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A MULTIPLE MORE ACCURATE HARDY-LITTLEWOOD-POLYA INEQUALITY

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By introducing multi-parameters and conjugate exponents and using Euler-Maclaurin's summation formula, we estimate the weight coefficient and prove a multiple more accurate Hardy-Littlewood-Polya (H-L-P) inequality, which is an extension of some earlier published results. We also prove that the constant factor in the new inequality is the best possible, and obtain its equivalent forms.

1. Introduction

After H. Weyl, published the well known Hilbert's inequality ([1]), by introducing one pair of conjugate exponents (p,q) $\left(\frac{1}{p}+\frac{1}{q}=1\right)$, in 1925, Hardy and Riese ([2]) gave an extension of the Hilbert's inequality as follows: If $p>1$, $a_n, b_n \geq 0$, such that $0<\sum_{n=1}^{\infty} a_n^p<\infty$ and $0<\sum_{n=1}^{\infty} b_n^q<\infty$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n}<\frac{\pi}{\sin(\pi/p)}\left(\sum_{n=1}^{\infty} a_n^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} b_n^q\right)^{\frac{1}{q}}, \tag{1}$$

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where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. The equivalent form of (1) is as follows:

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{a_m}{m+n}\right)^p < \left[\frac{\pi}{\sin(\pi/p)}\right]^p\sum_{n=1}^{\infty}a_n^p, \tag{2}$$

where the constant factor $\left[\frac{\pi}{\sin(\pi/p)}\right]^p$ is the best possible. In 1934, Hardy et *al.* [3] proved the following more accurate equivalent Hilbert's inequalities:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n-1} < \frac{\pi}{\sin(\pi/p)}\left(\sum_{n=1}^{\infty}a_n^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}b_n^q\right)^{\frac{1}{q}}, \tag{3}$$

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{a_m}{m+n-1}\right)^p < \left[\frac{\pi}{\sin(\pi/p)}\right]^p\sum_{n=1}^{\infty}a_n^p, \tag{4}$$

where the constant factors $\frac{\pi}{\sin(\pi/p)}$ and $\left[\frac{\pi}{\sin(\pi/p)}\right]^p$ are all the best possible. Huang [4] gave an extension of (1) and (3), which is a multiple more accurate Hilbert's. Many authors ([5]-[11]) proved more accurate Hilbert-type inequalities. Yang [12] also considered the multiple Hilbert-type integral inequality.

Hardy, et *al.* [3] also gave the following equivalent inequalities:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m,n\}} < pq\left(\sum_{m=1}^{\infty}a_m^p\right)^{\frac{1}{p}}\left(\sum_{m=1}^{\infty}b_m^q\right)^{\frac{1}{q}}, \tag{5}$$

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{a_m}{\max\{m,n\}}\right)^p < (pq)^p\sum_{n=1}^{\infty}a_n^p, \tag{6}$$

where the constant factors pq and $(pq)^p$ are the best possible. Inequalities (1) and (5) are important in mathematical analysis and its applications (see Mitrinovic et *al.* [13]). Recently, for $\beta \geq -\frac{1}{4}$, Yang proved the following more accurate equivalent Hardy-Littlewood-Pólya (H-L-P) inequalities (see Yang [7]):

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m,n\}+\beta} < pq\left(\sum_{m=1}^{\infty}a_m^p\right)^{\frac{1}{p}}\left(\sum_{m=1}^{\infty}b_m^q\right)^{\frac{1}{q}}, \tag{7}$$

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{a_m}{\max\{m,n\}+\beta}\right)^p < (pq)^p\sum_{n=1}^{\infty}a_n^p, \tag{8}$$

where the constant factors pq and $(pq)^p$ are the best possible.

This paper deals with the use of multi-parameters and conjugate exponents to estimate the weight coefficient and to prove a multiple more accurate Hardy-Littlewood-Pólya (H-L-P) inequality. It is shown that the Hardy-Littlewood-Pólya (H-L-P) inequality in an extension of inequalities (5) and (7). We also prove that the constant factor in the extended inequality is the best possible and then consider its equivalent forms.

2. Some lemmas

Lemma 2.1. *If* $n \in \mathbf{N} \setminus \{1\}$, p_i, $r_i > 1\,(i = 1, \cdots, n)$, $\sum_{i=1}^{n} \frac{1}{p_i} = \sum_{i=1}^{n} \frac{1}{r_i} = 1$, $0 < \lambda \le \min_{1 \le i \le n} \{r_i\}$, $\beta \ge \frac{\sqrt{33}}{12} - \frac{3}{4}$, *then*

$$A := \prod_{i=1}^{n} \left[(m_i + \beta)^{\left(\frac{\lambda}{r_i} - 1\right)(1 - p_i)} \prod_{j=1 (j \ne i)}^{n} (m_j + \beta)^{\frac{\lambda}{r_j} - 1} \right]^{\frac{1}{p_i}} = 1. \tag{9}$$

