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ON SOME APPLICATIONS OF SUBORDINATION AND SUPERORDINATION OF MULTIVALENT FUNCTIONS INVOLVING THE EXTENDED FRACTIONAL DIFFERINTEGRAL OPERATOR

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In this paper, we apply fractional differintegral operator and study various properties of differential subordination and superordination.

1. Introduction

Let $H(E)$ denote the class of analytic functions in the open unit disc $E = \{z \mid z \in \mathbb{C} \text{ and } |z| < 1\}$ and let $H[a, p]$ denote the subclass of the functions $f \in H(E)$ of the form

$$f(z) = a + a_p z^p + a_{p+1} z^{p+1} + \ldots, \quad (a \in \mathbb{C},\ p \in \mathbb{N} = \{1, 2, \ldots\}).$$

Also, let $\mathcal{A}(p)$ be the subclass of functions $f \in H(E)$ of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in \mathbb{N}), \tag{1}$$

and set $\mathcal{A} \equiv \mathcal{A}(1)$.

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If f and g are analytic in E, we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function w in E with $|w(0)| = 0$ and $|w(z)| < 1$, $z \in E$, such that $f(z) = g(w(z))$.

Suppose that h and k are two analytic functions in E, let

$$\varphi(r,s,t;z) : \mathbb{C}^3 \times E \longrightarrow \mathbb{C}.$$

If h and $\varphi(h(z),zh'(z),z^2h''(z);z)$ are univalent functions in E and if h satisfies the second order superordination

$$k(z) \prec \varphi(h(z),zh'(z),z^2h''(z);z), \tag{2}$$

then k is said to be a solution of the differential superordination (2). A function $q \in H(E)$ is called a subordinant to (2), if $q(z) \prec h(z)$ for all the functions h satisfying (2).

A univalent subordinant $\widetilde{q}$ that satisfies $q(z) \prec \widetilde{q}(z)$ for all of the subordinants q of (2), is said to be the best subordinant.

Miller and Mocanu [6] obtained sufficient conditions on the functions k, q and φ for which the following implications hold:

$$k(z) \prec \varphi(h(z),zh'(z),z^2h''(z);z) \Longrightarrow q(z) \prec h(z).$$

Using these results, the authors in [1] considered certain classes of first-order differential superordinations, see also [4], as well as superordination-preserving integral operators [3]. Aouf et al. [1,2], obtained sufficient conditions for certain normalized analytic functions f to satisfy

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

where q_1 and q_2 are given univalent normalized functions in E. Very recently, Shanmugam et al. [12] obtained the such called sandwich results for certain classes of analytic functions.

Definition 1.1. [8] The fractional integral of order $\lambda > 0$, is defined, for a function f, analytic in a simply-connected region of the complex plane containing the origin, by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(t)}{(z-t)^{1-\lambda}} dt, \tag{3}$$

where the multiplicity of $(z-t)^{\lambda-1}$ is removed by requiring $\log(z-t)$ to be real when $(z-t) > 0$.

Definition 1.2. [8] Under the hypothesis of Definition 1.1, the fractional derivative of f of order $\lambda > 0$ is defined by:

$$D_z^{\lambda} f(z) = \begin{cases} \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int\limits_0^z \frac{f(t)}{(z-t)^{\lambda}} dt, \; 0 \le \lambda < 1 \\ \frac{d^n}{dz^n} D_z^{\lambda-n} f(z) \quad n \le \lambda < n+1; \; n \in \mathbb{N}_0 = \{0,1,2,...\}, \end{cases} \tag{4}$$

where the multiplicity of $(z-t)^{\lambda-1}$ is removed as in Definition 1.1.

In [10], Patel and Mishra defined the extended fractional differintegrall operator $\Omega_z^{(\lambda,p)} : \mathcal{A}(p) \longrightarrow \mathcal{A}(p)$ for a function f of the form (1.1) (with $n = 1$) for a real number λ $(-\infty < \lambda < p+1)$ by :

$$\begin{aligned} \Omega_z^{(\lambda,p)} f(z) &= z^p + \sum_{k=1}^{\infty} \frac{\Gamma(p+k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(p+1-\lambda)} a_{k+p} z^{k+p} \\ &= z^p \; {}_2F_1(1, p+1; p+1-\lambda; z) * f(z), \\ &(-\infty < \lambda < p+1, \; z \in E), \end{aligned} \tag{5}$$

where ${}_2F_1$ is the Gaussian hypergeometric function and $(*)$ represent the Hadamard product (or convolution).

