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ON A CLASS OF GENERALIZED ANALYTIC FUNCTIONS

S. L. KALLA - N. VIRCHENKO - I. ALEXSANDROVICH

This paper deals with a new generalization of analytical functions. The p-wave functions are introduced and studied. We consider their theoretical aspect and applications. Some integral representations of $x^k y^l$-wave functions ($k, l - const. > 0$), and their inversion formulas are derived. As an application of the theory, a singular Cauchy problem is formulated and solved in terms of the Bessel function of the first kind and Gauss hypergeometric function.

1. Introduction

The generalized analytical functions of complex variables appear as a natural and rational generalization of analytical functions.
Picard in 1891 [11] noticed the connection between the theory of analytical functions and *elliptic* system of equations in partial derivatives,

$$\begin{cases} a_1 u_x + b_1 u_y + a_2 \upsilon_x + b_2 \upsilon_y = A_1 u + A_2 \upsilon, \\ c_1 u_x + d_1 u_y + c_2 \upsilon_x + d_2 \upsilon_y = B_1 u + B_2 \upsilon, \end{cases} \tag{1}$$

where $a_i, b_i, c_i, d_i, A_i, B_i$ $(i = 1,2)$ are the given functions of x and y. In this connection, it is worth mentioning the work of Beltrami [1].

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For many decades, the idea of Picard remained dormant, but during the middle of last century, researchers started to work on it. Bers and Gelbart [2, 3, 4] investigated the functions $f(z) = u + i\upsilon$ of $z = x + iy$ satisfying the following elliptic system of equations

$$\begin{cases} \sigma_1(x)u_x = \tau_1(y)\upsilon_y, \\ \sigma_2(x)u_y = -\tau_2(y)\upsilon_x, \end{cases} \tag{2}$$

where $\sigma_1, \sigma_2, \tau_1, \tau_2$ are given positive functions of their arguments.
Let

$$\Sigma = \begin{bmatrix} \sigma_1 \tau_1 \\ \sigma_2 \tau_2 \end{bmatrix}, \qquad \Sigma' = \begin{bmatrix} \frac{1}{\sigma_1}\tau_1 \\ \frac{1}{\sigma_2}\tau_2 \end{bmatrix}.$$

Then, the function $f(z) = u + iv$ is the Σ-*monogenic* as the system (2) is valid. They introduced the Σ-*integral*:

$$\Omega = \int \sigma_2 u dx - \tau_2 \upsilon dy + i \int \frac{\upsilon}{\upsilon_1} dx + \frac{u}{\tau_1} dy, \tag{3}$$

and the Σ-*derivative*:

$$f(z) = \sigma_1 u_x + i\frac{\upsilon_x}{\sigma_2} = \tau_1 \upsilon_y - i\frac{u_y}{\tau_2}. \tag{4}$$

Let us notice that the case $\sigma_1 = \sigma_2 = 1,\ \ \tau_1 = \tau_2 = y^{-n}\ \ (n = const.)$ was studied by Weinstein [15, 16].

Vekua [13, 14], Polozij [12], Jahanshahi and Aliev [6] , Manjavidze and Manjavidze [7] and others , have obtained important results in the generalization of the theory of analytical functions of elliptic type and their applications. Polozij [12] introduced the (p, q)–analytical functions, using the system:

$$\begin{cases} pu_x - qu_y - \upsilon_y = 0, \\ qu_x + pu_y + \upsilon_x = 0, \end{cases} \tag{5}$$

where p and q are the given real functions of x and y.
Later on the p–analytical and (p, q)–analytical functions found number of applications in different branches of the mathematics, mechanics etc.., for example, axial symmetric theory of elasticity, solution of the boundary value problems of the theory of rotating covers, in the theory of the filtration [9, 10].
In this paper, we introduce and study a new generalization of analytic functions. We consider some theoretical aspects of the *p-wave functions* $f(z) = u + i\upsilon$ as the solutions of the following system of the *hyperbolic type*:

$$\begin{cases} pu_x = \upsilon_y, \\ pu_y = \upsilon_x, \end{cases} \tag{6}$$

where $p = x^k y^l$ (k and l are positive constants). Some integral representations of p-wave functions and their inversion formulas are constructed.
The p-wave functions describe the processes of mechanics, hydromechanics, the theory of plastication, the supersonic stream of gas and they are useful for solving of the boundary-value problems of mathematical physics. The p-wave functions with the characteristic $p = x^k y^l$ are connected with Euler - Poisson - Darboux equation with two degenerate lines.

