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A STUDY ON CERTAIN CLASS OF HARMONIC FUNCTIONS OF COMPLEX ORDER ASSOCIATED WITH CONVOLUTION

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In this paper, we introduce a new class of harmonic functions of complex order associated with convolution. We also derive the coefficient inequality, distortion theorem, extreme points, convolution conditions and convex combination for this class.

1. Introduction

A continuous complex-valued function $f = u + iv$ defined in a simply connected complex domain D is said to be harmonic in D if both u and v are real harmonic in D. Let

$$f = h + \overline{g}$$

be defined in any simply connected domain, where h and g are analytic in D. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that

$$\left|h'(z)\right| > \left|g'(z)\right|,\ z \in D \text{ (see [2])}. \tag{1}$$

Let S_H denote the class of functions of the form:

$$f = h + \overline{g}$$

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which are harmonic univalent and sense-preserving in the open unit disc $\mathbb{U} = \{z : z \in \mathbb{C}, |z| < 1\}$, where

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n,\ g(z) = \sum_{n=1}^{\infty} b_n z^n,\ |b_1| < 1, \tag{2}$$

that, is that

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b_n z^n}. \tag{3}$$

Clunie and Shell-Small [2] investigated the class S_H as well as its geometric subclasses and obtained some coefficient bounds.

For $f(z) \in S_H$ of the form (3) and $F(z)$ given by

$$F(z) = z + \sum_{n=2}^{\infty} A_n z^n + \sum_{n=1}^{\infty} \overline{B_n z^n}, \tag{4}$$

the convolution $f * F$ of the functions f and F is defined by

$$(f * F)(z) = z + \sum_{n=2}^{\infty} a_n A_n z^n + \sum_{n=1}^{\infty} \overline{b_n B_n z^n}. \tag{5}$$

For $0 < \beta \le 1$, $0 \le \lambda \le 1$, $b \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$, $|b| \le 1$, $z' = \frac{\partial}{\partial\theta}\left(z = re^{i\theta}\right)$, $0 \le r < 1$, $0 \le \theta < 2\pi$ and $f'(z) = \frac{\partial}{\partial\theta} f(z)$, let $C_H(F, b, \lambda, \beta)$ be the subclass of S_H consisting of functions $f(z)$ and $F(z)$ of the form (3) and (4) respectively, and satisfying the analytic criterion:

$$\left| \frac{1}{b} \left[\frac{z(f*F)'(z)}{z'\left[(1-\lambda)z + \lambda(f*F)(z)\right]} - 1 \right] \right| < \beta, \tag{6}$$

or, equivalently,

$$\Re\left\{ \frac{z(f*F)'(z)}{z'\left[(1-\lambda)z + \lambda(f*F)(z)\right]} \right\} > 1 - \beta\,|b|. \tag{7}$$

Let $\overline{S}_H$ denote the class of functions of the form:

$$f = h + \overline{g}, \tag{8}$$

where

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n,\ g(z) = \sum_{n=1}^{\infty} |b_n| z^n,\ |b_1| < 1. \tag{9}$$

Let

$$C_{\overline{H}}(F, b, \lambda, \beta) = C_H(F, b, \lambda, \beta) \cap \overline{S}_H.$$

We note that:
(i) $C_{\overline{H}}(F,1-\gamma,\lambda,1)=T_H(F;\lambda,\gamma)$ $(0\le\gamma<1)$ (see Murugusundaramoorthy [5]);
(ii) For $F(z)=\frac{z}{1-z}+\overline{\left(\frac{z}{1-z}\right)}$, $C_{\overline{H}}(F,b,1,\beta)=\overline{H}S^*(b,\beta)$ (see Janteng [4]);
(iii) For $F(z)=\frac{z}{1-z}+\overline{\left(\frac{z}{1-z}\right)}$, $C_{\overline{H}}(F,1-\alpha,1,1)=\overline{S}^*_H(\alpha)$ $(0\le\alpha<1)$ (see Jahangiri [3]).

Also we note that:
(i) For $F(z)=\frac{z}{1-z}+\overline{\left(\frac{z}{1-z}\right)}$, $C_{\overline{H}}(F,b,\lambda,\beta)=C_{\overline{H}}(b,\lambda,\beta)$

$$=\left\{f(z)\in\overline{S}_H:\left|\frac{1}{b}\left[\frac{zf'(z)}{z'\left[(1-\lambda)z+\lambda f(z)\right]}-1\right]\right|<\beta\right\};$$

(ii) $C_{\overline{H}}(F,(1-\alpha)\cos\varphi e^{-i\varphi},\lambda,1)=C_{\overline{H}}(F,\alpha,\varphi,\lambda)$

$$=\left\{f(z)\in\overline{S}_H:\Re\left\{e^{i\varphi}\frac{z(f*F)'(z)}{z'\left[(1-\lambda)z+\lambda(f*F)(z)\right]}\right\}>\alpha\cos\varphi\right\},$$

where $|\varphi|<\frac{\pi}{2}$ and $0\le\alpha<1$;
(iii) $C_{\overline{H}}(z-\sum\limits_{n=2}^{\infty}\sigma_n(\alpha_1)z^n+\sum\limits_{n=1}^{\infty}\overline{\sigma_n(\alpha_1)z^n},b,\lambda,\beta)=C_{\overline{H}}(\alpha_1;b,\lambda,\beta)$

$$=\left\{f(z)\in\overline{S}_H:\left|\frac{1}{b}\left[\frac{z(W^p_q[\alpha_1]f(z))'}{z'\left[(1-\lambda)z+\lambda W^p_q[\alpha_1]f(z)\right]}-1\right]\right|<\beta\right\},$$

where $\sigma_n(\alpha_1)$ is defined by

$$\sigma_n(\alpha_1)=\frac{\left[\prod\limits_{m=0}^{q}\Gamma(\beta_m)\right]\Gamma(\alpha_1+A_1(n-1))\ldots\Gamma(\alpha_p+A_p(n-1))}{(n-1)!\left[\prod\limits_{m=0}^{p}\Gamma(\alpha_m)\right]\Gamma(\beta_1+B_1(n-1))\ldots\Gamma(\beta_q+B_q(n-1))},\tag{10}$$