It is easily seen from (5), see [10], that

$$z(\Omega_z^{(\lambda,p)} f(z))' = (p-\lambda)\Omega_z^{(\lambda+1,p)} f(z) + \lambda \Omega_z^{(\lambda,p)} f(z), \; (-\infty < \lambda < p+1, \; z \in E). \tag{6}$$

We also note that

$$\Omega_z^{(0,p)} f(z) = f(z), \;\; \Omega_z^{(1,p)} f(z) = \frac{zf'(z)}{p},$$

and in general

$$\Omega_z^{(\lambda,p)} f(z) = \frac{\Gamma(p+1-\lambda)}{\Gamma(p+1)} z^{\lambda} D_z^{\lambda} f(z) \;\; (-\infty < \lambda < p+1, \; z \in E), \tag{7}$$

where $D_z^{\lambda} f(z)$ is respectively, the fractional integral of f of order $-\lambda$ when $-\infty < \lambda < 0$ and the fractional derivative of f of order λ when $0 \le \lambda < p+1$. For integral value λ, (7) becomes

$$\Omega_z^{(j,p)} f(z) = \frac{(p-J)! f^{(j)}(z)}{p!} \quad (j \in \mathbb{N}; \; J < p+1).$$

and

$$\begin{aligned}\Omega_z^{(-m,p)}f(z) &= \frac{p+m}{z^m}\int_0^z t^{m-1}\Omega_z^{(-m+1,p)}f(t)dt \qquad m\in\mathbb{N}\\ &= F_{1,p}(f)\circ F_{2,p}(f)\circ F_{3,p}\circ\ldots\circ F_{m,p}(f)(z)\\ &= F_{1,p}*\left(\frac{z^p}{1-z}\right)*F_{2,p}*\left(\frac{z^p}{1-z}\right)*\ldots*F_{m,p}*\left(\frac{z^p}{1-z}\right)*f(z)'\end{aligned}$$

where $F_{\mu,p}$ is the familiar generalized Bernardi-Libra-Livingston operator and $\circ$ denotes the usual composition of functions.

The fractional differential operator $\Omega_z^{(\lambda,p)}$ with $0\le\lambda<1$ was investigated by Srivastava and Aouf [13]. More recently, Srivastava and Mishra [14] obtained several interesting properties and characteristics for certain subclasses of p-valent analytic functions involving the differintegral operator $\Omega_z^{(\lambda,p)}$ when $-\infty<\lambda<1$. The operator $\Omega_z^{(\lambda,1)}=\Omega_z^{\lambda}$ was introduced by Owa and Srivastava [9]. The interested reader are referred to the work done by research workers [1,8,15].

2. Preliminaries

Definition 2.1. ([7]) Let Q be the set of all functions f that are analytic and injective on $\overline{E}\backslash U(f)$, where

$$U(f)=\left\{\zeta\in\partial E:\lim_{z\to\zeta}f(z)=\infty\right\},$$

and are such that $f'(\zeta)\neq 0$ for $\zeta\in\partial E\backslash U(f)$.

To establish our main results we need the following Lemmas.

Lemma 2.2. (Miller and Mocanu [6]) *Let q be univalent in the unit disc E, and let θ and φ be analytic in a domain D containing $q(E)$, with $\varphi(w)\neq 0$ when $w\in q(E)$. Set $Q(z)=zq'(z)\varphi(q(z))$, $h(z)=\theta(q(z)+Q(z)$ and suppose that*

(i) Q is a starlike function in E,

(ii) $\operatorname{Re}\frac{zh'(z)}{Q(z)}>0$, $z\in E$.

If p is analytic in E with $p(0)=q(0)$, $p(E)\subseteq D$ and

$$\theta(p(z))+zp'(z)\varphi(p(z)\prec\theta(q(z))+zq'(z)\varphi(q(z),\tag{8}$$

then $p(z)\prec q(z)$, and q is the best dominant of (8).

Lemma 2.3. (Shanmugam et al. [12]) *Let μ, $\gamma \in \mathbb{C}$ with $\gamma \neq 0$, and let q be a convex function in E with*

$$\operatorname{Re}\left(1+\frac{zq''(z)}{q'(z)}\right) > \max\left\{0; -\operatorname{Re}\frac{\mu}{\gamma}\right\},\ z \in E.$$

If p is analytic in E and

$$\mu p(z)+\gamma zp'(z) \prec \mu q(z)+\gamma zq'(z), \tag{9}$$

then $p(z) \prec q(z)$, and q is the best dominant of (9).

Lemma 2.4. (Bulboacă [5]) *Let q be a univalent function in the unit disc E, and let θ and φ be analytic in a domain D containing $q(E)$. Suppose that*

(i) Re $\frac{\theta'(q(z))}{\varphi(q(z)}>0$ *for $z \in E$,*

(ii) *$h(z)=zq'(z)\varphi(q(z))$ is starlike in E.*

If $p \in H[q(0),1] \cap Q$ with $p(E) \subseteq D$, $\theta(p(z)+zp'(z))\varphi(p(z))$ is univalent in E, and

$$\theta(q(z))+zq'(z)\varphi(q(z)) \prec \theta(p(z))+zp'(z)\varphi(p(z)), \tag{10}$$

then $q(z) \prec p(z)$, and q is the best subordinant of (10).

Note that this result generalize a similar one obtained in [4].