2. Integral representations of the p-wave functions

Let us introduce the following differential operators

$$\frac{d\phi}{d\bar{z}} = \frac{1}{2}\left(\frac{\partial\phi(z)}{\partial x} - i\frac{\partial\bar{\phi}(z)}{\partial y}\right) = \frac{u_x - \upsilon_y}{2} + i\frac{\upsilon_x - \upsilon_y}{2}, \tag{7}$$

$$\frac{d_p\phi(z)}{d\bar{z}} = \frac{pu_x - \upsilon_y}{2} + i\frac{\upsilon_x - pu_y}{2}.$$

Remark 2.1. Observe that the system (6) is equivalent to the following equation:

$$\frac{d_p\phi(z)}{d\bar{z}} = 0. \tag{8}$$

For, $p \equiv 1$, p-wave function is the simple wave function and is a solution of the following equation:

$$\frac{d\phi_0(z)}{d\bar{z}} = 0. \tag{9}$$

The general solution of the equation (9) has the form:

$$\phi_0(z) = f_1(x-y) + f_2(x+y) + i(f_2(x+y) - f_1(x-y)), \tag{10}$$

where $f_1(z)$, $f_2(z)$ are arbitrary continuously differentiable functions.

Definition 2.2. The simply connected domain D in the complex plane z will be called *the axis – convex* if for any point $z = x + iy \in D \quad y > 0$;

1. the boundary of the domain D contains the segment $[a,b]$ of the real axis;
2. the segment connecting the points $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ $(\in D)$ belongs to the domain D.

Note that the domain D can coincide with the upper half plane.

Now we state and prove the following theorems related to integral representations of the p-wave functions.

Theorem 2.3. *If $\phi_0(z) = u_0(x,y) + i\upsilon_0(x,y)$ is an arbitrary wave function in D and*

$$\upsilon_0(x,y)|_{[a,b]} = 0, \quad (0 \le y < \infty), \tag{11}$$

then the function

$$\phi(z) = u(x,y) + i\upsilon(x,y) = \int_0^x \left[u_0(\xi,y)x^{1-k} + i\upsilon_0(\xi,y)\xi\right] \left(x^2 - \xi^2\right)^{\frac{k}{2}-1} d\xi \tag{12}$$

will be x^k-wave function ($k > 0$ is a const.) in D and satisfies the condition

$$Im\phi(z)|_{[a,b]} = \upsilon(0,y) = 0. \tag{13}$$

Proof. This theorem can be proved easily by the statement. Observe that the functions $u(x,y)$ and $\upsilon(x,y)$ satisfy the system (6) as $p = x^k$. □

Integral representation (12) establishes the one – to – one correspondence between x^k- wave functions and the wave functions in D as the imaginary parts of functions at the real axis are zero.

Theorem 2.4. *If $\phi(z) = u(x,y) + i\upsilon(x,y)$ is the x^k-wave function, continuous in the axis-convex region $D \cup [a,b]$ and*

$$\upsilon|_{[a,b]} = 0, \tag{14}$$

then the function

$$\tilde{\phi}(z) = \tilde{u}(x,y) + i\tilde{\upsilon}(x,y) = \int_0^y \left(y^{1-l}u(x,\tau) + i\tau\upsilon(x,\tau)\right) \left(y^2 - \tau^2\right)^{\frac{l}{2}-1} d\tau \tag{15}$$

will be $x^k y^l$ $(k, l - const. > 0)$ wave function and continuous in $D \cup [a,b]$ and

$$\tilde{\upsilon}(x,y)|_{[a,b]} = 0 \tag{16}$$

Proof. By using the conditions for $u(x,y)$ and $\upsilon(x,y)$

$$\begin{aligned} x^k u_x &= \upsilon_y, \\ x^k u_y &= \upsilon_x, \end{aligned} \tag{17}$$

we directly verify the validity of the conditions:

$$\begin{aligned} x^k y^l \tilde{u}_x &= \tilde{\upsilon}_y, \\ x^k y^l \tilde{u}_y &= \tilde{\upsilon}_x \end{aligned}$$

and (16). □

Let D be the domain in the first quadrant of the plane $z = x+iy$, restricted by segments $[a,b] \in \mathrm{O}x$, $[c,d] \in \mathrm{O}y$ and some curves, which are monotone with respect to x and y and such that any rectilinear segments retiring from any its point and orthogonal to Ox and Oy, belong to D.