Proof. We have

$$\begin{aligned} A &= \prod_{i=1}^{n} \left[(m_i + \beta)^{\left(\frac{\lambda}{r_i} - 1\right)(1 - p_i) + 1 - \frac{\lambda}{r_i}} \prod_{j=1}^{n} (m_j + \beta)^{\frac{\lambda}{r_j} - 1} \right]^{\frac{1}{p_i}} \\ &= \prod_{i=1}^{n} \left[(m_i + \beta)^{p_i \left(1 - \frac{\lambda}{r_i}\right)} \prod_{j=1}^{n} (m_j + \beta)^{\frac{\lambda}{r_j} - 1} \right]^{\frac{1}{p_i}} \\ &= \prod_{i=1}^{n} (m_i + \beta)^{1 - \frac{\lambda}{r_i}} \left[\prod_{j=1}^{n} (m_j + \beta)^{\frac{\lambda}{r_j} - 1} \right]^{\sum_{i=1}^{n} \frac{1}{p_i}} = 1 \end{aligned}$$

and thus (9) is valid. □

Lemma 2.2. *If* $r > 1$, $\frac{1}{r} + \frac{1}{s} = 1$, $0 < \lambda \le \min\{r, s\}$, $\beta \ge \frac{\sqrt{33}}{12} - \frac{3}{4}$, *then, for any* $n \in \mathbf{N}$, *we have the following inequality*

$$\frac{rs}{\lambda} \left[1 - O\left(\frac{1}{(n + \beta)^{\lambda / r}} \right) \right] < \sum_{m=1}^{\infty} \frac{(n + \beta)^{\frac{\lambda}{s}} (m + \beta)^{\frac{\lambda}{r} - 1}}{[\max\{n, m\} + \beta]^{\lambda}} < \frac{rs}{\lambda}. \tag{10}$$

Proof. We set $f(x) := \frac{(n+\beta)^{\frac{\lambda}{s}} (x+\beta)^{\frac{\lambda}{r} - 1}}{[\max\{n, x\} + \beta]^{\lambda}}$, $f_1(x) := (n + \beta)^{-\frac{\lambda}{r}} (x + \beta)^{\frac{\lambda}{r} - 1}$, $f_2(x) := (n + \beta)^{\frac{\lambda}{s}} (x + \beta)^{-\frac{\lambda}{s} - 1}$, $x \in (-\beta, \infty)$, we find $(-1)^i f_1^{(i)}(x) \ge 0$, $(-1)^i f_2^{(i)}(x) > 0$, $f_1^{(i)}(\infty) = f_2^{(i)}(\infty) = 0\,(i = 1, 2, 3, 4)$. Using the improved

Euler-Maclaurin's summation formula (see Yang [11]), we obtain

$$\sum_{m=1}^{n} f_1(m) \leq \int_1^n f_1(x)\,dx + \frac{1}{2}[f_1(1)+f_1(n)] + \frac{1}{12} f_1'(x)\Big|_1^n,$$
$$\sum_{m=n}^{\infty} f_2(m) < \int_n^{\infty} f_2(x)\,dx + \frac{1}{2} f_2(n) - \frac{1}{12} f_2'(n).$$

Since $f_1(n) = f_2(n)$, it follows that

$$\sum_{m=1}^{\infty} f(m) = \sum_{m=1}^{n} f_1(m) + \sum_{m=n}^{\infty} f_2(m) - f_2(n)$$
$$< \int_1^{\infty} f(x)\,dx + \frac{1}{2} f_1(1) - \frac{1}{12} f_1'(1) + \frac{1}{12}\left[f_1'(n) - f_2'(n)\right]. \tag{11}$$

By simple calculation, we find
$f_1(1) = (n+\beta)^{-\frac{\lambda}{r}}(1+\beta)^{\frac{\lambda}{r}-1}$, $f_1'(1) = \left(\frac{\lambda}{r}-1\right)(n+\beta)^{-\frac{\lambda}{r}}(1+\beta)^{\frac{\lambda}{r}-2}$,
$f_1'(n) = \left(\frac{\lambda}{r}-1\right)(n+\beta)^{-2}$, $f_2'(n) = -\left(\frac{\lambda}{s}+1\right)(n+\beta)^{-2}$, and

$$\int_1^{\infty} f(x)\,dx = \int_1^n f_1(x)\,dx + \int_n^{\infty} f_2(x)\,dx$$
$$= \frac{r}{\lambda}\left[1 - (n+\beta)^{-\frac{\lambda}{r}}(1+\beta)^{\frac{\lambda}{r}}\right] + \frac{s}{\lambda}$$
$$= \frac{rs}{\lambda} - \frac{r}{\lambda}(n+\beta)^{-\frac{\lambda}{r}}(1+\beta)^{\frac{\lambda}{r}}. \tag{12}$$

