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SOLUTION OF WAVE-LIKE EQUATION BASED ON HAAR WAVELET

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Wavelet transform and wavelet analysis are powerful mathematical tools for many problems. Wavelet also can be applied in numerical analysis. In this paper, we apply Haar wavelet method to solve wave-like equation with initial and boundary conditions known. The fundamental idea of Haar wavelet method is to convert the differential equations into a group of algebraic equations, which involves a finite number or variables. The results and graph show that the proposed way is quite reasonable when compared to exact solution.

1. Introduction

Wavelet transform and wavelet analysis is a recently developed mathematical tool for solution of ordinary differential equations and partial differential equations. In numerical analysis, wavelet based algorithms have become an important tools because of the properties of localization. One of the popular families of wavelet is Haar wavelet. Due to its simplicity, Haar wavelet had become an effective tool for solving ordinary differential equations and partial differential equations. In solving ordinary differential equations by using Haar related method, Chen and Hsiao[2] had derived an operational matrix of integration based on Haar wavelet. Phang Chang and PhangPiau [9] also gave a simple

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matrix method to solve ordinary differential equations. Lepik [5,6,7] had solved higher order as well as nonlinear ordinary differential equations by using Haar method. G. Hariharan [4] gave a simple method to solve partial differential equations.

The wave-like models are the integral part of applied sciences and arise in various physical phenomena. Several techniques [8, 10, 11] including spectral, characteristic, modified variational iteration and Adomian's decomposition have been used for solving these problems.Most of these techniques encounter a considerable size of difficulty. He [3] developed and formulated homotopy perturbation method (HPM) by merging the standard homotopy and perturbation. The homotopy perturbation method (HPM) proved to be compatible with the versatile nature of the physical problems and has been applied to a wide class of functional equations. In this technique, the solution is given in an infinite series usually converging to an accurate solution.

In the present paper, a new direct computational method for solving wave-like equation is introduced. This method consists of reducing the problem to a set of algebraic equations by first expanding the terms, which has maximum derivative, given in the equation as Haar functions with unknown coefficients. The operational matrix of integration and product operational matrix are utilized to evaluate the coefficients of the Haar functions. The differentiation of Haar wavelet always results in impulse functions which must be avoided, the integration of Haar wavelet is preferred.Since the integration of the Haar functions vector is continuous function, the solutions obtained are continuous. One main advantage of this method is that, we don't need to solve it manually it is fully computer supported.

In this paper we consider one-dimensional wave-like equation.

$$\frac{\partial^2 y}{\partial t^2} - \frac{x^2}{2}\frac{\partial^2 y}{\partial x^2} = 0, \;\; 0 < x < 1, \;\; t > 0, \tag{1}$$

Subject to the initial conditions

$$y(x,0) = x, \;\; \dot{y}(x,0) = x^2$$

and the boundary conditions

$$y(0,t) = 0, \;\; y(1,t) = 1 + \sinh t, \;\; t > 0.$$

2. Haar wavelet

The Haar wavelet was first introduced by Alfred Haar [1] in 1910. Haar wavelet is a certain sequence of rescaled "square-shaped" function which together forms a wavelet family or basis. Haar wavelet is defined for $x \in [0\ 1]$

$$\psi(x) = \begin{cases} 1 & 0 \le x < \frac{1}{2} \\ -1 & \frac{1}{2} \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Haar wavelet family for $x \in [0\ 1]$ is defined as

$$h_i(x) = \begin{cases} 1 & \text{for} \quad x \in [\eta_1, \eta_2) \\ -1 & \text{for} \quad x \in [\eta_2, \eta_3] \\ 0 & \quad\quad \text{otherwise} \end{cases} \tag{3}$$

Where $\eta_1 = \frac{K}{m}$, $\eta_2 = \frac{K+0.5}{m}$, $\eta_3 = \frac{K+1}{m}$. The integer $m = 2^j (j = 0, 1, \ldots, J)$ indicates the level of the wavelet; $k = 0, 1, \ldots, m-1$ is the translation parameter. The maximal level of relation is J. The index i is calculated according to the formula $i = m + k + 1$; In the case of minimal values $m = 1, k = 0$, we have $i = 2$. The maximum value of i is $i = 2^{J+1} = M$. It is assume that the value $i = 1$ corresponding to the scaling function for which $h_1 = 1$ for $x \in [0\ 1]$. The interval $[A, B]$ will be divided into M subintervals, hence $\Delta x = \frac{B-A}{M}$ and the matrices are in the dimension of $M \times M$.