Theorem 2.5. *If the function $\phi_0(z) = u_0(x,y) + i\upsilon_0(x,y)$ is the wave function in D and*

$$\upsilon_0|_{[a,b]} = 0, \quad \upsilon_0|_{[c,d]} = 0,$$

then the function

$$\tilde{\phi}(z) = \tilde{u}(x,y) + i\tilde{\upsilon}(x,y) = \int_0^y \left(y^2 - \eta^2\right)^{\frac{l}{2}-1} d\eta \int_0^x \Big[x^{1-k}y^{1-l}u_0(\xi,\eta) +$$

$$+ i\xi\eta\upsilon_0(\xi,\eta)\Big] \left(x^2 - \xi^2\right)^{\frac{k}{2}-1} d\xi \tag{18}$$

will be $x^k y^l$ - wave function in D and at $[a,b],[c,d]$ has continuous partial derivatives with respect to y,x respectively, and

$$\tilde{\upsilon}|_{[a,b]} = 0, \quad \tilde{\upsilon}|_{[c,d]} = 0. \tag{19}$$

Proof. We have

$$x^k y^{l-1}\left[\frac{\partial \tilde{u}}{\partial x} - \frac{1}{x^k y^l}\frac{\partial \tilde{\upsilon}}{\partial y}\right] = \int_0^x \int_0^y \left[\frac{\partial u_0(\xi,\eta)}{\partial \xi} - \frac{\partial \upsilon_0(\xi,\eta)}{\partial \eta}\right] \times$$

$$\times \left(x^2 - \xi^2\right)^{\frac{k}{2}-1} \left(y^2 - \eta^2\right)^{\frac{l}{2}-1} \xi d\xi d\eta;$$

$$x^{k-1} y^{l}\left[\frac{\partial \tilde{u}}{\partial y} - \frac{1}{x^k y^l}\frac{\partial \tilde{\upsilon}}{\partial x}\right] = \int_0^x \int_0^y \left[\frac{\partial u_0(\xi,\eta)}{\partial \eta} - \frac{\partial \upsilon_0(\xi,\eta)}{\partial \xi}\right] \times$$

$$\times \left(x^2 - \xi^2\right)^{\frac{k}{2}-1} \left(y^2 - \eta^2\right)^{\frac{l}{2}-1} \eta d\xi d\eta.$$

Using the fact that $\phi_0(z)$ is the wave function, we get that $\tilde{\phi}(z)$ is $x^k y^l$ - wave function.

Let $\xi = x\lambda,\ \eta = yt$ in (18), then we obtain,

$$\tilde{u}(x,y) = \int_0^1 \int_0^1 u_0(x\lambda, yt) \left(1-\lambda^2\right)^{\frac{k}{2}-1} \left(1-t^2\right)^{\frac{l}{2}-1} d\lambda dt,$$

$$\tilde{\upsilon}(x,y)=\int_0^1\int_0^1 x^k y^l \lambda t \upsilon_0(x\lambda,yt)\left(1-\lambda^2\right)^{\frac{k}{2}-1}\left(1-t^2\right)^{\frac{l}{2}-1}d\lambda dt,$$

taking into account the conditions of the theorem, we get (19) and that $\tilde{\phi}(z)$ has continuous partial derivatives. □

3. Inversion formulas for the representations of p-wave functions

Here, we derive some inversion formulas for the representations of p-wave functions.

Theorem 3.1. *Let $\phi(z)=u(x,y)+i\upsilon(x,y)$ is x^k-wave function in D and the condition that $\upsilon(x,y)|_{[a,b]}=0$, then function $\phi_0(z)=u_0(x,y)+i\upsilon_0(x,y)$ (see (9)) is the wave function defined by equality:*

$$u_0(x,y)+i\upsilon_0(x,y)=\begin{cases}\mu\frac{d}{dx}\int_0^x\frac{d^m\left[u(\xi,y)\xi^{k-1}+i\upsilon(\xi,y)\right]}{(d\xi^2)^m}\frac{\xi d\xi}{(x^2-\xi^2)^{\frac{k}{2}-m}}, & k\neq 2m,\\ \mu x\frac{d^m\left[u(x,y)x^{k-1}+i\upsilon(x,y)\right]}{(dx^2)^m}, & k=2m,\end{cases}\tag{20}$$

and $\upsilon(x,y)|_{[a,b]}=0$. Here $m=\left[\frac{k}{2}\right]$, $\mu=\frac{2}{\Gamma\left(m-\frac{k}{2}+1\right)\Gamma\left(\frac{k}{2}\right)}$.