Lemma 2.5. (Miller and Mocanu [7]) *Let q be convex in E and let $\gamma \in \mathbb{C}$, with* $\operatorname{Re}\gamma > 0$. *If $p \in H[q(0),1] \cap Q$ and $p(z)+\gamma zp'(z)$ is univalent in E, then*

$$q(z)+\gamma zq'(z) \prec p(z)+\gamma zp'(z), \tag{11}$$

implies $q(z) \prec p(z)$, and q is the best subordinant of (11).

Lemma 2.6. (Royster [11]) *The function $q(z)=\frac{1}{(1-z)^{2ab}}$ is univalent in E if and only if $|2ab-1| \leq 1$ or $|2ab+1| \leq 1$.*

3. Main Results

Theorem 3.1. *Let q be univalent in E, with $q(0)=1$, and suppose that*

$$\operatorname{Re}\left(1+\frac{zq''(z)}{q'(z)}\right) > \max\left\{0;\ -p(p-\lambda)\operatorname{Re}\frac{1}{\alpha}\right\},\quad z \in E, \tag{12}$$

where $-\infty < \lambda < p$, $\alpha \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$, $z \in E$ and $p \in \mathbb{N}$.

If $f \in \mathcal{A}(p)$ satisfies the subordination

$$\frac{(p+\alpha)}{p}\left(\frac{z^p}{\Omega_z^{(\lambda,p)} f(z)}\right) - \frac{\alpha}{p}\frac{z^p \Omega_z^{(\lambda+1,p)} f(z)}{\left(\Omega_z^{(\lambda,p)} f(z)\right)^2} \prec q(z) + \frac{\alpha}{p(p-\lambda)} zq'(z), \quad (13)$$

then

$$\left(\frac{z^p}{\Omega_z^{(\lambda,p)} f(z)}\right) \prec q(z),$$

and q is the best dominant of (13).

Proof. Set

$$\left(\frac{z^p}{\Omega_z^{(\lambda,p)} f(z)}\right) = h(z),$$

where $h(z)$ is analytic in E with $h(0) = 1$.

A simple computation along with identity (6) shows that

$$\frac{(p+\alpha)}{p}\left(\frac{z^p}{\Omega_z^{(\lambda,p)} f(z)}\right) - \frac{\alpha}{p}\frac{z^p \Omega_z^{(\lambda+1,p)} f(z)}{\left(\Omega_z^{(\lambda,p)} f(z)\right)^2} = h(z) + \frac{\alpha}{p(p-\lambda)} zh'(z),$$

hence the subordination (13) is equivalent to

$$h(z) + \frac{\alpha}{p(p-\lambda)} zh'(z) \prec q(z) + \frac{\alpha}{p(p-\lambda)} zq'(z).$$

Combining this last relation together with Lemma 2.3 for the special case $\gamma = \frac{\alpha}{p(p-\lambda)}$ and $\mu = 1$, we obtain our result. □

Taking $q(z) = \frac{1+Az}{1+Bz}$ in Theorem 3.1, where $-1 \le B < A \le 1$, the condition (12) reduces to

$$\operatorname{Re}\frac{1-Bz}{1+Bz} > \max\left\{0;\ -p(p-\lambda)\operatorname{Re}\frac{1}{\alpha}\right\},\ \ z \in E. \quad (14)$$

It is easy to verify that the function $\varphi(\zeta) = \frac{(1-\zeta)}{(1+\zeta)}$, $|\zeta| < B$, is convex in E, and since $\varphi(\overline{\zeta}) = \overline{\varphi(\zeta)}$ for all $|\zeta| < |B|$, it follows that $\varphi(E)$ is a convex domain symmetric with respect to the real axis, hence

$$\inf\left\{\operatorname{Re}\frac{1-Bz}{1+Bz} :\ z \in E\right\} = \frac{1-|B|}{1+|B|} > 0. \quad (15)$$

Then, the inequality (14) is equivalent to

$$p(p-\lambda)\operatorname{Re}\frac{1}{\alpha} \ge \frac{|B|-1}{|B|+1},$$

hence we have the following result.

Corollary 3.2. *Let* $-\infty < \lambda < p$, $p \in \mathbb{N}$, $\alpha \in \mathbb{C}^*$ *and* $-1 \le B < A \le 1$ *with*

$$\max\left\{0;\ -p(p-\lambda)\operatorname{Re}\frac{1}{\alpha}\right\} \le \frac{1-|B|}{1+|B|}.$$

If $f \in \mathcal{A}(p)$, *and*

$$\frac{(p+\alpha)}{p}\left(\frac{z^p}{\Omega_z^{(\lambda,p)}f(z)}\right) - \frac{\alpha}{p}\frac{z^p\Omega_z^{(\lambda+1,p)}f(z)}{\left(\Omega_z^{(\lambda,p)}f(z)\right)^2} \prec \frac{1+Az}{1+Bz} + \frac{\alpha}{p(p-\lambda)}\frac{(A-B)z}{(1+Bz)^2}, \tag{16}$$

then

$$\left(\frac{z^p}{\Omega_z^{(\lambda,p)}f(z)}\right) \prec \frac{1+Az}{1+Bz},$$

and $\frac{1+Az}{1+Bz}$ *is the best dominant of (16).*

Example 3.3. For $p = 1, A = 1$ and $B = -1$. Let $-\infty < \lambda < 1$ and $\alpha \in \mathbb{C}^*$ with