$\alpha_1,A_1,...,\alpha_p,A_p$ and $\beta_1,B_1,...,\beta_q,B_q$ $(p,q\in\mathbb{N})$ be positive and real parameters, $1+\sum\limits_{j=1}^{q}B_j-\sum\limits_{j=1}^{p}A_j\ge0$ and $W^p_q[\alpha_1]f(z)$ is the Wright generalized operator on harmonic functions (see Murugusundaramoorthy and Vijaya [6]), which is a generalization of many other linear operators considered earlier;
(iv) $C_{\overline{H}}(z-\sum\limits_{n=2}^{\infty}\Gamma_n(\alpha_1)z^n+\sum\limits_{n=1}^{\infty}\overline{\Gamma_n(\alpha_1)z^n},b,\lambda,\beta)=C\overline{H}_{q,s}(\alpha_1;b,\lambda,\beta)$

$$=\left\{f(z)\in\overline{S}_H:\left|\frac{1}{b}\left[\frac{z(H_{q,s}(\alpha_1,\beta_1)f(z))'}{z'\left[(1-\lambda)z+\lambda H_{q,s}(\alpha_1,\beta_1)f(z)\right]}-1\right]\right|<\beta\right\},$$

where $\Gamma_n(\alpha_1)$ is defined by

$$\Gamma_n(\alpha_1) = \frac{(\alpha_1)_{n-1}....(\alpha_q)_{n-1}}{(\beta_1)_{n-1}....(\beta_s)_{n-1}(1)_{n-1}}$$

$$(\alpha_i > 0, i = 1,...,q; \beta_j > 0, j = 1,...,s; q \leq s+1, q,s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}), \quad (11)$$

and the operator $H_{q,s}(\alpha_1, \beta_1)$ is the modified Dziok-Srivastava operator of the harmonic function (see [1]).

In this paper we obtain the coefficient inequality, distortion theorem, extreme points, convolution conditions and convex combination for functions of the class $C_{\overline{H}}(F, b, \lambda, \beta)$.

2. Main results

Unless otherwise mentioned, we assume throughout this paper that $0 < \beta \leq 1$, $0 \leq \lambda \leq 1$, $b \in \mathbb{C}^*$, $|b| \leq 1$, $z' = \frac{\partial}{\partial\theta}\left(z = re^{i\theta}\right)$, $0 \leq r < 1$, $0 \leq \theta < 2\pi$ $f'(z) = \frac{\partial}{\partial\theta} f(z)$ and $z \in \mathbb{U}$.

In the following theorem, we obtain the coefficient inequality for the class $C_H(F, b, \lambda, \beta)$.

Theorem 2.1. *Let $f = h + \overline{g}$, where h and g are given by* (2)*. Furthermore, let*

$$\sum_{n=2}^{\infty} \frac{[n - \lambda(1 - \beta|b|)]|A_n|}{\beta|b|}|a_n| + \sum_{n=1}^{\infty} \frac{[n + \lambda(1 - \beta|b|)]|B_n|}{\beta|b|}|b_n| \leq 1, \quad (12)$$

then f is sense-preserving, harmonic univalent in $\mathbb{U}$ and $f(z) \in C_H(F, b, \lambda, \beta)$.

Proof. Let $z_1 \neq z_2$, then

$$\left|\frac{f(z_1) - f(z_2)}{h(z_1) - h(z_2)}\right| \geq 1 - \left|\frac{g(z_1) - g(z_2)}{h(z_1) - h(z_2)}\right|$$

$$= 1 - \left|\frac{\sum\limits_{n=1}^{\infty} b_n\left(z_1^n - z_2^n\right)}{(z_1 - z_2) + \sum\limits_{n=2}^{\infty} a_n\left(z_1^n - z_2^n\right)}\right| > 1 - \frac{\sum\limits_{n=1}^{\infty} n|b_n|}{1 - \sum\limits_{n=2}^{\infty} n|a_n|}$$

$$\geq 1 - \frac{\sum\limits_{n=1}^{\infty} \frac{[n+\lambda(1-\beta|b|)]|B_n|}{\beta|b|}|b_n|}{1 - \sum\limits_{n=2}^{\infty} \frac{[n-\lambda(1-\beta|b|)]|A_n|}{\beta|b|}|a_n|} \geq 0.$$

This proves that f is univalent. We find that f is sense-preserving in $\mathbb{U}$, because

$$\begin{aligned}
\left|h'(z)\right| &\geq 1-\sum_{n=2}^{\infty} n\left|a_n\right||z|^{n-1}>1-\sum_{n=2}^{\infty} n\left|a_n\right| \\
&\geq 1-\sum_{n=2}^{\infty} \frac{[n-\lambda(1-\beta|b|)]\left|A_n\right|}{\beta|b|}\left|a_n\right| \geq \sum_{n=1}^{\infty} \frac{[n+\lambda(1-\beta|b|)]\left|B_n\right|}{\beta|b|}\left|b_n\right| \\
&> \sum_{n=1}^{\infty} \frac{[n+\lambda(1-\beta|b|)]\left|B_n\right|}{\beta|b|}\left|b_n\right||z|^{n-1} \geq \sum_{n=1}^{\infty} n\left|b_n\right||z|^{n-1} \geq \left|g'(z)\right|.
\end{aligned}$$