Proof. The real and imaginary parts of the integral representation (12) are integral equations of Abel' type. The solutions of these equations give (20). □

Remark 3.2. Note that integral representation of x^k-wave function (12) as $\upsilon_0(0,y)=0$ can be written in the following form:

$$\phi(z)=u(x,y)+i\upsilon(x,y)=\int_{y-x}^{y+x}f_2(t)\left(x^{1-k}+i(t-y)\left[x^2-(t-y)^2\right]^{\frac{k}{2}-1}\right)dt,$$

provided that $u_0(x,y)$ and $y_0(x,y)$ have the form (10).

Theorem 3.3. *Let be $\tilde{\varphi}(z)=\tilde{u}(x,y)+i\tilde{\upsilon}(x,y)$ is $x^k y^l$-wave function in D and the condition*

$$\tilde{\upsilon}(x,y)|_{[a,b]}=0,$$

then $\varphi(z)=u(x,y)+i\upsilon(x,y)$ is the x^k-wave function defined by equality:

$$\varphi(z) = u(x,y) + i\upsilon(x,y)$$
$$= \begin{cases} \frac{2}{\Gamma\left(\frac{l}{2}\right)\Gamma\left(m-\frac{l}{2}+1\right)} \left\{ \frac{d}{dy}\int_0^y \frac{d^m}{(d\tau^2)^m}\left(\tau^{l-1}\tilde{u}(x,\tau)\frac{\tau d\tau}{(y^2-\tau^2)^{\frac{l}{2}-m}}\right) \right. \\ \left. +i\frac{1}{y}\frac{d}{dy}\int_0^y \frac{d^m\tilde{\upsilon}(x,\tau)}{(d\tau^2)^m}\frac{\tau d\tau}{(y^2-\tau^2)^{\frac{l}{2}-m}} \right\}, \quad l \neq 2n, m = \left[\frac{l}{2}\right]; \\ \frac{2}{(n-1)!}\left\{ y\frac{d^n}{(dy^2)^n}\left[y^{l-1}\tilde{u}(x,y)\right] + i\frac{d^n\tilde{\upsilon}(x,y)}{(dy^2)^n} \right\}, \quad l = 2n, \end{cases} \tag{21}$$

and $\upsilon(x,y)|_{[a,b]} = 0.$

Proof. By virtue of (15), we have

$$\tilde{u}(x,y) = y^{1-l}\int_0^y u(x,\tau)(y^2-\tau^2)^{\frac{l}{2}-1}d\tau, \tag{22}$$

$$\tilde{\upsilon}(x,y) = \int_0^y \upsilon(x,\tau)\tau(y^2-\tau^2)^{\frac{l}{2}-1}d\tau. \tag{23}$$

Introducing notations,

$$L(x,y) = x^k u_x - \upsilon_y; \qquad \tilde{L}(x,y) = x^k y^l \tilde{u}_x - \tilde{\upsilon}_y,$$
$$M(x,y) = x^k u_y - \upsilon_x, \qquad \tilde{M}(x,y) = x^k y^l \tilde{u}_y - \tilde{\upsilon}_x,$$

we can rewrite (17) in the following form:

$$\tilde{L}(x,y) = y\int_0^y L(x,\tau)(y^2-\tau^2)^{\frac{l}{2}-1}d\tau, \tag{24}$$

$$\tilde{M}(x,y) = \int_0^y M(x,\tau)\tau(y^2-\tau^2)^{\frac{l}{2}-1}d\tau. \tag{25}$$

The equations (24) and (25) are Abel type integral equations. But $\tilde{L}(x,y)$=0, $\tilde{M}(x,y)$=0, then $L(x,y)$=0, $M(x,y)$=0. Therefore $\varphi(z)$ is x^k-wave function. The condition $\upsilon(x,y)|_{[a,b]} = 0$ can be verified directly. □