$$(1-\lambda)\operatorname{Re}\frac{1}{\alpha} \ge 0$$

If $f \in \mathcal{A}$, and

$$(1+\alpha)\left(\frac{z}{\Omega_z^{(\lambda,1)}f(z)}\right) - \frac{\alpha}{p}\frac{z\Omega_z^{(\lambda+1,1)}f(z)}{\left(\Omega_z^{(\lambda,1)}f(z)\right)^2} \prec \frac{1+z}{1-z} + \frac{\alpha}{(1-\lambda)}\frac{2z}{(1-z)^2}, \tag{17}$$

then

$$\left(\frac{z}{\Omega_z^{(\lambda,1)}f(z)}\right) \prec \frac{1+z}{1-z},$$

and $\frac{1+z}{1-z}$ is the best dominant of (17).

Theorem 3.4. *Let q be univalent in E, with $q(0) = 1$ and $q(z) \neq 0$ for all $z \in E$. Let γ, $\mu \in \mathbb{C}^*$ and v, $\eta \in \mathbb{C}$, with $v + \eta \neq 0$. Let $f \in \mathcal{A}(p)$ and suppose that f and q satisfy the following conditions:*

$$\frac{(v+\eta)z^p}{v\left(\Omega_z^{(\lambda+1,p)}f(z)\right) + \eta\left(\Omega_z^{(\lambda,p)}f(z)\right)} \neq 0,\ -\infty < \lambda < p,\ p \in \mathbb{N},\ z \in E, \tag{18}$$

and

$$\operatorname{Re}\left(1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right) > 0,\ z \in E. \tag{19}$$

If

$$1+\gamma\mu\left[p-\frac{vz(\left(\Omega_z^{(\lambda+1,p)}f(z)\right)'+\eta z\left(\Omega_z^{(\lambda,p)}f(z)\right)'}{v\left(\Omega_z^{(\lambda+1,p)}f(z)\right)+\eta\left(\Omega_z^{(\lambda,p)}f(z)\right)}\right]\prec 1+\gamma\frac{zq'(z)}{q(z)},\quad (20)$$

then

$$\left[\frac{(v+\eta)z^p}{v\left(\Omega_z^{(\lambda+1,p)}f(z)\right)+\eta\left(\Omega_z^{(\lambda,p)}f(z)\right)}\right]^{\mu}\prec q(z),$$

and q is the best dominant of (20). The power is the principal one.

Proof. We begin by setting

$$\left[\frac{(v+\eta)z^p}{v\left(\Omega_z^{(\lambda+1,p)}f(z)\right)+\eta\left(\Omega_z^{(\lambda,p)}f(z)\right)}\right]^{\mu}=h(z),\ z\in E,\quad (21)$$

where $h(z)$ is analytic in E with $h(0)=1$. Differentiating Equation (21) logarithmically with respect to z, we have

$$\mu\left[p-\frac{vz(\left(\Omega_z^{(\lambda+1,p)}f(z)\right)'+\eta z\left(\Omega_z^{(\lambda,p)}f(z)\right)'}{v\left(\Omega_z^{(\lambda+1,p)}f(z)\right)+\eta\left(\Omega_z^{(\lambda,p)}f(z)\right)}\right]=\frac{zh'(z)}{h(z)}.$$

To prove our result we use Lemma 2.2. Consider in this Lemma

$$\theta(w)=1 \text{ and } \varphi(w)=\frac{\gamma}{w},$$

then θ is analytic in $\mathbb{C}$ and $\varphi(w)\neq 0$ is analytic in $\mathbb{C}^*$. Also, if we let

$$Q(z)=zq'(z)\varphi(q(z))=\gamma\frac{zq'(z)}{q(z)},$$

and

$$g(z)=\theta(q(z))+Q(z)=1+\gamma\frac{zq'(z)}{q(z)},$$

then, since $Q(0)=0$ and $Q'(0)\neq 0$, the assumption (19) would yield that Q is a starlike function in E. From (19), we have

$$\operatorname{Re}\frac{zg'(z)}{Q(z)}=\operatorname{Re}\left(1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}\right)>0,\ z\in E,$$

and by using Lemma 2.2, we deduce that the subordination (20) implies that $h(z)\prec q(z)$, and the function q is the best dominant of (20). $\square$

In particular, $\nu = 0$, $\eta = \gamma = 1$ and $q(z) = \frac{1+Az}{1+Bz}$ in the above Theorem 3.4, it is easy to see that the assumption (19) holds whenever $-1 \le A < B \le 1$, which leads to the next result:

Corollary 3.5. *Let $-1 \le A < B \le 1$ and $\mu \in \mathbb{C}^*$. Let $f \in \mathcal{A}(p)$ and suppose that $\frac{z^p}{\Omega_z^{(\lambda,p)} f(z)} \neq 0$, $z \in E$.*

If

$$1 + \mu \left[p - \frac{z\left(\Omega_z^{(\lambda,p)} f(z)\right)'}{\left(\Omega_z^{(\lambda,p)} f(z)\right)} \right] \prec 1 + \frac{(A-B)z}{(1+Az)(1+Bz)}, \tag{22}$$

then

$$\left(\frac{z^p}{\Omega_z^{(\lambda,p)} f(z)} \right)^{\mu} \prec \frac{1+Az}{1+Bz},$$

and $\frac{1+Az}{1+Bz}$ is the best dominant of (22). The power is the principal one.