In view of inequality (11), it turns out that

$$\sum_{m=1}^{\infty} \frac{(n+\beta)^{\frac{\lambda}{s}}(m+\beta)^{\frac{\lambda}{r}-1}}{[\max\{n,m\}+\beta]^{\lambda}}$$
$$= \sum_{m=1}^{\infty} f(m) < \frac{rs}{\lambda} - \frac{r}{\lambda}(n+\beta)^{-\frac{\lambda}{r}}(1+\beta)^{\frac{\lambda}{r}} + \frac{1}{2}(n+\beta)^{-\frac{\lambda}{r}}(1+\beta)^{\frac{\lambda}{r}-1}$$
$$- \frac{1}{12}\left(\frac{\lambda}{r}-1\right)(n+\beta)^{-\frac{\lambda}{r}}(1+\beta)^{\frac{\lambda}{r}-2} + \frac{\lambda}{12}(n+\beta)^{-2}$$
$$= \frac{rs}{\lambda} - (n+\beta)^{-\frac{\lambda}{r}}(1+\beta)^{\frac{\lambda}{r}-2} R, \tag{13}$$

where

$$R := \frac{r}{\lambda}(1+\beta)^2 - \frac{1}{2}(1+\beta) + \frac{1}{12}\left(\frac{\lambda}{r}-1\right) - \frac{\lambda}{12}\left(\frac{n+\beta}{1+\beta}\right)^{\frac{\lambda}{r}-2}.$$

In view of $\frac{\lambda}{r}-2<0$, $\lambda \le \min\{r,s\}$, $1+\beta \ge \frac{3+\sqrt{33}}{12}$, we obtain

$$\begin{aligned}6R &\ge \frac{6r}{\lambda}(1+\beta)^2-3(1+\beta)+\frac{1}{2}\left(\frac{\lambda}{r}-1-\lambda\right)\\ &=\frac{6r}{\lambda}(1+\beta)^2-3(1+\beta)-\frac{1}{2}\left(\frac{\lambda}{s}+1\right)\\ &\ge 6(1+\beta)^2-3(1+\beta)-1\ge 0.\end{aligned}$$

By (13), we have the right-hand side of (10). On the other hand, it is obvious that $f(x)$ is decreasing in $(-\beta,\infty)$ and $f(x)$ is strictly decreasing in (n,∞), it follows from (12) that

$$\sum_{m=1}^{\infty} f(m) > \int_1^{\infty} f(x)\,dx = \frac{rs}{\lambda}-\frac{r}{\lambda}(n+\beta)^{-\frac{\lambda}{r}}(1+\beta)^{\frac{\lambda}{r}}.$$

Hence, the proof of the left-hand side of (10) is complete. □

Lemma 2.3. *In view of the assumption of Lemma 1, we define the weight coefficients* $\omega_i(m_i)=\omega(m_i;r_1,\cdots,r_n)$ *by*

$$\omega_i(m_i) := (m_i+\beta)^{\frac{\lambda}{r_i}}\sum_{m_n=1}^{\infty}\cdots\sum_{m_{i+1}=1}^{\infty}\sum_{m_{i-1}=1}^{\infty}\cdots\sum_{m_1=1}^{\infty}\frac{\prod_{j=1(j\neq i)}^{n}(m_j+\beta)^{\frac{\lambda}{r_j}-1}}{\left[\max_{1\le j\le n}\{m_j\}+\beta\right]^{\lambda}}, \tag{14}$$

where $i=1,\cdots,n$*, then, there exists* $\delta_n>0$*, such that*

$$\begin{aligned}&\frac{1}{\lambda^{n-1}}\prod_{i=1}^{n} r_i\left[1-O\left(\frac{1}{(m_n+\beta)^{\delta_n}}\right)\right]\\ &\quad< \omega_n(m_n) = (m_n+\beta)^{\frac{\lambda}{r_n}}\sum_{m_{n-1}=1}^{\infty}\cdots\sum_{m_1=1}^{\infty}\frac{\prod_{j=1}^{n-1}(m_j+\beta)^{\frac{\lambda}{r_j}-1}}{\left[\max_{1\le j\le n}\{m_j\}+\beta\right]^{\lambda}}\\ &\quad< \frac{1}{\lambda^{n-1}}\prod_{i=1}^{n} r_i.\end{aligned} \tag{15}$$

Moreover, for any $i\in\{1,\cdots,n\}$*, it follows that*

$$\omega_i(m_i) < \frac{1}{\lambda^{n-1}}\prod_{i=1}^{n} r_i. \tag{16}$$

Proof. We prove (15) by mathematical induction. For $n=2$, we set $r=r_1$ and $s=r_2$ satisfying $\frac{1}{r}+\frac{1}{s}=1$. Putting $m=m_1$, $\delta_2=\frac{\lambda}{r}>0$, we have

$$\omega_2(m_2)=\sum_{m_1=1}^{\infty}\frac{(m_1+\beta)^{\frac{\lambda}{r_1}-1}(m_2+\beta)^{\frac{\lambda}{r_2}}}{\left[\max_{1\le j\le 2}\{m_j\}+\beta\right]^{\lambda}}=\sum_{m=1}^{\infty}\frac{(m_2+\beta)^{\frac{\lambda}{s}}(m+\beta)^{\frac{\lambda}{r}-1}}{\left[\max\{m_2,m\}+\beta\right]^{\lambda}}.$$

Using inequality (10) shows that (15) is true.