Theorem 3.4. *The solution of the integral equation (18) has the form:*

$$1)\ u_0(x,y) + i\upsilon_0(x,y) = \frac{4xy}{m!n!}\frac{\partial^{m+n+2}\left(x^{k-1}y^{l-1}\tilde{u}(x,y)\right)}{\left(\partial x^2\right)^{m+1}\left(\partial y^2\right)^{n+1}} +$$
$$+i\frac{4}{m!n!}\frac{\partial^{m+n+2}\tilde{\upsilon}(x,y)}{\left(\partial x^2\right)^{m+1}\left(\partial y^2\right)^{n+1}}, \tag{26}$$

for $\frac{k}{2} - 1 = m,\ \frac{l}{2} - 1 = n.$

$$2)\ u_0(x,y)+i\upsilon_0(x,y)=\frac{4}{\Gamma\left(\frac{k}{2}\right)\Gamma\left(\frac{l}{2}\right)\Gamma\left(m+1-\frac{k}{2}\right)\Gamma\left(n+1-\frac{l}{2}\right)}\times$$

$$\times\left\{\frac{\partial^2}{\partial x\partial y}\int_0^x\int_0^y\frac{\partial^{m+n}\left(\xi^{k-1}\eta^{l-1}\tilde{u}(\xi,\eta)\right)}{(\partial\xi^2)^m(\partial\eta^2)}\frac{\xi\eta d\xi d\eta}{(x^2\xi^2)^{\frac{k}{2}-m}(y^2-\eta^2)^{\frac{l}{2}-n}}+\right. \quad (27)$$

$$\left.+\frac{i}{xy}\frac{\partial^2}{\partial x\partial y}\int_0^x\int_0^y\frac{\partial^{m+n}\tilde{\upsilon}(\xi,\eta)}{(\partial\xi^2)^m(\partial\eta^2)^n}\frac{\xi\eta d\xi d\eta}{(x^2-\xi^2)^{\frac{k}{2}-m}(y^2-\eta^2)^{\frac{l}{2}-n}}\right\},$$

for $\frac{k}{2}-1\neq m,\ \frac{l}{2}-1\neq n.$

Proof. It is sufficient to consider the integral equation:

$$\int_{x_0}^x\int_{y_0}^y\varphi(\xi,\eta)\left[\omega(x)-\omega(\xi)\right]^\alpha\left[\gamma(y)-\gamma(\eta)\right]^\beta d\xi d\eta=\psi(x,y), \quad (28)$$

where $x_0,y_0\in R$; x,y are independent variables; $\alpha,\beta>-1$ are constants; $\omega(x)$, $\gamma(y)$ are given functions with the following properties:
for $x_0<x$ $\omega(x)>\omega(\xi)$, $\xi\in(x_0,x)$;
for $x_0>x$ $\omega(\xi)>\omega(x)$, $\xi\in(x,x_0)$;
for $y_0<y$ $\gamma(y)>\gamma(\eta)$, $\eta\in(y_0,y)$;
for $y_0>y$ $\gamma(\eta)>\gamma(y)$, $\eta\in(y,y_0)$;
$\psi(x,y)$ is the given, continuously differentiable function; $\varphi(x,y)$ is an unknown function.

If α,β are integers and $\alpha\equiv m,\beta\equiv n$ the solution of equation (28) has the following form:

$$\varphi(x,y)=\frac{\omega'(x)\gamma'(y)}{m!n!}\frac{\partial^{m+n+2}\psi(x,y)}{(\partial\omega(x))^{m+1}(\partial\gamma(y))^{n+1}}. \quad (29)$$

When α,β are not integers, then the solution of equation (28) has the form

$$\varphi(x,y)=\frac{1}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(k-\alpha)\Gamma(\alpha-\beta)}\times$$

$$\times\frac{\partial^2}{\partial x\partial y}\int_{x_0}^x\int_{y_0}^y\frac{\partial^{k+l}\psi(\xi,\eta)}{(\partial\omega(\xi))^k(\partial\gamma(\eta))^l}\frac{\omega'(\xi)\gamma'(t)d\xi d\eta}{(\omega(x)-\omega(\xi))^{\alpha+1-k}}\times$$