Putting $\nu = 0$, $\eta = p = 1, \lambda = 0$, $\gamma = \frac{1}{ab}$, $a, b \in \mathbb{C}^*$, $\mu = a$, and $q(z) = \frac{1}{(1-z)^{2ab}}$ in Theorem 3.4, then combining this together with Lemma 2.6, we have the next result.

Corollary 3.6. *Let $a, b \in \mathbb{C}^*$ such that $|2ab - 1| \le 1$ or $|2ab + 1| \le 1$. Let $f \in \mathcal{A}$ and let $\frac{z}{f(z)} \neq 0$ for all $z \in E$. If*

$$1 + \frac{1}{b}\left(1 - \frac{zf'(z)}{f(z)}\right) \prec \frac{1+z}{1-z},$$

then

$$\left(\frac{z}{f(z)}\right)^{a} \prec \frac{1}{(1-z)^{2ab}}, \tag{23}$$

and $\frac{1}{(1-z)^{2ab}}$ is the best dominant of (23). The power is the principal one.

Putting $\nu = 0$, $\eta = \gamma = p = 1$, $\lambda = 0$, and $q(z) = (1+Bz)^{\frac{\mu(A-B)}{B}}$, $-1 \le B < A \le 1$, $B \neq 0$ in Theorem 3.4, and using Lemma 2.6, we have the next result.

Corollary 3.7. *Let $-1 \le B < A \le 1$ with $B \neq 0$, and suppose that $\left|\frac{\mu(A-B)}{B-1}\right| \le 1$ or $\left|\frac{\mu(A-B)}{B+1}\right| \le 1$. Let $f \in \mathcal{A}$ such that $\frac{z}{f(z)} \neq 0$ for all $z \in E$, and let $\mu \in \mathbb{C}^*$. If*

$$1 + \mu\left(1 - \frac{zf'(z)}{f(z)}\right) \prec \frac{1 + [B + \mu(A-B)]z}{1+Bz}, \tag{24}$$

then

$$\left(\frac{z}{f(z)}\right)^{\mu} \prec (1+Bz)^{\frac{\mu(A-B)}{B}},$$

and $(1+Bz)^{\frac{\mu(A-B)}{B}}$ *is the best dominant of (24). Here the power is the principal one.*

By taking $v=0$, $\eta=\gamma=p=1$, $\lambda=0$, $\gamma=\frac{e^{i\lambda}}{ab\cos\lambda}$, a, $b\in\mathbb{C}^*$, $|\lambda|<\frac{\pi}{2}$, $\mu=a$ and $q(z)=\frac{1}{(1-z)^{2ab\cos\lambda e^{-i\lambda}}}$ in Theorem 3.4, we obtain the following result.

Corollary 3.8. *Let $a,b\in\mathbb{C}^*$ and $|\lambda|<\frac{\pi}{2}$, and suppose that $\left|ab\cos\lambda e^{-i\lambda}-1\right|\leq 1$ or $\left|ab\cos\lambda e^{-i\lambda}+1\right|\leq 1$. Let $f\in\mathcal{A}$ such that $\frac{z}{f(z)}\neq 0$ for all $z\in E$. If*

$$1+\frac{e^{i\lambda}}{b\cos\lambda}\left(1-\frac{zf'(z)}{f(z)}\right)\prec\frac{1+z}{1-z}, \tag{25}$$

then

$$\left(\frac{z}{f(z)}\right)^{a}\prec\frac{1}{(1-z)^{2ab\cos\lambda e^{-i\lambda}}},$$

and $\frac{1}{(1-z)^{2ab\cos\lambda e^{-i\lambda}}}$ *is the best dominant of (25). The power is the principal one.*

Theorem 3.9. *Let q be univalent in E with $q(0)=1$, let $\mu,\gamma\in\mathbb{C}^*$ and let δ, Ω, v, $\eta\in\mathbb{C}$ with $v+\eta\neq 0$. Let $f\in\mathcal{A}(p)$ and suppose that f and q satisfy the next two conditions:*

$$\frac{(v+\eta)z^p}{v\left(\Omega_z^{(\lambda+1,p)}f(z)\right)+\eta\left(\Omega_z^{(\lambda,p)}f(z)\right)}\neq 0,\ z\in E, \tag{26}$$

and

$$\operatorname{Re}\left(1+\frac{zq''(z)}{q'(z)}\right)>\max\left\{0;-\operatorname{Re}\frac{\delta}{\gamma}\right\},\ z\in E. \tag{27}$$

