+1}q^{-\nu-(n+1)}(\frac{x}{\widetilde{[2]}_{\sqrt{q}}})^{\nu+2n+1}}{\widetilde{\Gamma}_q(\nu+1+n+1)\widetilde{[n]}_q!}\\
&= q^{-\frac{\nu-1}{2}}\sum_{n=0}^{\infty}\frac{(-1)^n(\frac{\sqrt{q}x}{\widetilde{[2]}_{\sqrt{q}}})^{\nu-1+2n}}{\widetilde{\Gamma}_q(\nu-1+n+1)\widetilde{[n]}_q!} - q^{-\frac{\nu+1}{2}}\sum_{n=0}^{\infty}\frac{(-1)^n(\frac{x}{\widetilde{[2]}_{\sqrt{q}}\sqrt{q}})^{\nu+2n+1}}{\widetilde{\Gamma}_q(\nu+1+n+1)\widetilde{[n]}_q!}.
\end{aligned}$$

Thus

$$\widetilde{[2]}_{\sqrt{q}}\widetilde{D}_q\widetilde{J}_\nu(x,q) = q^{-\frac{\nu-1}{2}}\widetilde{J}_{\nu-1}(\sqrt{q}x,q) - q^{-\frac{\nu+1}{2}}\widetilde{J}_{\nu+1}(\frac{x}{\sqrt{q}},q). \tag{40}$$

The relation (38) can be obtained from (37) by changing q by q^{-1}. □

5. The symmetric q-$\widetilde{j}_\nu$ Bessel function

We define the symmetric q-$\widetilde{j}_\nu$ Bessel function by

$$\widetilde{j}_\nu(x;q^2) = \sum_{n=0}^{\infty}\frac{(-1)^n\widetilde{\Gamma}_{q^2}(\nu+1)(\frac{x}{\widetilde{[2]}_q})^{2n}}{\widetilde{\Gamma}_{q^2}(n+\nu+1)\widetilde{\Gamma}_{q^2}(n+1)}. \tag{41}$$

We define a symmetric q-trigonometric functions by

$$\widetilde{\cos}(x,q^2) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{\widetilde{[2n]}_q!} \tag{42}$$

$$\widetilde{\sin}(x,q^2) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{\widetilde{[2n+1]}_q!}. \tag{43}$$

One can see, after simple computation that

$$\widetilde{j}_{-\frac{1}{2}}(x,q^2) = \widetilde{\cos}(x,q^2),$$

$$\widetilde{j}_{\frac{1}{2}}(x,q^2) = \frac{\widetilde{\sin}(x,q^2)}{x}.$$

Proposition 5.1. *The symmetric q-$\widetilde{j}_\nu$ Bessel function satisfies the following recurrence relation:*

$$(\frac{1}{x}\widetilde{D}_q)\widetilde{j}_\nu(x,q^2) = \frac{-1}{\widetilde{[2(\nu+1)]}_q}\widetilde{j}_{\nu+1}(x,q^2). \tag{44}$$

Proof. We have

$$\begin{aligned}
\widetilde{D}_q\widetilde{j}_\nu(x,q^2) &= \widetilde{\Gamma}_{q^2}(\nu+1)\sum_{n=1}^{\infty}\frac{(-1)^n}{\widetilde{\Gamma}_{q^2}(n+1)\widetilde{\Gamma}_{q^2}(n+\nu+1)}\,(\frac{1}{\widetilde{[2]}_q})^{2n}\,\widetilde{[2n]}_q\,x^{2n-1}\\
&= \widetilde{\Gamma}_{q^2}(\nu+1)\sum_{n=1}^{\infty}\frac{(-1)^n}{\widetilde{\Gamma}_{q^2}(n)\widetilde{\Gamma}_{q^2}(n+\nu+1)}\,(\frac{x}{\widetilde{[2]}_q})^{2n-1}\\
&= -x\,\frac{1}{\widetilde{[2(\nu+1)]}_q}\,\widetilde{j}_{\nu+1}(x,q^2).
\end{aligned}$$

□

By induction, we deduce

Proposition 5.2. *For all integers n, the symmetric $q-\widetilde{j}_\nu$ Bessel function verifies*

$$(\frac{1}{x}\widetilde{D}_q)^n\widetilde{j}_\nu(x,q^2) = (\frac{-1}{\widetilde{[2]}_q})^n\frac{\widetilde{\Gamma}_{q^2}(\nu+1)}{\widetilde{\Gamma}_{q^2}(n+\nu+1)}\,\widetilde{j}_{\nu+n}(x,q^2). \tag{45}$$