$$\times \frac{1}{(\gamma(y)-\gamma(\eta))^{\beta+1-l}}. \tag{30}$$

Indeed, the formula (29) is evident. Let α, β are not integers, $k=[\alpha+1]$, $l=[\beta+1]$. Let be put $\mu \equiv k-\alpha$, $v=l-\beta$ $(0<\mu<1, 0<v<1)$. By differentiating both sides of (28) k times with respect to $\omega(x)$ and l times with respect to $\gamma(y)$, we have

$$\frac{\partial^{k+l}\psi(x,y)}{(\partial\omega(x))^k(\partial\gamma(y))^l} = \\ = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha-k+1)\Gamma(\beta-l+1)} \int_{x_0}^{x}\int_{y_0}^{y} \frac{\phi(\xi,\eta)d\xi d\eta}{(\omega(x)-\omega(\xi))^{\mu}(\gamma(y)-\gamma(\eta))^{v}}.$$

In the above expression, replace x by t and y by τ, and multiply by

$$\frac{\omega'(t)\gamma'(t)dtd\tau}{(\omega(x)-\omega(t))^{1-\mu}(\gamma(y)-\gamma(\tau))^{1-v}}.$$

Then, integrate with respect to t and τ to get,

$$\int_{x_0}^{x}\int_{y_0}^{y}\left\{\int_{x_0}^{t}\int_{y_0}^{\tau}\frac{\varphi(\xi,\eta)d\xi d\eta}{(\omega(t)-\omega(\xi))^{\mu}(\gamma(\tau)-\gamma(\eta))^{v}}\right\}\times \\ \times \frac{\omega'(t)\gamma'(\tau)dtd\tau}{(\omega(x)-\omega(t))^{1-\mu}(\gamma(y)-\gamma(\tau))^{1-v}} =$$

$$= \frac{\Gamma(\alpha-k+1)\Gamma(\beta-l+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)}\times \\ \times \int_{x_0}^{x}\int_{y_0}^{y}\frac{\partial^{k+l}\psi(t,\tau)}{(\partial\omega(t))^k(\partial\gamma(\tau))^l}\cdot\frac{\omega'(t)\gamma'(\tau)dtd\tau}{(\omega(x)-\omega(t))^{1-\mu}(\gamma(y)-\gamma(\tau))^{1-v}}.$$

Then transform the left part of equality,

$$\int_{x_0}^{x}\int_{y_0}^{y}\varphi(\xi,\eta)d\xi d\eta\int_{\xi}^{x}\frac{\omega'(t)dt}{(\omega(t)-\omega(\xi))^{\mu}(\mu(x)-\omega(t))^{1-\mu}}\times \\ \times\int_{\eta}^{y}\frac{\gamma'(\tau)d\tau}{(\gamma(\tau)-\gamma(\eta))^{v}(\gamma(y)-\gamma(\tau))^{1-v}} =$$

$$= \left| r=\frac{\omega(t)-\omega(\xi)}{\omega(x)-\omega(\xi)},\ s=\frac{\gamma(\tau)-\gamma(\eta)}{\gamma(y)-\gamma(\eta)}\right| = \int_{x_0}^{x}\int_{y_0}^{y}\varphi(\xi,\eta)d\xi d\eta\times$$

$$\times \int_0^1 \frac{dr}{r^{\mu}(1-r)^{1-\mu}} \int_0^1 \frac{ds}{s^{\nu}(1-s)^{1-\nu}} =$$

$$= \Gamma(1+\alpha-k)\Gamma(k-\alpha)\Gamma(1+\beta-l)\Gamma(l-\beta) \int_{x_0}^{x} \int_{y_0}^{y} \varphi(\xi,\eta)d\xi d\eta.$$

□

Let us note that by the help of formulas (18), (26) and (28), the one–to–one correspondence between $x^k y^l$- wave functions and wave functions in D is established as their imaginary parts on $[a,b]$ and $[c,d]$ are zero.