Now we show that $f(z) \in C_H(F,b,\lambda,\beta)$. We only need to show that if (12) holds then the condition (7) is satisfied. Since $\Re(w)>\delta$ if and only if $|1-\delta+w|>|1+\delta-w|$, it suffices to show that

$$\begin{aligned}
&\left|(1+\beta|b|)\left(z'\left[(1-\lambda)z+\lambda(f*F)(z)\right]\right)\right. \\
&\qquad\left.+\left[z(f*F)'(z)-z'\left[(1-\lambda)z+\lambda(f*F)(z)\right]\right]\right| \\
&\qquad -\left|(1-\beta|b|)\left(z'\left[(1-\lambda)z+\lambda(f*F)(z)\right]\right)\right. \\
&\qquad\qquad\left.-\left[z(f*F)'(z)-z'\left[(1-\lambda)z+\lambda(f*F)(z)\right]\right]\right|>0.
\end{aligned}$$

Substituting for $(f*F)(z)$ and $z(f*F)'(z)$, we obtain

$$\begin{aligned}
&\left|(1+\beta|b|)\left(z'\left[(1-\lambda)z+\lambda(f*F)(z)\right]\right)\right. \\
&\qquad\left.+\left[z(f*F)'(z)-z'\left[(1-\lambda)z+\lambda(f*F)(z)\right]\right]\right| \\
&\qquad -\left|(1-\beta|b|)\left(z'\left[(1-\lambda)z+\lambda(f*F)(z)\right]\right)\right. \\
&\qquad\left.-\left[z(f*F)'(z)-z'\left[(1-\lambda)z+\lambda(f*F)(z)\right]\right]\right| \\
&=\left|(1+\beta|b|)z+\sum_{n=2}^{\infty}(n+\lambda\beta|b|)a_nA_nz^n-\sum_{n=1}^{\infty}(n-\lambda\beta|b|)b_nB_n\bar{z}^n\right| \\
&-\left|(1-\beta|b|)z-\sum_{n=2}^{\infty}(n+\lambda\beta|b|-2\lambda)a_nA_nz^n+\sum_{n=1}^{\infty}(n-\lambda\beta|b|+2\lambda)b_nB_n\bar{z}^n\right| \\
&\geq(1+\beta|b|)|z|-\sum_{n=2}^{\infty}(n+\lambda\beta|b|)\left|a_nA_n\right||z|^n-\sum_{n=1}^{\infty}(n-\lambda\beta|b|)\left|b_nB_n\right||z|^n \\
&\qquad -(1-\beta|b|)|z|-\sum_{n=2}^{\infty}(n+\lambda\beta|b|-2\lambda)\left|a_nA_n\right||z|^n \\
&\qquad\qquad -\sum_{n=1}^{\infty}(n-\lambda\beta|b|+2\lambda)\left|b_nB_n\right||z|^n
\end{aligned}$$

$$= 2\beta |b| - \sum_{n=2}^{\infty} 2 [n - \lambda (1 - \beta |b|)] |a_n A_n| |z|^{n-1}$$
$$- \sum_{n=1}^{\infty} 2 [n + \lambda (1 - \beta |b|)] |b_n B_n| |z|^{n-1}$$
$$> 2\beta |b| \left\{ 1 - \sum_{n=2}^{\infty} \frac{[n - \lambda (1 - \beta |b|)] |a_n A_n|}{\beta |b|} - \sum_{n=1}^{\infty} \frac{[n + \lambda (1 - \beta |b|)] |b_n B_n|}{\beta |b|} \right\}. \tag{13}$$

This last expression is non-negative by (12), which completes the proof of the Theorem.

The harmonic univalent functions

$$f(z) = z + \sum_{n=2}^{\infty} \frac{\beta |b|}{[n - \lambda (1 - \beta |b|)] |A_n|} X_n z^n + \sum_{n=1}^{\infty} \frac{\beta |b|}{[n + \lambda (1 - \beta |b|)] |B_n|} \overline{Y_n z^n}, \tag{14}$$

where $\sum_{n=2}^{\infty} |X_n| + \sum_{n=1}^{\infty} |Y_n| = 1$, show that the coefficient bound given by (12) is sharp. This is because

$$\sum_{n=2}^{\infty} \frac{[n - \lambda (1 - \beta |b|)] |A_n|}{\beta |b|} |a_n| + \sum_{n=1}^{\infty} \frac{[n + \lambda (1 - \beta |b|)] |B_n|}{\beta |b|} |b_n|$$
$$= \sum_{n=2}^{\infty} \frac{[n - \lambda (1 - \beta |b|)] |A_n|}{\beta |b|} \cdot \frac{\beta |b|}{[n - \lambda (1 - \beta |b|)] |A_n|} |X_n|$$
$$+ \sum_{n=1}^{\infty} \frac{[n + \lambda (1 - \beta |b|)] |B_n|}{\beta |b|} \cdot \frac{\beta |b|}{[n + \lambda (1 - \beta |b|)] |B_n|} |Y_n|$$
$$= \sum_{n=2}^{\infty} |X_n| + \sum_{n=1}^{\infty} |Y_n| = 1.$$

□

Now, we need to prove that the condition (12) is also necessary for functions of the form (8) to be in the class $C_{\overline{H}}(F, b, \lambda, \beta)$.