Proposition 5.3. *The symmetric $q-\widetilde{j}_\nu$ Bessel function verifies the following relation*

$$(\frac{1}{x}\widetilde{D}_q)(x^{2\nu}\widetilde{j}_\nu(x,q^2)) = \widetilde{[2\nu]}_q\,x^{2(\nu-1)}\widetilde{j}_{\nu-1}(x,q^2). \tag{46}$$

Proof. We have

$$\begin{aligned}
&(\frac{1}{x}\widetilde{D}_q)(x^{2\nu}\widetilde{j}_\nu(x,q^2))\\
&\quad = x^{2(\nu-1)}\sum_{n=0}^{\infty}\frac{(-1)^n\widetilde{\Gamma}_{q^2}(\nu+1)}{\widetilde{\Gamma}_{q^2}(n+1)\widetilde{\Gamma}_{q^2}(n+\nu+1)}\,(\frac{x}{\widetilde{[2]}_q})^{2n}\widetilde{[2(\nu+n)]}_q\\
&\quad = \widetilde{[2]}_q\widetilde{[\nu]}_{q^2}\sum_{n=0}^{\infty}\frac{(-1)^n\widetilde{\Gamma}_{q^2}(\nu+1)}{\widetilde{\Gamma}_{q^2}(n+1)\widetilde{\Gamma}_{q^2}(n+\nu)}\,(\frac{x}{\widetilde{[2]}_q})^{2n}\\
&\quad = \widetilde{[2\nu]}_q\,x^{2(\nu-1)}\widetilde{j}_{\nu-1}(x,q^2).
\end{aligned}$$

□

Proposition 5.4.

$$(\frac{1}{x}\widetilde{D}_q)^n(x^{2\nu}\widetilde{j}_\nu(x,q^2)) = \widetilde{[2]}_q^{\,n}\,\frac{\widetilde{\Gamma}_{q^2}(\nu+1)}{\widetilde{\Gamma}_{q^2}(\nu-n+1)}\,x^{2(\nu-n)}\widetilde{j}_{\nu-n}(x,q^2). \tag{47}$$

We introduce the q-Bessel operator

$$\widetilde{\triangle}_{\nu,q}f(x) = \frac{1}{x^{2\nu+1}}\,\widetilde{D}_q[x^{2\nu+1}\,\widetilde{D}_qf](x) = q^{2\nu+1}\widetilde{D}_q^2f(x) + \frac{\widetilde{[2\nu+1]}_q}{x}\widetilde{D}_qf(q^{-1}x). \tag{48}$$

Theorem 5.5. *For $\lambda\in\mathbb{C}$, the function $x\mapsto\widetilde{j}_\nu(\lambda x,q^2)$ is the unique solution of the problem*

$$\widetilde{\triangle}_{\nu,q}f(x) = -\lambda^2 f(x) \tag{49}$$

$$f(0)=1,\quad f'(0)=0. \tag{50}$$

The proof is straightforward.
In [3], the $q-j_\nu$ Bessel function is defined for $0<q<1$ by

$$j_\nu(x,q^2) = \Gamma_{q^2}(\nu+1)\sum_{n=0}^{\infty}(-1)^n\frac{q^{n(n-1)}}{\Gamma_{q^2}(\nu+n+1)\Gamma_{q^2}(n+1)}(\frac{x}{1+q})^{2n} \tag{51}$$

Then, thanks to the following relation

$$\widetilde{j}_\nu(x,q) = j_\nu(q^{\frac{\nu+1}{2}}x,q^2), \tag{52}$$

we obtain

Theorem 5.6. *For $\nu > -\frac{1}{2}$ and $0 < q < 1$, the symmetric $q - \widetilde{j}_\nu$ Bessel function has the following q-integral representation of Mehler type*

$$\widetilde{j}_\nu(x,q) = \widetilde{C}(\nu,q)\int_0^1 W_\nu(t,q)\widetilde{\cos}(q^{\frac{2\nu+1}{4}}xt,q)d_qt \tag{53}$$

where W_ν is the q-binominal function defined by

$$W_\nu(t,q) = \frac{(t^2q^2,q^2)_\infty}{(t^2q^{2\nu+1},q^2)_\infty}. \tag{54}$$

and

$$\widetilde{C}(\nu,q) = \frac{1+q}{q^{\frac{1}{2}(\frac{3}{2}-\nu)}\widetilde{B}_q(\nu+\frac{1}{2},\frac{1}{2})}. \tag{55}$$