We assume that, for $n(\geq 2)$, (15) is valid, then, for $n+1$, setting $m_{j_0} = \max_{2\leq j\leq n+1}\{m_j\}$ $(\geq m_{n+1})$, $s_1 = \left(1-\frac{1}{r_1}\right)^{-1}$, by (10), we have the following

$$\frac{r_1 s_1}{\lambda}\left[1-O_1\left(\frac{1}{(m_{j_0}+\beta)^{\lambda/r_1}}\right)\right] < \sum_{m_1=1}^{\infty}\frac{(m_{j_0}+\beta)^{\frac{\lambda}{s_1}}(m_1+\beta)^{\frac{\lambda}{r_1}-1}}{\left[\max\{m_{j_0},m_1\}+\beta\right]^{\lambda}} < \frac{r_1 s_1}{\lambda}. \tag{17}$$

Setting $\widetilde{\lambda} = \frac{\lambda}{s_1}$, $\widetilde{r}_j = \frac{r_{j+1}}{s_1}$, $\widetilde{m}_j = m_{j+1}\,(j = 1,\cdots,n)$, we find $\sum_{j=1}^{n}\frac{1}{\widetilde{r}_j} = 1$, $\widetilde{\lambda} < \min_{1\leq i\leq n}\{\widetilde{r}_i\}$. In view of (17) and the assumption of induction, it follows that

$$\begin{aligned}
\omega_{n+1}(m_{n+1}) &= (\widetilde{m}_n+\beta)^{\frac{\widetilde{\lambda}}{\widetilde{r}_n}}\sum_{\widetilde{m}_{n-1}=1}^{\infty}\cdots\sum_{\widetilde{m}_1=1}^{\infty}\frac{\prod_{j=1}^{n-1}(\widetilde{m}_j+\beta)^{\widetilde{\lambda}/\widetilde{r}_j-1}}{\left[\max_{1\leq j\leq n}\{\widetilde{m}_j\}+\beta\right]^{\widetilde{\lambda}}} \\
&\quad\times\left\{\sum_{m_1=1}^{\infty}\frac{(m_{j_0}+\beta)^{\frac{\lambda}{s_1}}(m_1+\beta)^{\frac{\lambda}{r_1}-1}}{\left[\max\{m_{j_0},m_1\}+\beta\right]^{\lambda}}\right\} \\
&< (\widetilde{m}_n+\beta)^{\frac{\widetilde{\lambda}}{\widetilde{r}_n}}\sum_{\widetilde{m}_{n-1}=1}^{\infty}\cdots\sum_{\widetilde{m}_1=1}^{\infty}\frac{\prod_{j=1}^{n-1}(\widetilde{m}_j+\beta)^{\widetilde{\lambda}/\widetilde{r}_j-1}}{\left[\max_{1\leq j\leq n}\{\widetilde{m}_j\}+\beta\right]^{\widetilde{\lambda}}}\cdot\frac{r_1 s_1}{\lambda} \\
&< \frac{1}{\widetilde{\lambda}^{n-1}}\prod_{i=1}^{n}\widetilde{r}_i\cdot\frac{r_1 s_1}{\lambda} = \frac{1}{\lambda^n}\prod_{i=1}^{n+1}r_i, \qquad (18)
\end{aligned}$$

$$\begin{aligned}
\omega_{n+1}(m_{n+1}) &> (\widetilde{m}_n+\beta)^{\frac{\widetilde{\lambda}}{\widetilde{r}_n}}\sum_{\widetilde{m}_{n-1}=1}^{\infty}\cdots\sum_{\widetilde{m}_1=1}^{\infty}\frac{\prod_{j=1}^{n-1}(\widetilde{m}_j+\beta)^{\widetilde{\lambda}/\widetilde{r}_j-1}}{\left[\max_{1\leq j\leq n}\{\widetilde{m}_j\}+\beta\right]^{\widetilde{\lambda}}}\cdot\frac{r_1 s_1}{\lambda} \\
&\quad\times\left[1-O_1\left(\frac{1}{(m_{j_0}+\beta)^{\lambda/r_1}}\right)\right] \\
&\geq \frac{r_1 s_1}{\lambda}\left[(\widetilde{m}_n+\beta)^{\frac{\widetilde{\lambda}}{\widetilde{r}_n}}\sum_{\widetilde{m}_{n-1}=1}^{\infty}\cdots\sum_{\widetilde{m}_1=1}^{\infty}\frac{\prod_{j=1}^{n-1}(\widetilde{m}_j+\beta)^{\widetilde{\lambda}/\widetilde{r}_j-1}}{\left[\max_{1\leq j\leq n}\{\widetilde{m}_j\}+\beta\right]^{\widetilde{\lambda}}}-\gamma\right] \\
&> \frac{1}{\lambda^n}\prod_{i=1}^{n+1}r_i\left[1-\widetilde{O}_2\left(\frac{1}{(\widetilde{m}_n+\beta)^{\widetilde{\delta}_n}}\right)\right]-\frac{r_1 s_1}{\lambda}\gamma, \qquad (19)
\end{aligned}$$