Theorem 2.2. *Let $f = h + \overline{g}$, where h and g are given by* (9)*, then $f(z) \in C_{\overline{H}}(F, b, \lambda, \beta)$ if and only if*

$$\sum_{n=2}^{\infty} \frac{[n - \lambda (1 - \beta |b|)] |A_n|}{\beta |b|} |a_n| + \sum_{n=1}^{\infty} \frac{[n + \lambda (1 - \beta |b|)] |B_n|}{\beta |b|} |b_n| \leq 1. \tag{15}$$

Proof. Since $C_{\overline{H}}(F,b,\lambda) \subset C_H(F,b,\lambda,\beta)$, we only need to prove the "only if" part of this theorem. Let $f(z) \in C_{\overline{H}}(F,b,\lambda,\beta)$, then

$$\Re\left\{\frac{z(f*F)'(z)}{z'\left[(1-\lambda)z+\lambda(f*F)(z)\right]}\right\} \geq 1-\beta\,|b|,$$

that, is that

$$\Re\left\{\frac{z(f*F)'(z)-(1-\beta\,|b|)\,z'\left[(1-\lambda)z+\lambda(f*F)(z)\right]}{z'\left[(1-\lambda)z+\lambda(f*F)(z)\right]}\right\}$$

$$\geq \Re\left\{\frac{\beta|b|z-\sum\limits_{n=2}^{\infty}[n-\lambda(1-\beta|b|)]a_nA_nz^n-\sum\limits_{n=1}^{\infty}[n+\lambda(1-\beta|b|)]b_nB_n\bar{z}^n}{z-\sum\limits_{n=2}^{\infty}\lambda a_nA_nz^n+\sum\limits_{n=1}^{\infty}\lambda b_nB_n\bar{z}^n}\right\} \geq 0. \tag{16}$$

By choosing the values of z on the positive real axis where $0 \leq z = r < 1$, we have

$$\frac{\beta\,|b|-\sum\limits_{n=2}^{\infty}\left[n-\lambda\left(1-\beta\,|b|\right)\right]a_nA_nr^{n-1}-\sum\limits_{n=1}^{\infty}\left[n+\lambda\left(1-\beta\,|b|\right)\right]b_nB_nr^{n-1}}{1-\sum\limits_{n=2}^{\infty}\lambda a_nA_nr^{n-1}+\sum\limits_{n=1}^{\infty}\lambda b_nB_nr^{n-1}} \geq 0. \tag{17}$$

If the condition (15) does not hold, then the numerator in (17) is negative for $r \to 1$. This contradicts (17), then the proof of Theorem 2.2 is completed. □

Putting $F(z) = \frac{z}{1-z} + \overline{\left(\frac{z}{1-z}\right)}$ and $\lambda = 1$ in Theorem 2.2, we obtain the following corollary:

Corollary 2.3. *Let $f = h+\overline{g}$, where h and g are given by* (9)*. Then $f(z) \in \overline{H}S^*(b,\beta)$ if and only if*

$$\sum_{n=2}^{\infty}\frac{[n-1+\beta\,|b|]}{\beta\,|b|}\,|a_n| + \sum_{n=1}^{\infty}\frac{[n+1-\beta\,|b|]}{\beta\,|b|}\,|b_n| \leq 1.$$

Also f is univalent sense-preserving and harmonic in $\mathbb{U}$.

Remark 2.4. Corollary 2.3 corrects the result obtained by Janteng [4, Th. 2.1].

Putting $b = (1-\alpha)\cos\varphi e^{-i\varphi}$ ($|\varphi| < \frac{\pi}{2}, 0 \leq \alpha < 1$) and $\beta = 1$ in Theorem 2.2, we obtain the following corollary:

Corollary 2.5. *Let $f = h+\overline{g}$, where h and g are given by* (9)*. Then $f(z) \in C_{\overline{H}}(F,\alpha,\varphi,\lambda)$ if and only if*

$$\sum_{n=2}^{\infty}\frac{\left[n-\lambda\left(1-(1-\alpha)\cos\varphi\right)\right]|A_n|}{(1-\alpha)\cos\varphi}\,|a_n| + \sum_{n=1}^{\infty}\frac{\left[n+\lambda\left(1-(1-\alpha)\cos\varphi\right)\right]|B_n|}{(1-\alpha)\cos\varphi}\,|b_n| \leq 1. \tag{18}$$

Also f is univalent sense-preserving and harmonic in $\mathbb{U}$.

Distortion bounds for the class $C_{\overline{H}}(F,b,\lambda,\beta)$ are given in the following theorem.