Corollary 3.5. *Let $\frac{k}{2}-1$ is integer ($\frac{k}{2}-1 \equiv m$), $\frac{l}{2}-1$ is not an integer. Introducing the notation, $\left[\frac{l}{2}\right] \equiv n$ we get the following inversion formula for (18):*

$$u_0(x,y)+i\upsilon_0(x,y) = \frac{2}{m!\Gamma\left(\frac{l}{2}\right)\Gamma\left(n+1-\frac{l}{2}\right)} \left\{ 2x\frac{\partial}{\partial y}\int_0^y \frac{\partial^{m+n+1}}{(\partial x^2)^{m+1}(\partial t^2)^n} \right.$$

$$\frac{\left(x^{k-1}\eta^{l-1}\tilde{u}(x,\eta)\right)\eta d\eta}{(y^2-\eta^2)^{\frac{l}{2}-n}} + i\frac{2}{y}\frac{\partial}{\partial y}\int_0^y \frac{\partial^{m+n+1}\left(\tilde{\upsilon}(x,\eta)\right)}{(\partial x^2)^{m+1}(\partial\eta^2)^n} \times \tag{31}$$

$$\left. \times \frac{\eta d\eta}{(y^2-\eta^2)^{\frac{l}{2}-n}} \right\}.$$

Corollary 3.6. *Let be $\frac{k}{2}-1$ is not integer, $\frac{l}{2}-1$ is integer ($\frac{l}{2}-1 \equiv n$). Introducing notation, $\left[\frac{k}{2}\right] \equiv m$ we get the following inversion formula for (18):*

$$u_0(x,y)+i\upsilon_0(x,y) =$$

$$= \frac{2}{n!\Gamma\left(\frac{k}{2}\right)\Gamma\left(m+1-\frac{k}{2}\right)} \left\{ 2y\frac{\partial}{\partial x}\int_0^x \frac{\partial^{m+n+1}\left(\xi^{k-1}y^{l-1}\tilde{u}(\xi,y)\right)}{(\partial\xi^2)^m(\partial y^2)^{n+1}} \times \right.$$

$$\left. \times \frac{\xi d\xi}{(x^2-\xi^2)^{\frac{k}{2}-m}} + i\frac{2}{x}\frac{\partial}{\partial x}\int_0^x \frac{\partial^{m+n+1}\left(\tilde{\upsilon}(\xi,y)\right)}{(\partial\xi^2)^m(\partial y^2)^{n+1}} \frac{\xi d\xi}{(x^2-\xi^2)^{\frac{k}{2}-m}} \right\}. \tag{32}$$

Theorem 3.7. (special integral representation of the $x^k y^l$-wave function) *Let be domain G is restricted by characteristics:*
$x-y=a,\ x+y=b,\ b>a\geq 0$, *and by segment* $[a,b]$ *of* Ox. *Then the function*

$$\phi(z) = u(x,y)+i\upsilon(x,y) = \frac{1}{\sqrt{x^k y^l}}\int_0^{\infty} \sqrt{xr}J_{\frac{k-1}{2}}(rx)J_{\frac{l-1}{2}}(ry)A(r)\frac{dr}{r^{\frac{k+l}{2}}} -$$

$$-i\sqrt{x^k y^l}\int_0^\infty \sqrt{xr}J_{\frac{k+1}{2}}(rx)J_{\frac{l+1}{2}}(ry)A(r)\frac{dr}{r^{\frac{k+l}{2}}} \tag{33}$$

is the $x^k y^l$- wave function in G with the condition: $A(r)$ is restricted on every finite interval and

$$\int_0^\infty |A(r)|\frac{dr}{r^{\frac{k+l}{2}}} < \infty, \quad (r \to \infty); \;\; A(r) \in C((0,+\infty)). \tag{34}$$

Proof. Follows from the definition of the $x^k y^l$- wave function, asymptotic formulae for the Bessel function of the first kind, and Bessel functions differentiation formulae [8]. □

4. Singular Cauchy' problem

As an example we consider here a singular Cauchy problem.

In the domain $G = \{(x,y) : 0 < x < \infty, 0 < y < x\}$ find the $x^k y^l$-wave function
$\phi(z) = u(x,y) + i\upsilon(x,y)$, which satisfy the following conditions:

$$\phi(z) \in C(\bar{G}),$$

$$\phi(x) = f(x), x \in (0,+\infty). \tag{35}$$

Here, $f(x) \in D(0,\infty)$. Let be $D(0,\infty)$ is the set of the finitely differentiable functions on $(0,\infty)$.