where $\widetilde{\delta}_n > 0$ and

$$0 < \gamma = (\widetilde{m}_n + \beta)^{\frac{\widetilde{\lambda}}{\widetilde{r}_n}} \sum_{\widetilde{m}_{n-1}=1}^{\infty} \cdots \sum_{\widetilde{m}_1=1}^{\infty} \frac{\prod_{j=1}^{n-1} (\widetilde{m}_j + \beta)^{\widetilde{\lambda}/\widetilde{r}_j - 1}}{\left[\max_{1 \le j \le n} \{\widetilde{m}_j\} + \beta\right]^{\widetilde{\lambda}}} \widetilde{O}_1 \left(\frac{1}{(m_{n+1} + \beta)^{\frac{\lambda}{r_1}}} \right)$$
$$< \frac{1}{s_1 \lambda^{n-1}} \prod_{i=2}^{n+1} r_i \widetilde{O}_1 \left(\frac{1}{(m_{n+1} + \beta)^{\lambda/r_1}} \right)$$

Setting $\delta_{n+1} = \min\left\{\widetilde{\delta}_n, \frac{\lambda}{r_1}\right\} > 0$, and using (19), we obtain

$$\omega_{n+1}(m_{n+1}) > \frac{1}{\lambda^n} \prod_{i=1}^{n+1} r_i \left[1 - O\left(\frac{1}{(m_{n+1} + \beta)^{\delta_{n+1}}} \right) \right]. \tag{20}$$

Hence, (15) is valid by (18) and (20) and the mathematical induction.

Setting $\widetilde{m}_j = m_j$, $\widetilde{r}_j = r_j\,(j = 1, \cdots, i-1)$, $\widetilde{m}_j = m_{j+1}$, $\widetilde{r}_j = r_{j+1}\,(j = i, \cdots, n-1)$, $\widetilde{m}_n = m_i, \widetilde{r}_n = r_i$, we obtain

$$\omega_i(m_i) = \omega(\widetilde{m}_n; \widetilde{r}_1, \cdots, \widetilde{r}_n) < \frac{1}{\lambda^{n-1}} \prod_{i=1}^{n} \widetilde{r}_i = \frac{1}{\lambda^{n-1}} \prod_{i=1}^{n} r_i.$$

Thus, inequality (16) follows. □

3. Main results

Theorem 3.1. *Suppose that* $n \in \mathbf{N} \backslash \{1\}$, p_i, $r_i > 1\,(i = 1, \cdots, n)$, $\sum_{i=1}^{n} \frac{1}{p_i} = \sum_{i=1}^{n} \frac{1}{r_i} = 1$, $\frac{1}{q_n} = 1 - \frac{1}{p_n}$, $0 < \lambda \le \min_{1 \le i \le n}\{r_i\}$, $\beta \ge \frac{\sqrt{33}}{12} - \frac{3}{4} = -0.27^+$, $a_{m_i}^{(i)} \ge 0\,(m_i \in \mathbf{N})$, *such that*

$$0 < \sum_{m_i=1}^{\infty} (m_i + \beta)^{p_i\left(1 - \frac{\lambda}{r_i}\right) - 1} \left(a_{m_i}^{(i)}\right)^{p_i} < \infty \qquad (i = 1, \cdots, n), \tag{21}$$

then, the following equivalent inequalities hold:

$$I=\sum_{m_n=1}^{\infty}\cdots\sum_{m_1=1}^{\infty}\frac{1}{\left[\max_{1\le j\le n}\left\{m_j\right\}+\beta\right]^{\lambda}}\prod_{i=1}^{n}a_{m_i}^{(i)}$$
$$<\frac{1}{\lambda^{n-1}}\prod_{i=1}^{n}r_i\left\{\sum_{m_i=1}^{\infty}(m_i+\beta)^{p_i\left(1-\frac{\lambda}{r_i}\right)-1}\left(a_{m_i}^{(i)}\right)^{p_i}\right\}^{\frac{1}{p_i}},\tag{22}$$

$$J=\left\{\sum_{m_n=1}^{\infty}(m_n+\beta)^{\frac{\lambda q_n}{r_n}-1}\left[\sum_{m_{n-1}=1}^{\infty}\cdots\sum_{m_1=1}^{\infty}\frac{\prod_{i=1}^{n-1}a_{m_i}^{(i)}}{\left[\max_{1\le j\le n}\left\{m_j\right\}+\beta\right]^{\lambda}}\right]^{q_n}\right\}^{\frac{1}{q_n}}$$
$$<\frac{r_n}{\lambda^{n-1}}\prod_{i=1}^{n-1}r_i\left\{\sum_{m_i=1}^{\infty}(m_i+\beta)^{p_i\left(1-\frac{\lambda}{r_i}\right)-1}\left(a_{m_i}^{(i)}\right)^{p_i}\right\}^{\frac{1}{p_i}}.\tag{23}$$