Theorem 2.6. *Let the function $f(z)$ of the form* (8) *be in the class $C_{\overline{H}}(F,b,\lambda,\beta)$. Then for $|z|=r<1$, we have*

$$|f(z)| \le (1+|b_1|)\,r+\left(\frac{\beta\,|b|}{[2-\lambda\,(1-\beta\,|b|)]\,|A_2|}-\frac{[1+\lambda\,(1-\beta\,|b|)]\,|B_1|}{[2-\lambda\,(1-\beta\,|b|)]\,|A_2|}\,|b_1|\right) r^2 \tag{19}$$

and

$$|f(z)| \ge (1-|b_1|)\,r - \left(\frac{\beta\,|b|}{[2-\lambda\,(1-\beta\,|b|)]\,|A_2|}-\frac{[1+\lambda\,(1-\beta\,|b|)]\,|B_1|}{[2-\lambda\,(1-\beta\,|b|)]\,|A_2|}\,|b_1|\right) r^2. \tag{20}$$

The equalities in (19) *and* (20) *are attained for the functions $f(z)$ given by*

$$f(z)=(1+b_1)\,\overline{z}+\left(\frac{\beta\,|b|}{[2-\lambda\,(1-\beta\,|b|)]\,A_2}-\frac{[1+\lambda\,(1-\beta\,|b|)]\,B_1}{[2-\lambda\,(1-\beta\,|b|)]\,A_2}b_1\right)\overline{z}^2$$

and

$$f(z)=(1-b_1)\,\overline{z}-\left(\frac{\beta\,|b|}{[2-\lambda\,(1-\beta\,|b|)]\,A_2}-\frac{[1+\lambda\,(1-\beta\,|b|)]\,B_1}{[2-\lambda\,(1-\beta\,|b|)]\,A_2}b_1\right)\overline{z}^2.$$

Proof. Let $f(z)\in C_{\overline{H}}(F,b,\lambda,\beta)$. Then, we have

$$\begin{aligned}
|f(z)| &= \left|z-\sum_{n=2}^{\infty} a_n z^n+\sum_{n=1}^{\infty}\overline{b_n z^n}\right| \\
&\le (1+|b_1|)\,|z|+\sum_{n=2}^{\infty}(|a_n|+|b_n|)\,|z|^n \le (1+|b_1|)\,r+r^2\sum_{n=2}^{\infty}(|a_n|+|b_n|)
\end{aligned}$$

$$\begin{aligned}
&= (1+|b_1|)\,r+\frac{\beta\,|b|}{[2-\lambda\,(1-\beta\,|b|)]\,|A_2|}\cdot \\
&\quad \cdot\sum_{n=2}^{\infty}\left(\frac{[2-\lambda\,(1-\beta\,|b|)]\,|A_2|}{\beta\,|b|}\,|a_n|+\frac{[2-\lambda\,(1-\beta\,|b|)]\,|A_2|}{\beta\,|b|}\,|b_n|\right) r^2 \\
&\le (1+|b_1|)\,r+\frac{\beta\,|b|}{[2-\lambda\,(1-\beta\,|b|)]\,|A_2|}\cdot \\
&\quad \cdot\sum_{n=2}^{\infty}\left(\frac{[n-\lambda\,(1-\beta\,|b|)]\,|A_n|}{\beta\,|b|}\,|a_n|+\frac{[n+\lambda\,(1-\beta\,|b|)]\,|B_n|}{\beta\,|b|}\,|b_n|\right) r^2
\end{aligned}$$

$$\begin{aligned}
&\leq (1+|b_1|)\,r+\frac{\beta\,|b|}{[2-\lambda\,(1-\beta\,|b|)]\,|A_2|}\left(1-\frac{[1+\lambda\,(1-\beta\,|b|)]\,|B_1|}{\beta\,|b|}\,|b_1|\right)r^2\\
&= (1+|b_1|)\,r+\left(\frac{\beta\,|b|}{[2-\lambda\,(1-\beta\,|b|)]\,|A_2|}-\frac{[1+\lambda\,(1-\beta\,|b|)]\,|B_1|}{[2-\lambda\,(1-\beta\,|b|)]\,|A_2|}\,|b_1|\right)r^2.
\end{aligned}$$

Similarly, since

$$|f(z)|\geq(1-|b_1|)\,|z|-\sum_{n=2}^{\infty}(|a_n|+|b_n|)\,|z|^n, \tag{21}$$

we can prove (20). Thus the proof of Theorem 2.6 is completed. □

Putting $b=1-\gamma\,(0\leq\gamma<1)$, $\beta=1$ and replacing A_n, B_n by $C_n\,(n\geq 1)$ in Theorem 2.6, we obtain the following corollary:

Corollary 2.7. *Let the function $f(z)$ of the form* (8) *be in the class $T_H(F;\lambda,\gamma)$. Then for $|z|=r<1$, we have*

$$|f(z)|\leq(1+|b_1|)\,r+\left(\frac{1-\gamma}{(2-\gamma\lambda)\,|C_2|}-\frac{(1+\gamma\lambda)\,|C_1|}{(2-\gamma\lambda)\,|C_2|}\,|b_1|\right)r^2 \tag{22}$$

and

$$|f(z)|\geq(1-|b_1|)\,r-\left(\frac{1-\gamma}{(2-\gamma\lambda)\,|C_2|}-\frac{(1+\gamma\lambda)\,|C_1|}{(2-\gamma\lambda)\,|C_2|}\,|b_1|\right)r^2. \tag{23}$$

The equalities in (22) *and* (23) *are attained for the functions $f(z)$ given by*

$$f(z)=(1+b_1)\,\overline{z}+\left(\frac{1-\gamma}{(2-\gamma\lambda)\,C_2}-\frac{(1+\gamma\lambda)\,C_1}{(2-\gamma\lambda)\,C_2}b_1\right)\overline{z}^2$$

and

$$f(z)=(1-b_1)\,\overline{z}-\left(\frac{1-\gamma}{(2-\gamma\lambda)\,C_2}-\frac{(1+\gamma\lambda)\,C_1}{(2-\gamma\lambda)\,C_2}b_1\right)\overline{z}^2.$$

Remark 2.8. Corollary 2.7 corrects the result obtained by Murugusundaramoorthy [5, Theorem 3.1].