Proof. Using the q-integral representation of Mehler type of $q - j_\nu$ (see [3])

$$j_\nu(x,q^2) = (1+q)C(\nu,q^2)\int_0^1 W_\nu(t,q^2)j_{-\frac{1}{2}}(xt,q^2)d_qt, \tag{56}$$

where $(1+q)C(\nu,q^2) = \frac{(1+q)\Gamma_{q^2}(\nu+1)}{\Gamma_{q^2}(\frac{1}{2})\Gamma_{q^2}(\nu+\frac{1}{2})} = \widetilde{C}(\nu,q)$, and from the relation (52) we obtain

$$\begin{aligned}
\widetilde{j}_\nu(x,q) &= j_\nu(q^{\frac{\nu+1}{2}}x,q^2) = (1+q)C(\nu,q^2)\int_0^1 W_\nu(t,q^2)j_{-\frac{1}{2}}(q^{\frac{\nu+1}{2}}xt,q^2)d_qt \\
&= (1+q)C(\nu,q^2)\int_0^1 W_\nu(t,q^2)j_{-\frac{1}{2}}(q^{\frac{1}{4}}q^{\frac{2\nu+1}{4}}xt,q^2)d_qt \\
&= (1+q)C(\nu,q^2)\int_0^1 W_\nu(t,q^2)\widetilde{cos}(q^{\frac{2\nu+1}{4}}xt,q)d_qt.
\end{aligned}$$

The proof is complete. □

Proposition 5.7. *For $\nu > -\frac{1}{2}$, $p \geq 0$ and $0 < q < 1$ the $q - \widetilde{j}_{\nu+p}$ symmetric Bessel function have the following q-integral representation of Sonine type*

$$\widetilde{j}_{\nu+p}(x,q) = \widetilde{C}_1(\nu,q)\int_0^1 t^{2\nu+1}W_{p-\frac{1}{2}}(t,q)\widetilde{j}_\nu(q^{\frac{p}{2}}xt,q)d_qt \tag{57}$$

where

$$\widetilde{C}_1(\nu,q) = \frac{1+q}{q^{1-p(\nu+1)}}\,\frac{\widetilde{\Gamma}_q(\nu+p+1)}{\widetilde{\Gamma}_q(\nu+1)\widetilde{\Gamma}_q(p)}. \tag{58}$$

Proof. The result follows from the q-integral representation of sonine type of $q - j_\nu$ (see [3]) and the relation (52). □

5.1. q-Transmutation

The q-Bessel operator of the $q-j_\nu$ function is defined by (see [3])

$$\triangle_{q,\nu} f(x) = q^{2\nu+1} \triangle_q f(x) + [2\nu+1]_q \frac{1}{q^{-1}x} D_q f(q^{-1}x) \tag{59}$$

where

$$\triangle_q f(x) = (D_q^2 f)(q^{-1}x). \tag{60}$$

We design by $\mathcal{D}_{*,q}$ the space of functions defined in $\widetilde{\mathbb{R}}_{q,+}$ which are the restriction of the even function with compact support in $\widetilde{\mathbb{R}}_q$. This space is equipped with the topology of uniform convergence. For $\nu > -\frac{1}{2}$, and $f \in \mathcal{D}_{*,q}$, the q-analogue of the Kober-Erdelyi transform is defined by

$$\chi_{\nu,q}(f)(x) \quad = \quad C(\nu,q^2)(1+q)\int_0^1 W_\nu(t,q^2) f(xt) d_q t. \tag{61}$$

Theorem 5.8. *The operator* $\chi_{\nu,q}$ *is an isomorphism on* $\mathcal{D}_{*,q}$. *Moreover, it transmutes the q-operator* $\widetilde{\triangle}_{\nu,q}$ *and* $\widetilde{D}_q^2$ *in the following sense*

$$\widetilde{\triangle}_{\nu,q}\chi_{\nu,q^2} = q^{-2\nu-1}\chi_{\nu,q^2}\widetilde{D}_q^2 \tag{62}$$

Proof. In [3], the authors proved that the operator $\chi_{\nu,q}$ is an isomorphism on $D_{*,q}$ and verifies the relation

$$\Delta_{\nu,q}\chi_{\nu,q} = \chi_{\nu,q}\Delta_q. \tag{63}$$

From the relations,

$$\widetilde{D}_q^2 f(x) = q^{-1} D_{q^2}^2 f(q^{-2}x),\ \widetilde{D}_q f(x) = D_{q^2} f(q^{-1}x) \ \text{ and } \ \widetilde{[x]}_q = q^{-(x-1)}[x]_{q^2} \tag{64}$$

it follows that

$$\widetilde{\triangle}_{\nu,q} = q^{-2\nu-2} \triangle_{\nu,q^2} \quad \textit{and} \quad \triangle_{q^2} = q\widetilde{D}_q^2. \tag{65}$$

Now, using the relations (63) and (65), we obtain

$$\widetilde{\triangle}_{\nu,q}\chi_{\nu,q^2} = q^{-2\nu-1}\chi_{\nu,q^2}\widetilde{D}_q^2. \tag{66}$$

The proof is complete. □

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