Proof. Since $\frac{1}{p_n}+\frac{1}{q_n}=1$, we use (9) and Hölder's inequality (see Kuang [14]) to find

$$\left[\sum_{m_{n-1}=1}^{\infty}\cdots\sum_{m_1=1}^{\infty}\frac{\prod_{i=1}^{n-1}a_{m_i}^{(i)}}{\left[\max_{1\le j\le n}\left\{m_j\right\}+\beta\right]^{\lambda}}\right]^{q_n}$$
$$=\left\{\sum_{m_{n-1}=1}^{\infty}\cdots\sum_{m_1=1}^{\infty}\frac{1}{\left[\max_{1\le j\le n}\left\{m_j\right\}+\beta\right]^{\lambda}}\right.$$
$$\times\left[(m_n+\beta)^{\left(\frac{\lambda}{r_n}-1\right)(1-p_n)}\prod_{j=1}^{n-1}(m_j+\beta)^{\frac{\lambda}{r_j}-1}\right]^{\frac{1}{p_n}}$$
$$\left.\times\prod_{i=1}^{n-1}\left[(m_i+\beta)^{\left(\frac{\lambda}{r_i}-1\right)(1-p_i)}\prod_{j=1(j\ne i)}^{n}(m_j+\beta)^{\frac{\lambda}{r_j}-1}\right]^{\frac{1}{p_i}}a_{m_i}^{(i)}\right\}^{q_n}$$
$$\le\left\{\omega_n(m_n)(m_n+\beta)^{p_n\left(1-\frac{\lambda}{r_n}\right)-1}\right\}^{\frac{q_n}{p_n}}\sum_{m_{n-1}=1}^{\infty}\cdots\sum_{m_1=1}^{\infty}\frac{1}{\left[\max_{1\le j\le n}\left\{m_j\right\}+\beta\right]^{\lambda}}$$
$$\times\prod_{i=1}^{n-1}\left[(m_i+\beta)^{\left(\frac{\lambda}{r_i}-1\right)(1-p_i)}\prod_{j=1(j\ne i)}^{n}(m_j+\beta)^{\frac{\lambda}{r_j}-1}\right]^{\frac{q_n}{p_i}}\left(a_{m_i}^{(i)}\right)^{q_n}$$
$$\le\left(\frac{\prod_{i=1}^{n}r_i}{\lambda^{n-1}}\right)^{\frac{q_n}{p_n}}(m_n+\beta)^{1-\frac{\lambda q_n}{r_n}}\sum_{m_{n-1}=1}^{\infty}\cdots\sum_{m_1=1}^{\infty}\frac{1}{\left[\max_{1\le j\le n}\left\{m_j\right\}+\beta\right]^{\lambda}}$$
$$\times\prod_{i=1}^{n-1}\left[(m_i+\beta)^{\left(\frac{\lambda}{r_i}-1\right)(1-p_i)}\prod_{j=1(j\ne i)}^{n}(m_j+\beta)^{\frac{\lambda}{r_j}-1}\right]^{\frac{q_n}{p_i}}\left(a_{m_i}^{(i)}\right)^{q_n}.$$

Thus,

$$J \leq \left(\frac{\prod_{i=1}^{n} r_i}{\lambda^{n-1}}\right)^{\frac{1}{p_n}} \left\{ \sum_{m_n=1}^{\infty} \sum_{m_{n-1}=1}^{\infty} \cdots \sum_{m_1=1}^{\infty} \frac{1}{\left[\max_{1\leq j\leq n}\left\{m_j\right\}+\beta\right]^{\lambda}} \right.$$
$$\left. \times \prod_{i=1}^{n-1} \left[(m_i+\beta)^{\left(\frac{\lambda}{r_i}-1\right)(1-p_i)} \prod_{j=1(j\neq i)}^{n} (m_j+\beta)^{\frac{\lambda}{r_j}-1} \right]^{\frac{q_n}{p_i}} \left(a_{m_i}^{(i)}\right)^{q_n} \right\}^{\frac{1}{q_n}}$$
$$= \left(\frac{\prod_{i=1}^{n} r_i}{\lambda^{n-1}}\right)^{\frac{1}{p_n}} \left\{ \sum_{m_{n-1}=1}^{\infty} \cdots \sum_{m_1=1}^{\infty} \left[\sum_{m_n=1}^{\infty} \frac{(m_n+\beta)^{\frac{\lambda}{r_n}-1}}{\left[\max_{1\leq j\leq n}\left\{m_j\right\}+\beta\right]^{\lambda}} \right] \right.$$
$$\times \prod_{i=1}^{n-1} \left[(m_i+\beta)^{p_i\left(1-\frac{\lambda}{r_i}\right)-1} (m_i+\beta)^{\frac{\lambda}{r_i}} \prod_{j=1(j\neq i)}^{n-1} (m_j+\beta)^{\frac{\lambda}{r_j}-1} \right]^{\frac{q_n}{p_i}}$$
$$\left. \times \left(a_{m_i}^{(i)}\right)^{q_n} \right\}^{\frac{1}{q_n}}. \tag{24}$$