Putting $F(z)=\frac{z}{1-z}+\overline{\left(\frac{z}{1-z}\right)}$ and $\lambda=1$ in Theorem 2.6, we obtain the following corollary:

Corollary 2.9. *Let the function $f(z)$ of the form* (8) *be in the class $\overline{H}S^*(b,\beta)$. Then for $|z|=r<1$, we have*

$$|f(z)|\leq(1+|b_1|)\,r+\left(\frac{\beta\,|b|}{1+\beta\,|b|}-\frac{2-\beta\,|b|}{1+\beta\,|b|}\,|b_1|\right)r^2 \tag{24}$$

and

$$|f(z)| \geq (1-|b_1|)\, r - \left(\frac{\beta\,|b|}{1+\beta\,|b|} - \frac{2-\beta\,|b|}{1+\beta\,|b|}\,|b_1| \right) r^2. \tag{25}$$

The equalities in (24) *and* (25) *are attained for the functions* $f(z)$ *given by*

$$f(z) = (1+b_1)\,\overline{z} + \left(\frac{\beta\,|b|}{1+\beta\,|b|} - \frac{2-\beta\,|b|}{1+\beta\,|b|} b_1 \right) \overline{z}^2$$

and

$$f(z) = (1-b_1)\,\overline{z} - \left(\frac{\beta\,|b|}{1+\beta\,|b|} - \frac{2-\beta\,|b|}{1+\beta\,|b|} b_1 \right) \overline{z}^2.$$

Remark 2.10. Corollary 2.9 corrects the result obtained by Janteng [4, Theorem 2.2].

Putting $b = (1-\alpha)\cos\varphi e^{-i\varphi}$ ($|\varphi| < \frac{\pi}{2}, 0 \leq \alpha < 1$) and $\beta = 1$ in Theorem 2.6, we obtain the following corollary:

Corollary 2.11. *Let the function* $f(z)$ *of the form* (8) *be in the class* $C_{\overline{H}}(F,\alpha,\varphi,\lambda)$. *Then for* $|z| = r < 1$, *we have*

$$|f(z)| \leq (1+|b_1|)\, r + \left(\frac{(1-\alpha)\cos\varphi}{[2-\lambda(1-(1-\alpha)\cos\varphi)]|A_2|} - \frac{[1+\lambda(1-(1-\alpha)\cos\varphi)]|B_1|}{[2-\lambda(1-(1-\alpha)\cos\varphi)]|A_2|}\,|b_1| \right) r^2 \tag{26}$$

and

$$|f(z)| \geq (1-|b_1|)\, r - \left(\frac{(1-\alpha)\cos\varphi}{[2-\lambda(1-(1-\alpha)\cos\varphi)]|A_2|} - \frac{[1+\lambda(1-(1-\alpha)\cos\varphi)]|B_1|}{[2-\lambda(1-(1-\alpha)\cos\varphi)]|A_2|}\,|b_1| \right) r^2. \tag{27}$$

The equalities in (26) *and* (27) *are attained for the functions* $f(z)$ *given by*

$$f(z) = (1+b_1)\,\overline{z} + \left(\frac{(1-\alpha)\cos\varphi}{[2-\lambda(1-(1-\alpha)\cos\varphi)]A_2} - \frac{[1+\lambda(1-(1-\alpha)\cos\varphi)]B_1}{[2-\lambda(1-(1-\alpha)\cos\varphi)]A_2} b_1 \right) \overline{z}^2$$

and

$$f(z) = (1-b_1)\,\overline{z} - \left(\frac{(1-\alpha)\cos\varphi}{[2-\lambda(1-(1-\alpha)\cos\varphi)]A_2} - \frac{[1+\lambda(1-(1-\alpha)\cos\varphi)]B_1}{[2-\lambda(1-(1-\alpha)\cos\varphi)]A_2} b_1 \right) \overline{z}^2.$$

The covering result for the class $C_{\overline{H}}(F,b,\lambda,\beta)$ is given by the following corollary.

Corollary 2.12. *If* $f(z) \in C_{\overline{H}}(F,b,\lambda,\beta)$ *then we have*

$$\left\{ w : |w| < \frac{[2-\lambda(1-\beta|b|)]|A_2| - \beta|b|}{[2-\lambda(1-\beta|b|)]|A_2|} - \frac{[2-\lambda(1-\beta|b|)]|A_2| - [1+\lambda(1-\beta|b|)]|B_1|}{[2-\lambda(1-\beta|b|)]|A_2|}\,|b_1| \right\} \subset f(\mathbb{U}). \tag{28}$$

Proof. From (20) and letting $r \to 1$, we have

$$(1-|b_1|)-\left(\frac{\beta|b|}{[2-\lambda(1-\beta|b|)]|A_2|}-\frac{[1+\lambda(1-\beta|b|)]|B_1|}{[2-\lambda(1-\beta|b|)]|A_2|}|b_1|\right)$$
$$=(1-|b_1|)-\frac{1}{[2-\lambda(1-\beta|b|)]|A_2|}\{\beta|b|-[1+\lambda(1-\beta|b|)]|B_1b_1|\}$$
$$=\frac{(1-|b_1|)[2-\lambda(1-\beta|b|)]|A_2|-\{\beta|b|-[1+\lambda(1-\beta|b|)]|B_1b_1|\}}{[2-\lambda(1-\beta|b|)]|A_2|}$$

$$=\left\{\frac{[2-\lambda(1-\beta|b|)]|A_2|-\beta|b|}{[2-\lambda(1-\beta|b|)]|A_2|}\right.$$
$$\left.-\frac{[2-\lambda(1-\beta|b|)]|A_2|-[1+\lambda(1-\beta|b|)]|B_1|}{[2-\lambda(1-\beta|b|)]|A_2|}|b_1|\right\}\subset f(\mathbb{U}).$$

□

Our next theorem is on the extreme points of convex hulls of $C_{\overline{H}}(F,b,\lambda,\beta)$ denoted by *clco* $C_{\overline{H}}(F,b,\lambda,\beta)$.