3. Computing $p_{i,v}(x)$

By Hsiao-Chen method [1]

$$p_{i,1}(x) = \int_0^x h_i(x)\, dx, \tag{4}$$

$$p_{i,v}(x) = \int_0^x p_{i,\ v-1}(x)\, dx, \quad v = 2, 3, \ldots \tag{5}$$

Carrying out these integrations with the aid equation (3), we have

$$p_{i,1}(x) = \begin{cases} x - \eta_1 & \text{for} \quad x \in [\eta_1, \eta_2) \\ \eta_3 - x & \text{for} \quad x \in [\eta_2, \eta_3] \\ 0 & \quad\quad \text{elsewhere} \end{cases} \tag{6}$$

$$p_{i,2}(x) = \begin{cases} \frac{1}{2}(x - \eta_1)^2 & \text{for} \quad x \in [\eta_1, \eta_2), \\ \frac{1}{4m^2} - \frac{1}{2}(\eta_3 - x)^2 & \text{for} \quad x \in [\eta_2, \eta_3), \\ \frac{1}{4m^2} & \text{for} \quad x \in [\eta_3, 1], \\ 0 & \quad\quad \text{elsewhere}, \end{cases} \tag{7}$$

4. Method for solving wave - like equation

Consider the wave-like equation (1) with the initial conditions

$$y(x,0) = x = f(x), \quad \dot{y}(x,0) = x^2, \quad 0 < x < 1$$

and the boundary conditions

$$y(0,t) = 0 = \lambda_0(t)$$

and

$$y(1,t) = 1 + \sinh t = \lambda_1(t), \quad t > 0.$$

In terms of Haar wavelet $\ddot{y}''(x, t)$ can be expanded as

$$\ddot{y}''(x, t) = \sum_{i=1}^{M} a_i h_i(x) \tag{8}$$

Where '$\cdot$' and '$''$' means differentiation with respect to t and x, respectively, Haar wavelet coefficient is constant in the subinterval $t \in [t_{n,} \, t_{n+1}]$.
On twice integration of equation (8) with respect to t from t_n to t and with respect to x from 0 to x, following equations are obtained

$$\dot{y}''(x, t) = (t - t_n) \sum_{i=1}^{M} a_i h_i(x) + \dot{y}''(x, t_n) \tag{9}$$

$$y''(x, t) = \frac{1}{2}\left(t^2 - 2tt_n + t_n^2\right) \sum_{i=1}^{M} a_i h_i(x) + (t - t_n)\dot{y}''(x, t_n) + y''(x, t_n) \tag{10}$$

$$\begin{aligned} y'(x, t) = {} & \frac{1}{2}\left(t^2 - 2\,tt_n + t_n^2\right) \sum_{i=1}^{M} a_i P_{i,1}(x) \\ & + (t - t_n)\left[\dot{y}'(x, t_n) - \dot{y}'(0, t_n)\right] + y'(x, t_n) - y'(0, t_n) + y'(0, t) \end{aligned} \tag{11}$$

$$\begin{aligned} y(x,t) = {} & \frac{1}{2}\left(t^2 - 2\,tt_n + t_n^2\right) \sum_{i=1}^{M} a_i P_{i,2}(x) \\ & + (t - t_n)\left[\dot{y}(x, t_n) - \dot{y}(0, t_n) - x\dot{y}'(0, t_n)\right] \\ & + y(x, t_n) - y(0, t_n) - x\left[y'(0, t_n) - y'(0, t)\right] + y(0,t) \end{aligned} \tag{12}$$