If

$$\begin{aligned}\psi(z)\equiv&\left[\frac{(v+\eta)z^p}{v\left(\Omega_z^{(\lambda+1,p)}f(z)\right)+\eta\left(\Omega_z^{(\lambda,p)}f(z)\right)}\right]^{\mu} \\ &\cdot\left[\delta+\gamma\mu\left(p-\frac{vz(\left(\Omega_z^{(\lambda+1,p)}f(z)\right)'+\eta z\left(\Omega_z^{(\lambda,p)}f(z)\right)'}{v\left(\Omega_z^{(\lambda+1,p)}f(z)\right)+\eta\left(\Omega_z^{(\lambda,p)}f(z)\right)}\right)\right]+\Omega,\end{aligned} \tag{28}$$

and

$$\psi(z)\prec\delta q(z)+\gamma zq'(z)+\Omega, \tag{29}$$

then

$$\left[\frac{(v+\eta)z^p}{v\left(\Omega_z^{(\lambda+1,p)}f(z)\right)+\eta\left(\Omega_z^{(\lambda,p)}f(z)\right)}\right]^\mu \prec q(z),$$

and q is the best dominant of (29). All the powers are the principal ones.

Proof. We begin by setting

$$\left[\frac{(v+\eta)z^p}{v\left(\Omega_z^{(\lambda+1,p)}f(z)\right)+\eta\left(\Omega_z^{(\lambda,p)}f(z)\right)}\right]^\mu = h(z). \tag{30}$$

Then $h(z)$ is analytic in E with $h(0)=1$. Logarithmic differentiating of (30) yields

$$\mu\left(p-\frac{vz(\left(\Omega_z^{(\lambda+1,p)}f(z)\right)'+\eta z\left(\Omega_z^{(\lambda,p)}f(z)\right)'}{v\left(\Omega_z^{(\lambda+1,p)}f(z)\right)+\eta\left(\Omega_z^{(\lambda,p)}f(z)\right)}\right)=\frac{zh'(z)}{h(z)},$$

and hence

$$\mu h(z)\left(p-\frac{vz(\left(\Omega_z^{(\lambda+1,p)}f(z)\right)'+\eta z\left(\Omega_z^{(\lambda,p)}f(z)\right)'}{v\left(\Omega_z^{(\lambda+1,p)}f(z)\right)+\eta\left(\Omega_z^{(\lambda,p)}f(z)\right)}\right)=zh'(z).$$

Let us consider the functions:

$$\begin{aligned}\theta(w) &= \delta w+\Omega, \quad \varphi(w)=\gamma,\ w\in\mathbb{C},\\ Q(z) &= zq'(z)\varphi(q(z)=\gamma zq'(z),\ z\in E,\end{aligned}$$

and

$$g(z)=\theta(q(z)+Q(z)=\delta q(z)+\gamma zq'(z)+\Omega,\ z\in E.$$

From the assumption (27) we see that Q is starlike in E and, that

$$\operatorname{Re}\frac{zg'(z)}{Q(z)}=\operatorname{Re}\left(\frac{\delta}{\gamma}+1+\frac{zq''(z)}{q'(z)}\right)>0,\ z\in E,$$

thus, by applying Lemma 2.2 this completes the proof. □

Taking $q(z)=\frac{(1+Az)}{(1+Bz)}$ in Theorem 3.9, where $-1\le B<A\le 1$ and according to (15), the condition (27) becomes

$$\max\left\{0;-\operatorname{Re}\frac{\delta}{\gamma}\right\}\le\frac{1-|B|}{1+|B|}.$$

Hence, for the special case $v=1=\gamma$, $\eta=0$, we have the following result:

Corollary 3.10. *Let* $-1 \le B < A \le 1$ *and let* $\delta \in \mathbb{C}$ *with*

$$\max\{0; -\operatorname{Re}\delta\} \le \frac{1-|B|}{1+|B|}.$$

Let $f \in \mathcal{A}(p)$ *and suppose that*

$$\frac{z^p}{\Omega_z^{(\lambda+1,p)} f(z)} \neq 0,\ z \in E,\ -\infty < \lambda < p,\ p \in \mathbb{N},$$

and let $\mu \in \mathbb{C}^*$. *If*

$$\left[\frac{z^p}{\Omega_z^{(\lambda+1,p)} f(z)}\right]^{\mu}\left[\delta+\mu\left(p-\frac{z(\Omega_z^{(\lambda,p)} f(z))'}{\Omega_z^{(\lambda,p)} f(z)}\right)\right]+\Omega \prec \delta\frac{1+Az}{1+Bz}+\Omega+\frac{z(A-B)}{(1+Bz)^2}, \tag{31}$$

then

$$\left(\frac{z^p}{\Omega_z^{(\lambda+1,p)} f(z)}\right)^{\mu} \prec \frac{1+Az}{1+Bz},$$

and $\frac{1+Az}{1+Bz}$ *is the best dominant of (31). All the powers are the principal ones.*

Taking $v = 0$, $\eta = \gamma = p = 1$, $\lambda = 0$ and $q(z) = \frac{1+z}{1-z}$ in Theorem 3.9, we obtain the next result.