Theorem 2.13. *Let $f(z)$ be given by* (8)*. Then $f\in C_{\overline{H}}(F,b,\lambda,\beta)$ if and only if $f(z)=\sum_{n=1}^{\infty}(X_nh_n(z)+Y_ng_n(z))$, where*

$$h_1(z)=z,\ h_n(z)=z-\frac{\beta|b|}{[n-\lambda(1-\beta|b|)]A_n}z^n\quad(n=2,3,\dots),\tag{29}$$

and

$$g_n(z)=z+\frac{\beta|b|}{[n+\lambda(1-\beta|b|)]B_n}\overline{z}^n\ (n=1,2,\dots)$$
$$\left(X_n\geq 0;Y_n\geq 0;\sum_{n=1}^{\infty}(X_n+Y_n)=1\right).\tag{30}$$

Proof. Let

$$f(z)=\sum_{n=1}^{\infty}(X_nh_n+Y_ng_n)$$
$$=\sum_{n=1}^{\infty}(X_n+Y_n)z-\sum_{n=2}^{\infty}\frac{\beta|b|}{[n-\lambda(1-\beta|b|)]|A_n|}X_nz^n$$
$$+\sum_{n=1}^{\infty}\frac{\beta|b|}{[n+\lambda(1-\beta|b|)]|B_n|}Y_n\overline{z}^n$$
$$=z-\sum_{n=2}^{\infty}\frac{\beta|b|}{[n-\lambda(1-\beta|b|)]A_n}X_nz^n+\sum_{n=1}^{\infty}\frac{\beta|b|}{[n+\lambda(1-\beta|b|)]B_n}Y_n\overline{z}^n.\tag{31}$$

From (15) we have

$$\sum_{n=2}^{\infty}\frac{[n-\lambda(1-\beta|b|)]|A_n|}{\beta|b|}|a_n|+\sum_{n=1}^{\infty}\frac{[n+\lambda(1-\beta|b|)]|B_n|}{\beta|b|}|b_n|$$
$$=\sum_{n=2}^{\infty}\frac{[n-\lambda(1-\beta|b|)]|A_n|}{\beta|b|}\cdot\frac{\beta|b|}{[n-\lambda(1-\beta|b|)]|A_n|}X_n$$
$$+\sum_{n=1}^{\infty}\frac{[n+\lambda(1-\beta|b|)]|B_n|}{\beta|b|}\cdot\frac{\beta|b|}{[n+\lambda(1-\beta|b|)]|B_n|}Y_n$$
$$=\sum_{n=2}^{\infty}X_n+\sum_{n=1}^{\infty}Y_n=\sum_{n=1}^{\infty}(X_n+Y_n)-X_1\leq 1,$$

then $f\in C_{\overline{H}}(F,b,\lambda,\beta)$.

Conversely, if $f\in C_{\overline{H}}(F,b,\lambda,\beta)$, then

$$X_n=\frac{[n-\lambda(1-\beta|b|)]|A_n|}{\beta|b|}|a_n|\quad(n=2,3,\ldots) \tag{32}$$

and

$$Y_n=\frac{[n+\lambda(1-\beta|b|)]|B_n|}{\beta|b|}|b_n|\quad(n=1,2,\ldots), \tag{33}$$

where $\sum\limits_{n=1}^{\infty}(X_n+Y_n)=1$. Then

$$f(z)=h(z)+\overline{g(z)}=z-\sum_{n=2}^{\infty}|a_n|z^n+\sum_{n=1}^{\infty}|b_n|\overline{z}^n$$
$$=z-\sum_{n=2}^{\infty}\frac{\beta|b|}{[n-\lambda(1-\beta|b|)]A_n}X_nz^n+\sum_{n=1}^{\infty}\frac{\beta|b|}{[n+\lambda(1-\beta|b|)]B_n}Y_n\overline{z}^n$$

$$=z+\sum_{n=2}^{\infty}(h_n(z)-z)X_n+\sum_{n=1}^{\infty}(g_n(z)-z)Y_n$$
$$=\sum_{n=1}^{\infty}(X_nh_n(z)+Y_ng_n(z)).$$

This completes the proof of Theorem 2.13. □

Now we wish to prove that the class $C_{\overline{H}}(F,b,\lambda,\beta)$ is closed under convex combinations.

Theorem 2.14. *Let* $0 \le c_i \le 1$ *for* $i = 1,2,\ldots$ *and* $\sum_{i=1}^{\infty} c_i = 1$. *If the functions* $f_i(z)$ *defined by*

$$f_i(z) = z - \sum_{n=2}^{\infty} |a_{n,i}| z^n + \sum_{n=1}^{\infty} |b_{n,i}| \overline{z}^n \ (z \in \mathbb{U}; i = 1,2,3,...) \tag{34}$$

are in the class $C_{\overline{H}}(F,b,\lambda,\beta)$, *then* $\sum_{i=1}^{\infty} c_i f_i(z)$ *of the form*

$$\sum_{i=1}^{\infty} c_i f_i(z) = z - \sum_{n=2}^{\infty} \left(\sum_{i=1}^{\infty} c_i |a_{n,i}| \right) z^n + \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} c_i |b_{n,i}| \right) \overline{z}^n \tag{35}$$

is in the class $C_{\overline{H}}(F,b,\lambda,\beta)$.