For $n \geq 3$, since $\sum_{i=1}^{n-1} \frac{q_n}{p_i} = 1$, by Hölder's inequality again in (24), it follows that

$$J \leq \left(\frac{\prod_{i=1}^{n} r_i}{\lambda^{n-1}}\right)^{\frac{1}{p_n}} \prod_{i=1}^{n-1} \left\{ \sum_{m_{n-1}=1}^{\infty} \cdots \sum_{m_1=1}^{\infty} \sum_{m_n=1}^{\infty} \frac{(m_n+\beta)^{\frac{\lambda}{r_n}-1}}{\left[\max_{1\leq j\leq n}\left\{m_j\right\}+\beta\right]^{\lambda}} \right.$$
$$\left. \times \left[(m_i+\beta)^{p_i\left(1-\frac{\lambda}{r_i}\right)-1} (m_i+\beta)^{\frac{\lambda}{r_i}} \prod_{j=1(j\neq i)}^{n-1} (m_j+\beta)^{\frac{\lambda}{r_j}-1} \right] \left(a_{m_i}^{(i)}\right)^{p_i} \right\}^{\frac{1}{p_i}}$$
$$= \left(\frac{\prod_{i=1}^{n} r_i}{\lambda^{n-1}}\right)^{\frac{1}{p_n}} \prod_{i=1}^{n-1} \left\{ \sum_{m_i=1}^{\infty} \omega_i(m_i)(m_i+\beta)^{p_i\left(1-\frac{\lambda}{r_i}\right)-1} \left(a_{m_i}^{(i)}\right)^{p_i} \right\}^{\frac{1}{p_i}}. \tag{25}$$

When $n = 2$, we directly get (25) from (24). Hence, (23) is valid by (25) and (16).

Since $\frac{1}{q_n} + \frac{1}{p_n} = 1$, by Hölder's inequality once again, it follows that

$$I = \sum_{m_n=1}^{\infty} \left[(m_n+\beta)^{\frac{\lambda}{r_n}-\frac{1}{q_n}} \sum_{m_{n-1}=1}^{\infty} \cdots \sum_{m_1=1}^{\infty} \frac{\prod_{i=1}^{n-1} a_{m_i}^{(i)}}{\left[\max_{1\leq j\leq n}\left\{m_j\right\}+\beta\right]^{\lambda}} \right]$$
$$\times \left[(m_n+\beta)^{\frac{1}{q_n}-\frac{\lambda}{r_n}} a_{m_n}^{(n)} \right]$$
$$\leq J \left\{ \sum_{m_n=1}^{\infty} (m_n+\beta)^{p_n\left(1-\frac{\lambda}{r_n}\right)-1} \left(a_{m_n}^{(n)}\right)^{p_n} \right\}^{\frac{1}{p_n}}. \tag{26}$$

Using (23) gives (22).

On the other hand, assuming that (22) is valid and setting

$$a_{m_n}^{(n)} = (m_n+\beta)^{\frac{\lambda q_n}{r_n}-1}\left[\sum_{m_{n-1}=1}^{\infty}\cdots\sum_{m_1=1}^{\infty}\frac{\prod_{i=1}^{n-1}a_{m_i}^{(i)}}{\left[\max_{1\le j\le n}\{m_j\}+\beta\right]^{\lambda}}\right]^{q_n-1},$$

it turns out that

$$J=\left\{\sum_{m_n=1}^{\infty}(m_n+\beta)^{p_n\left(1-\frac{\lambda}{r_n}\right)-1}\left(a_{m_n}^{(n)}\right)^{p_n}\right\}^{\frac{1}{q_n}} = I^{\frac{1}{q_n}}. \tag{27}$$

It follows from (22) that $J<\infty$. If $J=0$, then (23) is naturally valid. Suppose that $J>0$, by (22) we find

$$\begin{aligned}0 &< \sum_{m_n=1}^{\infty}(m_n+\beta)^{p_n\left(1-\frac{\lambda}{r_n}\right)-1}\left(a_{m_n}^{(n)}\right)^{p_n} = J^{q_n}=I\\ &< \frac{\prod_{i=1}^{n}r_i}{\lambda^{n-1}}\prod_{i=1}^{n}\left\{\sum_{m_i=1}^{\infty}(m_i+\beta)^{p_i\left(1-\frac{\lambda}{r_i}\right)-1}\left(a_{m_i}^{(i)}\right)^{p_i}\right\}^{\frac{1}{p_i}}<\infty. \end{aligned}\tag{28}$$

Dividing by $J^{\frac{q_n}{p_n}}$ in both sides of (28), we obtain

$$\begin{aligned}&\left\{\sum_{m_n=1}^{\infty}(m_n+\beta)^{p_n\left(1-\frac{\lambda}{r_n}\right)-1}\left(a_{m_n}^{(n)}\right)^{p_n}\right\}^{\frac{1}{q_n}}\\ &\qquad = J < \frac{\prod_{i=1}^{n}r_i}{\lambda^{n-1}}\prod_{i=1}^{n-1}\left\{\sum_{m_i=1}^{\infty}(m_i+\beta)^{p_i\left(1-\frac{\lambda}{r_i}\right)-1}\left(a_{m_i}^{(i)}\right)^{p_i}\right\}^{\frac{1}{p_i}}.\end{aligned}$$

Thus, (23) is valid, which is equivalent to (22). □

Theorem 3.2. *Using the assumption of Theorem 1, the same constant factor* $\frac{\prod_{i=1}^{n}r_i}{\lambda^{n-1}}$ *in (22) and (23) is the best possible.*