Corollary 3.11. *Let* $f \in \mathcal{A}$ *such that* $\frac{z}{f(z)} \neq 0$ *for all* $z \in E$, *and let* $\mu \in \mathbb{C}^*$. *If*

$$\left[\frac{z}{f(z)}\right]^{\mu}\left[\delta+\mu\left(1-\frac{zf'(z)}{f(z)}\right)\right]+\Omega \prec \delta\frac{1+z}{1-z}+\Omega+\frac{2z}{(1-z)^2}, \tag{32}$$

then

$$\left[\frac{z}{f(z)}\right]^{\mu} \prec \frac{1+z}{1-z},$$

and $\frac{1+z}{1-z}$ *is the best dominant of (32). All the powers are the principal ones.*

4. Superordination and Sandwich results

Theorem 4.1. *Let* q *be convex in* E *with* $q(0) = 1$, $-\infty < \lambda < p$, $p \in \mathbb{N}$. *Let* $\alpha \in \mathbb{C}^*$ *with* $(p-\lambda)\operatorname{Re}\alpha > 0$. *Let* $f \in \mathcal{A}(p)$ *and suppose that*

$$\frac{z^p}{\Omega_z^{(\lambda,p)} f(z)} \in H[q(0), 1] \cap Q.$$

If the function

$$\frac{(p+\alpha)}{p}\left(\frac{z^p}{\Omega_z^{(\lambda,p)}f(z)}\right)-\frac{\alpha}{p}\frac{z^p\Omega_z^{(\lambda+1,p)}f(z)}{\left(\Omega_z^{(\lambda,p)}f(z)\right)^2}$$

is univalent in the unit disc E, and

$$q(z)+\frac{\alpha}{p(p-\lambda)}zq'(z)\prec\frac{(p+\alpha)}{p}\left(\frac{z^p}{\Omega_z^{(\lambda,p)}f(z)}\right)-\frac{\alpha}{p}\frac{z^p\Omega_z^{(\lambda+1,p)}f(z)}{\left(\Omega_z^{(\lambda,p)}f(z)\right)^2},\quad(33)$$

then

$$q(z)\prec\left(\frac{z^p}{\Omega_z^{(\lambda,p)}f(z)}\right),$$

and q is the best subordinant of (33).

Proof. Set

$$\left(\frac{z^p}{\Omega_z^{(\lambda,p)}f(z)}\right)=h(z),\ z\in E.$$

Then $h(z)$ is analytic in E with $h(0)=1$.Taking logarithmic differentiation with respect z, we have

$$p-z\left(\frac{(\Omega_z^{(\lambda,p)}f(z))'}{\Omega_z^{(\lambda,p)}f(z)}\right)=\frac{zh'(z)}{h(z)}.\quad(34)$$

A simple computation together with (6) show that

$$h(z)+\frac{\alpha}{p(p-\lambda)}zh'(z)=\frac{(p+\alpha)}{p}\left(\frac{z^p}{\Omega_z^{(\lambda,p)}f(z)}\right)-\frac{\alpha}{p}\frac{z^p\Omega_z^{(\lambda+1,p)}f(z)}{\left(\Omega_z^{(\lambda,p)}f(z)\right)^2},$$

and now, by using Lemma 2.5, we obtain the desired result. □

Taking $q(z)=\frac{1+Az}{1+Bz}$ in Theorem 4.1, where $-1\le B<A\le 1$, we obtain the next result.

Corollary 4.2. *Let q be convex in E with $q(0)=1$, let $\alpha\in\mathbb{C}^*$ with $(p-\lambda)\operatorname{Re}\alpha>0$. Let $f\in\mathcal{A}(p)$ suppose that $\frac{z^p}{\Omega_z^{(\lambda,p)}f(z)}\in H[q(0),1]\cap Q$. If the function*

$$\frac{(p+\alpha)}{p}\left(\frac{z^p}{\Omega_z^{(\lambda,p)}f(z)}\right)-\frac{\alpha}{p}\frac{z^p\Omega_z^{(\lambda+1,p)}f(z)}{\left(\Omega_z^{(\lambda,p)}f(z)\right)^2}$$

is univalent in the unit disc E, and

$$\frac{1+Az}{1+Bz}+\frac{\alpha(A-B)z}{p(p-\lambda)(1+Bz)^2}\prec\frac{(p+\alpha)}{p}\left(\frac{z^p}{\Omega_z^{(\lambda,p)}f(z)}\right)-\frac{\alpha}{p}\frac{z^p\Omega_z^{(\lambda+1,p)}f(z)}{\left(\Omega_z^{(\lambda,p)}f(z)\right)^2},\tag{35}$$

then

$$\frac{1+Az}{1+Bz}\prec\left(\frac{z^p}{\Omega_z^{(\lambda,p)}f(z)}\right),$$

and $\frac{1+Az}{1+Bz}$is the best subordinant of (35), where $-1\leq B<A\leq 1$.

Using the same techniques as in Theorem 3.9, and then applying Lemma 2.4, we have the following theorem.