We seek the solution of the problem in the form of (33). It is easy to note that (35) is valid. Using the second condition of (35) for the finding of function $A(r)$,we get equality

$$\frac{2^{\frac{1-l}{2}}}{\Gamma\left(\frac{l+1}{2}\right)}\int_0^\infty (rx)^{\frac{1-k}{2}}J_{\frac{k-1}{2}}(rx)A(r)dr = f(x). \tag{36}$$

But $f(x) \in D(0,\infty)$, and Hankel transform (36) (see [5]) is an automorphism in the space of the basic functions. Thus we have

$$A(r) = 2^{\frac{l-1}{2}}\Gamma\left(\frac{l+1}{2}\right)\int_0^\infty (r\xi)^{\frac{k+1}{2}}J_{\frac{k-1}{2}}(r\xi)f(\xi)d\xi. \tag{37}$$

Substituting (37) into (33) we get,

$$\phi(z) = 2^{\frac{l-1}{2}}\Gamma\left(\frac{l+1}{2}\right)\times$$
$$\times\left[\int_0^\infty (rx)^{\frac{1-k}{2}} J_{\frac{k-1}{2}}(ry)^{\frac{1-l}{2}} J_{\frac{l-1}{2}}(ry)\left(\int_0^\infty (r\xi)^{\frac{k+1}{2}} J_{\frac{k-1}{2}}(r\xi) f(\xi)d\xi\right)dr-\right.$$
$$-i\int_0^\infty r^{-k-l}(rx)^{\frac{k+1}{2}} J_{\frac{k+1}{2}}(rx)(ry)^{\frac{1+l}{2}} J_{\frac{l+1}{2}}(ry)\times$$
$$\left.\times\left(\int_0^\infty (r\xi)^{\frac{k+1}{2}} J_{\frac{k-1}{2}}(r\xi) f(\xi)d\xi\right)dr\right].$$

After some transformation and simplification, we get the solution of the singular Cauchy' problem in the class of the $x^k y^l$- wave functions, involving Gauss hypergeometric function [8]

$$\phi(z) = \int_{x-y}^{x+y} f(\xi)W_1(x,y,\xi)d\xi + i\int_{x-y}^{x+y} f(\xi)W_2(x,y,\xi)d\xi, \tag{38}$$

where

$$W_1(x,y,\xi) = \frac{\Gamma\left(\frac{l+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{l}{2}\right)} y^{1-l}\left(\frac{\xi}{x}\right)^{\frac{k}{2}}\left[y^2-(x-\xi)^2\right]^{\frac{l}{2}-1}\times$$
$$\times\ {}_2F_1\left(1-\frac{k}{2},\frac{k}{2};\frac{l}{2};\frac{y^2-(x-\xi)^2}{4x\xi}\right),$$

$$W_2(x,y,\xi) = -\frac{\Gamma\left(\frac{l+1}{2}\right)(x\xi)^{\frac{k}{2}}}{2\sqrt{\pi}\Gamma\left(1+\frac{l}{2}\right)}\left\{\frac{k}{2\xi}\left[y^2-(x^2-\xi^2)\right]^{\frac{l}{2}}\times\right. \tag{39}$$
$$\left.\times\ {}_2F_1\left(-\frac{k}{2},\frac{k}{2}+1;\frac{l}{2}+1;\frac{y^2-(x-\xi)^2}{4x\xi}\right)\Big]\right\}+$$

$$+\frac{d}{d\xi}\left[\left[y^2-(x-\xi)^2\right)^{\frac{l}{2}}\right]{}_2F_1\left(-\frac{k}{2},\frac{k}{2}+1;\frac{l}{2}+1;\frac{y^2-(x-\xi)^2}{4x\xi}\right)\Big]\Big\}$$

as $k \geq 0, l > 2$.

Formula (37) gives the solution of the singular Cauchy' problem in G with the condition, that $f(x)$ can be continued on all axis in such way that its continuation belongs to class $D(0,\infty)$.

Note, that for (37) is valid and on weaker condition for $f(x)$.

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SHYAM L. KALLA
Department of Computer Engineering
Vyas Institute of Higher Education
Jodhpur 342001, India
e-mail: shyamkalla@yahoo.com

NINA VIRCHENKO
Department of Mathematics and Physics
National Technical University of Ukraine
"KPI", Kyiv, Ukraine
e-mail: nvirchenko@hotmail.com

IRYNA ALEXSANDROVICH
Department of Computational Mathematics
National Kyiv T. Shevchenko University
Ukraine
e-mail: ms123@ukr.net