Proof. In view of (15) and

$$\lim_{N\to\infty}(m_n+\beta)^{\frac{\lambda}{r_n}}\sum_{m_{n-1}=1}^{N}\cdots\sum_{m_1=1}^{N}\frac{\prod_{j=1}^{n-1}(m_j+\beta)^{\frac{\lambda}{r_j}-1}}{\left[\max_{1\le j\le n}\{m_j\}+\beta\right]^{\lambda}}=\omega_n(m_n),$$

there exists $N_0\in\mathbf{N}$, such that when $N>N_0$,

$$\begin{aligned}&(m_n+\beta)^{\frac{\lambda}{r_n}}\sum_{m_{n-1}=1}^{N}\cdots\sum_{m_1=1}^{N}\frac{\prod_{j=1}^{n-1}(m_j+\beta)^{\frac{\lambda}{r_j}-1}}{\left[\max_{1\le j\le n}\{m_j\}+\beta\right]^{\lambda}}\\ &\qquad > \frac{1}{\lambda^{n-1}}\prod_{i=1}^{n}r_i\times\left[1-O\left(\frac{1}{(m_n+\beta)^{\delta_n}}\right)\right],\end{aligned}$$

where $\delta_n > 0$. We next set

$$\widetilde{a}_{m_i}^{(i)} = \begin{cases} (m_i+\beta)^{\frac{\lambda}{r_i}-1}, & m_i \leq N \\ 0, & m_i > N \end{cases} \quad (i=1,\cdots,n), \tag{29}$$

obtaining

$$\begin{aligned}
\widetilde{I} &= \sum_{m_n=1}^{\infty}\cdots\sum_{m_1=1}^{\infty}\frac{1}{\left[\max_{1\leq j\leq n}\{m_j\}+\beta\right]^{\lambda}}\prod_{i=1}^{n}\widetilde{a}_{m_i}^{(i)} \\
&= \sum_{m_n=1}^{N}\frac{(m_n+\beta)^{\frac{\lambda}{r_n}}}{m_n+\beta}\sum_{m_{n-1}=1}^{N}\cdots\sum_{m_1=1}^{N}\frac{\prod_{i=1}^{n-1}(m_i+\beta)^{\frac{\lambda}{r_i}-1}}{\left[\max_{1\leq j\leq n}\{m_j\}+\beta\right]^{\lambda}} \\
&> \sum_{m_n=1}^{N}\frac{1}{m_n+\beta}\times\frac{1}{\lambda^{n-1}}\prod_{i=1}^{n}r_i\times\left[1-O\left(\frac{1}{(m_n+\beta)^{\delta_n}}\right)\right] \\
&= \frac{1}{\lambda^{n-1}}\prod_{i=1}^{n}r_i\times\left(\sum_{m_n=1}^{N}\frac{1}{m_n+\beta}\right) \\
&\qquad\times\left\{1-\left(\sum_{m_n=1}^{N}\frac{1}{m_n+\beta}\right)^{-1}\sum_{m_n=1}^{N}O\left(\frac{1}{(m_n+\beta)^{\delta_n+1}}\right)\right\}.
\end{aligned}\tag{30}$$

If there exists a constant $k \leq \frac{1}{\lambda^{n-1}}\prod_{i=1}^{n}r_i$, such that (22) is still valid if we replace $\frac{1}{\lambda^{n-1}}\prod_{i=1}^{n}r_i$ by k, then, in particular, we have

$$\widetilde{I} < k\prod_{i=1}^{n}\left\{\sum_{m_i=1}^{\infty}(m_i+\beta)^{p_i(1-\frac{\lambda}{r_i})-1}\left(\widetilde{a}_{m_i}^{(i)}\right)^{p_i}\right\}^{\frac{1}{p_i}} = k\sum_{m_n=1}^{N}\frac{1}{m_n+\beta}. \tag{31}$$

In view of (30) and (31), it follows that

$$\frac{1}{\lambda^{n-1}}\prod_{i=1}^{n}r_i\left\{1-\left(\sum_{m_n=1}^{N}\frac{1}{m_n+\beta}\right)^{-1}\sum_{m_n=1}^{N}O\left(\frac{1}{(m_n+\beta)^{\delta_n+1}}\right)\right\} < k.$$

For $N \to \infty$, we have $\frac{1}{\lambda^{n-1}}\prod_{i=1}^{n}r_i \leq k$. Hence $k = \frac{1}{\lambda^{n-1}}\prod_{i=1}^{n}r_i$ is the best value of (22). We confirm that the constant factor $\frac{1}{\lambda^{n-1}}\prod_{i=1}^{n}r_i$ in (23) is the best possible, otherwise we can get a contradiction by (26) that the constant factor in (22) is not the best possible. □

Remark 3.3. When $n=2$, $r_1=q$, $r_2=p$, $p_1=p$, $p_2=q$, setting $\lambda=1$, (22) and (23) reduce to (7) and (8) respectively. Moreover, setting $\beta=0$, we obtain (5) and (6).

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