$$\dot{y}(x\ ,t) = (t - t_n)\sum_{i=1}^{M} a_i P_{i,2}(x) + \left[\dot{y}(x,\ t_n)\ -\ \dot{y}(0,\ t_n)\ - x\dot{y}^{'}(0,\ \ t_n)\right] + x\dot{y}^{'}(0,\ t) + \dot{y}(0\ ,t) \quad (13)$$

$$\ddot{y}(x\ ,\ t) = \sum_{i=1}^{M} a_i P_{i,2}(x) + x\,\ddot{y}^{'}(0,\ t) + \ddot{y}(0\ ,\ t) \quad (14)$$

From the initial and boundary conditions, we have the following equations as

$$\begin{aligned} y(x,0) = f(x),\quad y(0,t) = \lambda_0(t),\quad y(1,t) = \lambda_1(t),\\ y(0,t_n) = \lambda_0(t_n),\quad y(1,t_n) = \lambda_1(t_n), \end{aligned}$$

$$\begin{aligned} \dot{y}(0,\ t_n) = \lambda_0^{'}(t_n),\quad \dot{y}(1,\ t_n) = \lambda_1^{'}(t_n),\\ \ddot{y}(0,\ t_n) = \lambda_0^{''}(t_n),\quad \ddot{y}(1,\ t_n) = \lambda_1^{''}(t_n) \end{aligned}$$

Put $x = 1$ in the formula (12) and (14) and by using condition, we have

$$\begin{aligned} y^{'}(0,\ \ t) - y^{'}(0,t_n) \ \ = \ \ & -\tfrac{1}{2}\left(t^2 -\ 2tt_n + t_n{}^2\right)\textstyle\sum_{i=1}^{M} a_i P_{i,2}(1) - \\ & (t - t_n)\left[\lambda_1^{'}(t_n) - \lambda_0^{'}(t_n) - \dot{y}^{'}(0,\ \ t_n)\right] + \\ & \lambda_1(t) - \lambda_1(t_n) +\ \lambda_0(t_n) - \lambda_0(t) \end{aligned} \quad (15)$$

$$\ddot{y}^{'}(0,\ \ t) =\ -\sum_{i=1}^{M} a_i P_{i,2}(1) - \lambda_0^{''}(t) +\ \lambda_1^{''}(t) \quad (16)$$

If the equations (15) and (16) are substituted into the equations (10)– (12) and the results are discriticised by assuming $x \to x_l,\ \ t \to t_{n+1}$, we obtain

$$y^{''}(x_l\ ,t_{n+1}) = \frac{1}{2}\left(t_{n+1}^2 -\ 2t_{n+1}t_n + t_n^2\right)\sum_{i=1}^{M} a_i h_i(x_l) + (t_{n+1} - t_n)\dot{y}^{''}(x_l,\ \ t_n)\ \ +\ y^{''}(x_l\ ,t_n) \quad (17)$$

$$\begin{aligned} y^{'}(x_l\ ,t_{n+1}) = & \frac{1}{2}\left(t_{n+1}^2 -\ 2t_{n+1}t_n + t_n^2\right)\sum_{i=1}^{M} a_i P_{i,1}(x_l) \\ & + (t_{n+1} - t_n)\dot{y}^{'}(x_l,\ t_n) + y^{'}(x_l\ ,t_n) \\ & - \frac{1}{2}\left(t_{n+1}^2 -\ 2t_{n+1}t_n + t_n^2\right)\sum_{i=1}^{M} a_i P_{i,2}(1) - (t_{n+1} - t_n)\left[\lambda_1^{'}(t_n) - \lambda_0^{'}(t_n)\right] \\ & + \lambda_1(t_{n+1}) - \lambda_1(t_n) + \lambda_0(t_n) - \lambda_0(t_{n+1}) \end{aligned} \quad (18)$$