Theorem 4.3. *Let q be convex in E with $q(0)=1$, let μ, $\gamma\in\mathbb{C}^*$, and let δ, Ω, v, $\eta\in\mathbb{C}$ with $v+\eta\neq 0$ and $\operatorname{Re}\frac{\delta}{\gamma}>0$. Let $f\in\mathcal{A}(p)$ and suppose that f satisfies the following conditions:*

$$\frac{(v+\eta)z^p}{v\left(\Omega_z^{(\lambda+1,p)}f(z)\right)+\eta\left(\Omega_z^{(\lambda,p)}f(z)\right)}\neq 0,\ -\infty<\lambda<p,\ p\in\mathbb{N},\ z\in E,$$

and

$$\left[\frac{(v+\eta)z^p}{v\left(\Omega_z^{(\lambda+1,p)}f(z)\right)+\eta\left(\Omega_z^{(\lambda,p)}f(z)\right)}\right]^{\mu}\in H[q(0),1]\cap Q.$$

If the function ψ given by equation (28) is univalent in E, and

$$\delta q(z)+\gamma zq'(z)+\Omega\prec\psi(z),\tag{36}$$

then

$$q(z)\prec\left[\frac{(v+\eta)z^p}{v\left(\Omega_z^{(\lambda+1,p)}f(z)\right)+\eta\left(\Omega_z^{(\lambda,p)}f(z)\right)}\right]^{\mu},$$

and q is the best subordinate of (36) (all powers are the principal ones).

Note that by combining Theorem 3.1 with Theorem 4.1 and Theorem 3.9 with Theorem 4.3, we have, respectively, the following two sandwich theorems:

Theorem 4.4. *Let q_1 and q_2 be two convex functions in E with $q_1(0)=q_2(0)=1$, let $\alpha\in\mathbb{C}^*$ with $(p-\lambda)\operatorname{Re}\alpha>0$, $-\infty<\lambda<p$, $p\in\mathbb{N}$. Let $f\in\mathcal{A}(p)$ and suppose that $\frac{z^p}{\Omega_z^{(\lambda,p)}f(z)}\in H[q(0),1]\cap Q$. If the function*

$$\frac{(p+\alpha)}{p}\left(\frac{z^p}{\Omega_z^{(\lambda,p)}f(z)}\right)-\frac{\alpha}{p}\frac{z^p\Omega_z^{(\lambda+1,p)}f(z)}{\left(\Omega_z^{(\lambda,p)}f(z)\right)^2}$$

is univalent in the unit disc E, and

$$q_1(z)+\frac{\alpha}{p(p-\lambda)}zq_1'(z)\prec\frac{(p+\alpha)}{p}\left(\frac{z^p}{\Omega_z^{(\lambda,p)}f(z)}\right)-\frac{\alpha}{p}\frac{z^p\Omega_z^{(\lambda+1,p)}f(z)}{\left(\Omega_z^{(\lambda,p)}f(z)\right)^2}$$
$$\prec q_2(z)+\frac{\alpha}{p(p-\lambda)}zq_2'(z), \quad (37)$$

then

$$q_1(z)\prec\left(\frac{z^p}{\Omega_z^{(\lambda,p)}f(z)}\right)\prec q_2(z),$$

and q_1 and q_2 are, respectively, the best subordinate and the best dominant of (37).

Theorem 4.5. *Let q_1 and q_2 be two convex functions in E with $q_1(0)=q_2(0)=1$ $-\infty<\lambda<p$, $p\in\mathbb{N}$, let $\mu,\gamma\in\mathbb{C}^*$, and let $\delta,\Omega,v,\eta\in\mathbb{C}$ with $v+\eta\neq 0$ and* $\operatorname{Re}\frac{\delta}{\gamma}>0$. *Let $f\in\mathcal{A}(p)$ satisfy the following conditions:*

$$\left[\frac{(v+\eta)z^p}{v\left(\Omega_z^{(\lambda+1,p)}f(z)\right)+\eta\left(\Omega_z^{(\lambda,p)}f(z)\right)}\right]\neq 0,\ z\in E,$$

and

$$\left[\frac{(v+\eta)z^p}{v\left(\Omega_z^{(\lambda+1,p)}f(z)\right)+\eta\left(\Omega_z^{(\lambda,p)}f(z)\right)}\right]^{\mu}\in H[q(0),1]\cap Q.$$

If the function ψ given by (28) is univalent in E, and

$$\delta q_1(z)+\gamma zq_1'(z)+\Omega\prec\psi(z)\prec\delta q_2(z)+\gamma zq_2'(z)+\Omega, \quad (38)$$

then

$$q_1(z)\prec\left[\frac{(v+\eta)z^p}{v\left(\Omega_z^{(\lambda+1,p)}f(z)\right)+\eta\left(\Omega_z^{(\lambda,p)}f(z)\right)}\right]^{\mu}\prec q_2(z),$$

and q_1 and q_2 are, respectively, the best subordinate and the best dominant of (38) (all the powers are the principal ones).

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