Proof. Since $f_i(z) \in C_{\overline{H}}(F,b,\lambda,\beta)$, it follows from Theorem 2.2 that

$$\sum_{n=2}^{\infty} \frac{[n - \lambda (1 - \beta |b|)] |A_n|}{\beta |b|} |a_{n,i}| + \sum_{n=1}^{\infty} \frac{[n + \lambda (1 - \beta |b|)] |B_n|}{\beta |b|} |b_{n,i}| \le 1 \tag{36}$$

for every $i = 1,2,3,\ldots$ Hence

$$\begin{aligned} &\sum_{n=2}^{\infty} \left(\frac{[n-\lambda(1-\beta|b|)]|A_n|}{\beta|b|} \sum_{i=1}^{\infty} c_i |a_{n,i}| \right) + \sum_{n=1}^{\infty} \left(\frac{[n+\lambda(1-\beta|b|)]|B_n|}{\beta|b|} \sum_{i=1}^{\infty} c_i |b_{n,i}| \right) \\ &= \sum_{i=1}^{\infty} c_i \left(\sum_{n=2}^{\infty} \frac{[n-\lambda(1-\beta|b|)]|A_n|}{\beta|b|} |a_{n,i}| + \sum_{n=1}^{\infty} \frac{[n+\lambda(1-\beta|b|)]|B_n|}{\beta|b|} |b_{n,i}| \right) \\ &\le \sum_{i=1}^{\infty} c_i \le 1. \end{aligned}$$

By Theorem 2.2, it follows that $\sum_{i=1}^{\infty} c_i f_i(z) \in C_{\overline{H}}(F,b,\lambda,\beta)$. This proves that the class $C_{\overline{H}}(F,b,\lambda,\beta)$ is closed under convex combinations. □

Theorem 2.15. *If* $f \in C_{\overline{H}}(F,b,\lambda,\beta)$. *Then* f *is convex in the disc*

$$|z| \le \min_{n\ge2} \left\{ \frac{(1 - |b_1|)\, \beta\, |b|}{n [\beta |b| - [1 + \lambda (1 - \beta |b|)] |B_1 b_1|]} \right\}^{\frac{1}{n-1}}. \tag{37}$$

Proof. Since $f \in C_{\overline{H}}(F,b,\lambda,\beta)$ and $0 \le r < 1$, then $r^{-1} f(r,z) \in C_{\overline{H}}(F,b,\lambda,\beta)$ and

$$\sum_{n=2}^{\infty} n^2 (|a_n| + |b_n|)\, r^{n-1} = \sum_{n=2}^{\infty} n (|a_n| + |b_n|) \left(nr^{n-1}\right)$$

$$\begin{aligned}
&\leq \sum_{n=2}^{\infty}\left[\frac{[n-\lambda(1-\beta|b|)]|A_n|}{\beta|b|}|a_n|+\frac{[n+\lambda(1-\beta|b|)]|B_n|}{\beta|b|}|b_n|\right]nr^{n-1}\\
&\leq 1-|b_1|,
\end{aligned}$$

provided that

$$nr^{n-1}\leq \frac{1-|b_1|}{1-\dfrac{[1+\lambda(1-\beta|b|)]|B_1|}{\beta|b|}|b_1|}, \tag{38}$$

which mean that

$$r\leq \min_{n\geq 2}\left\{\frac{(1-|b_1|)\beta|b|}{n[\beta|b|-[1+\lambda(1-\beta|b|)]|B_1b_1|]}\right\}^{\frac{1}{n-1}}.$$

Thus the proof of the Theorem is completed. □

Remark 2.16. (i) Putting $F(z)=\frac{z}{1-z}+\overline{\left(\frac{z}{1-z}\right)}$ in the above results, we obtain the corresponding results for the class $C_{\overline{H}}(b,\lambda,\beta)$.
(ii) Putting $b=(1-\alpha)\cos\varphi e^{-i\varphi}$ $\left(0\leq\alpha<1,|\varphi|<\frac{\pi}{2}\right)$ and $\beta=1$ in the above results, we obtain the corresponding results for the class $C_{\overline{H}}(F,\alpha,\varphi,\lambda)$;
(iii) Putting $F(z)=z-\sum\limits_{n=2}^{\infty}\sigma_n(\alpha_1)z^n+\sum\limits_{n=1}^{\infty}\overline{\sigma_n(\alpha_1)z^n}$, where $\sigma_n(\alpha_1)$ is given by (10) in the above results, we obtain the corresponding results for the class $C_{\overline{H}}(\alpha_1;b,\lambda,\beta)$;
(iv) Putting $F(z)=z-\sum\limits_{n=2}^{\infty}\Gamma_n(\alpha_1)z^n+\sum\limits_{n=1}^{\infty}\overline{\Gamma_n(\alpha_1)z^n}$, where $\Gamma_n(\alpha_1)$ is given by (11) in the above results, we obtain the corresponding results for the class $C\overline{H}_{q,s}(\alpha_1;b,\lambda,\beta)$.

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