$$
\begin{aligned}
y(x_l\ ,t_{n+1}) = {} & \frac{1}{2}\left(t_{n+1}^2 -\ 2t_{n+1}t_n + t_n^2\right)\sum_{i=1}^{M} a_i P_{i,2}(x_l) \\
& + (t_{n+1}-t_n)\left[\dot{y}(x_l\ ,\ t_n) - \dot{y}(0\ ,\ t_n)\right] + y(x_l\ ,t_n) -\ y(0\ ,t_n) \\
& - \frac{x_l}{2}\left(t_{n+1}^2 -\ 2t_{n+1}t_n + t_n^2\right)\sum_{i=1}^{M} a_i P_{i,2}(1) \\
& -\ x_l(t_{n+1}-t_n)\left[\lambda_1^{'}(t_n) - \lambda_0^{'}(t_n)\right] \\
& -\ x_l\ \left[\lambda_1(t_n) - \lambda_0(t_n) +\ \lambda_0(t_{n+1}) - \lambda_1(t_{n+1})\right] + \lambda_0(t_{n+1})
\end{aligned} \tag{19}
$$

$$
\begin{aligned}
\dot{y}(x_l,\ t_{n+1}) = {} & (t_{n+1}-t_n)\sum_{i=1}^{M} a_i P_{i,2}(x_l) + \left[\dot{y}(x_l\ ,\ t_n) - \dot{y}(0\ ,\ t_n)\right] \\
& -\ x_l(t_{n+1}-t_n)\sum_{i=1}^{M} a_i P_{i,2}(1) - x_l\left[\lambda_1^{'}(t_n) - \lambda_0^{'}(t_n)\right] \\
& - x_l\left[\lambda_0^{'}(t_{n+1}) -\ \lambda_1^{'}(t_{n+1})\right] + \lambda_0^{'}(t_{n+1})
\end{aligned} \tag{20}
$$

$$
\begin{aligned}
\ddot{y}(x_l,\ t_{n+1}) = {} & \sum_{i=1}^{M} a_i\left[P_{i,2}(x_l)\ - x_l P_{i,2}(1)\right] \\
& - x_l\left[\lambda_0^{''}(t_{n+1}) - \lambda_1^{''}(t_{n+1})\right] + \lambda_0^{''}(t_{n+1})
\end{aligned} \tag{21}
$$

From equation (7) we obtain

$$
p_{i,2}(1) = \begin{cases} 0.5 & \text{if} \quad i=1 \\ \frac{1}{4m^2} & \text{if} \quad i>1 \end{cases} \tag{22}
$$

5. Numerical Solution of Wave-like equation

From equation (1)

$$
\frac{\partial^2 y}{\partial t^2} - \frac{x^2}{2}\frac{\partial^2 y}{\partial x^2} = 0, 0 < x < 1, t > 0.
$$

By Variational iteration method the exact solution is

$$
Y(x,t) =\ x + x^2 \sinh t \tag{23}
$$

After substituting value from equation (21) in Wave-like equation, we have

$$
\begin{aligned}
\textstyle\sum_{i=1}^{M} a_i\left[P_{i,2}(x_l) - x_l P_{i,2}(1)\right] \quad = \quad & \tfrac{x_l^2}{2} y^{''}(x_l\ ,t_n) + \\
& x_l\left[\lambda_0^{''}(t_{n+1}) - \lambda_1^{''}(t_{n+1})\right] - \lambda_0^{''}(t_{n+1})
\end{aligned} \tag{24}
$$

Equation (24) is algebraic form of wave-like equation. After solving these algebraic equations we can compute Haar coefficients $a_i^{\prime}s$.Then from equation (19), we obtain the value of y, which is very near to the exact solution.
This solution process is started with

$$\begin{aligned} y(x_l\ ,0) &= f(x_l) \\ y^{'}(x_l\ ,\ 0) &= f^{'}(x_l) \\ y^{''}(x_l\ ,\ 0) &= f^{''}(x_l) \end{aligned}$$

The solution of the resulting algebraic linear system of equations, which was constructed by applying the grid points, was obtained by the Matlab software.

6. Comparison between exact and Haar solution:

All the results, given in the following, J is taken as 3.

Table 1
For $t = 0.1$

x /32	**Exact solution**	**Haar solution**	**Absolute error**
1	0.03134781909182	0.03135286718812	5.0481E-06
3	0.09463037182635	0.09464453906436	1.4167E-05
5	0.15869547729541	0.15871746094060	2.1984E-05
7	0.22354313549900	0.22357163281684	2.8497E-05
9	0.28917334643712	0.28920705469308	3.3708E-05
11	0.35558611010977	0.35562372656932	3.7616E-05
13	0.42278142651695	0.42282164844556	4.0222E-05
15	0.49075929565866	0.49080082032180	4.1525E-05
17	0.55951971753490	0.55956124219804	4.1525E-05
19	0.62906269214567	0.62910291407428	4.0222E-05
21	0.69938821949097	0.69942583595052	3.7616E-05
23	0.77049629957080	0.77053000782676	3.3708E-05
25	0.84238693238516	0.84241542970300	2.8497E-05
27	0.91506011793405	0.91508210157924	2.1984E-05
29	0.98851585621747	0.98853002345548	1.4167E-05
31	1.06275414723542	1.06275919533172	5.0481E-06

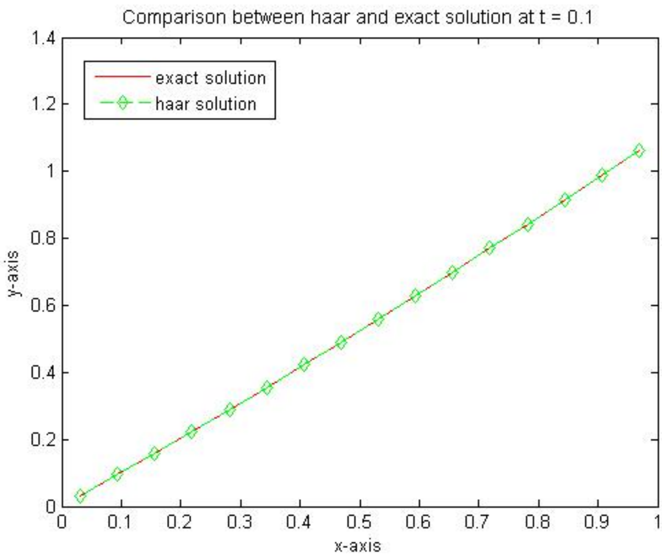


Table 2

For $t = 0.2$

x /32	Exact solution	Haar solution	Absolute error
1	0.03144661718998	0.03150905265981	6.2435E-05
3	0.09551955470983	0.09570001003031	1.8046 E-04
5	0.16116542974954	0.16145477013535	2.8934 E-04
7	0.22838424230910	0.22868514139433	3.0090 E-04
9	0.29717599238850	0.29755333801904	3.7735 E-04
11	0.36754067998777	0.36797675954323	4.3608 E-04
13	0.43947830510688	0.43997115698040	4.9285 E-04
15	0.51298886774585	0.51351501079519	5.2614 E-04
17	0.58807236790466	0.58863190632280	5.5954 E-04
19	0.66472880558333	0.66530847298528	5.7967 E-04
21	0.74295818078186	0.74368100259674	7.2282 E-04
23	0.82276049350023	0.82335714596786	5.9665 E-04
25	0.90413574373846	0.90472718677171	5.9144 E-04
27	0.98708393149654	0.98701737303047	6.6558E-05
29	1.07160505677447	1.07215867631477	5.5362 E-04
31	1.15769911957226	1.15822012518425	5.2101 E-04

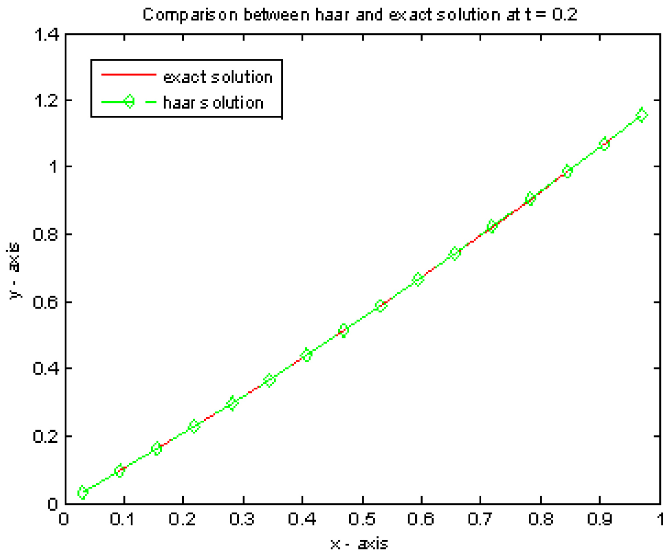
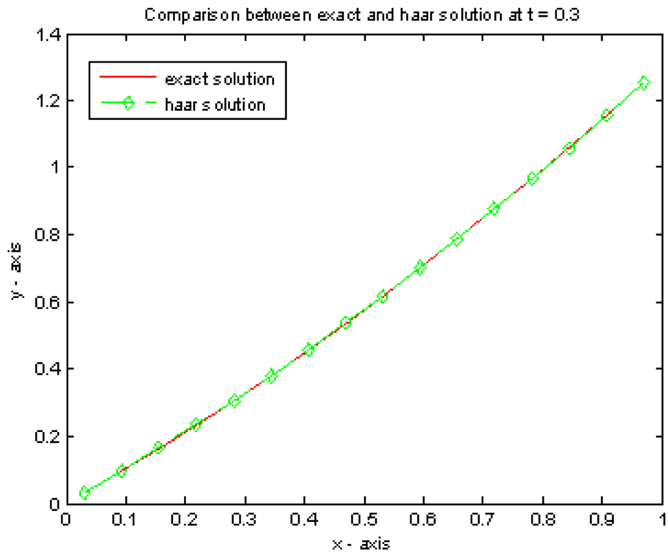


Table 3

For $t = 0.3$

x /32	Exact solution	Haar solution	Absolute error
1	0.03154738309907	0.03166642632323	1.1904 E-04
3	0.09642644789163	0.09676491960958	3.3847 E-04
5	0.16368457747674	0.16421760041579	5.3302 E-04
7	0.23332177185440	0.23384808558068	5.2631 E-04
9	0.30533803102463	0.30598080352781	6.4277 E-04
11	0.37973335498741	0.38045055332471	7.1720 E-04
13	0.45650774374274	0.45728883699836	7.8109 E-04
15	0.53566119729063	0.53645261547804	7.9142 E-04
17	0.61719371563108	0.61798905943414	7.9534 E-04
19	0.70110529876408	0.70187142771077	7.6613 E-04
21	0.78739594668964	0.78837230393617	9.7636 E-04
23	0.87606565940775	0.87670698973167	6.4133 E-04
25	0.96711443691842	0.96765605244344	5.4162 E-04
27	1.06054227922165	1.05967198811581	8.7029 E-04
29	1.15634918631743	1.15661673989004	2.6755 E-04
31	1.25453515820577	1.25462836488545	9.3207 E-05



Conclusion

In this paper, Haar wavelet approach is proposed for the Wave-like equation. Approximate solution of the wave-like equation, obtain by Matlab, are compared with exact solution.This calculation demonstrates the accuracy of the Haar wavelet solution. Applications of this method are very simple, and also it gives the implicit form of the approximate solutions of the problems. This method is also very convenient for solving the boundary value problems.Hence, the present method is a very reliable, simple, fast, minimal computation costs and flexible.

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