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SERIES SOLUTIONS FOR INITIAL-VALUE PROBLEMS OF TIME FRACTIONAL GENERALIZED ANOMALOUS DIFFUSION EQUATIONS

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In this paper, we formulate an initial-value problem of generalized anomalous diffusion equation consisting second order in two dimensional space and fractional order in time derivatives and then solve it by applying a series involving bilinear Eigen functions. Also, we evaluate a numerical approximation formula of this solution and also discuss its some special cases.

1. Introduction

Oldham and Spanier [14] presented a general solution of the diffusion equation for semi-infinite geometries. Oldham [13] again discussed diffusive transport to planar, cylindrical and spherical electrodes through a general equation consisting first order in one dimensional space and half order in time derivative and solved it through Laplace transform method.

Oldham and Spanier [15] also obtained various applications of the fractional calculus to the problems whose formulations and solutions are normally couched in terms of integrals or derivatives alone.

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The anomalous diffusion is a phenomenon that occurs in complex and non homogeneous mediums. The phenomenon of anomalous diffusion may be based on generalized diffusion equation which contains fractional order space and / or time derivatives (see Metzler and Klafter [12]).

Turski et al [25] presented the recurrence of anomalous diffusion from the physical point of view and also described the importance of fractional derivatives in space and / or time to diffusion propagation. Agarwal [1] presented an analytical technique by applying Eigen functions to solve a fractional diffusion-wave system. Again, Agarwal [1] also formulated a general solution by using finite sine transform techniques for a fractional diffusion-wave equation in a bounded domain.

In our investigations, we make an appeal to the following definitions: (see Kilbas et al. [9], Podlubny [16] , Samko et al. [17] and Diethelm [3]).

Let $n \in \mathbb{R}$ (set of real numbers). The operator $J_a^n.$ defined on $L_1[a,b] := \{f : [a,b] \to \mathbb{R}; f$ is measurable on $[a,b]$ and $\int_a^b f(x)dx < \infty\}$ by

$$J_a^n f(x) := \frac{1}{\Gamma(n)} \int_a^x (x-t)^{n-1} f(t)\, dt \tag{1}$$

for $a \le x \le b$, is called the Riemann-Liouville fractional integral operator of order n.

Let $n \in \mathbb{R}$, and $m = [n]$. The operator $D_a^n.$ is defined by

$$\begin{aligned} &D_a^n f := D^m J_a^{m-n} f, \\ &\left(Df(x) := \frac{d}{dx} f(x),\, D^n f(x) := DD^{n-1} f(x) := \frac{d}{dx}\frac{d^{n-1}}{dx^{n-1}}\, f(x) \right), \end{aligned} \tag{2}$$

and is called the Riemann- Liouville fractional differential operator of order n, again for $n = 0$, $D_a^0 f = If = f$.

For $n \ge 0$ and f is such that $D_a^n[f - T_{m-1}[f:a]]$ exists; where $m = [n]$, then there is a function $D_{*a}^n f$ defined by

$$D_{*a}^n f := D_a^n [f - T_{m-1}[f:a]] := J_a^{m-n} D^m f \tag{3}$$

The operator D_{*a}^n is called the Caputo differential operator of order n. As usual $T_{m-1}[f:a]$ denotes the Taylor polynomial of degree $m-1$ for the function f, centered at a; in the case $m = 0$, $T_{m-1}[f:a] := 0$ and that is

$$T_{m-1}[f:a](x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(a)}{k!}(x-a)^k \tag{4}$$

Here n denotes the order of the Caputo-type differential operator. We shall only consider the case $n > 0$ and $n \notin \mathbb{N}$ (the set of natural numbers) and we use the notation $m = [n]$ to denote the smallest integer greater than (or equal to) n. For $n \in \mathbb{N}$, the Caputo differential operator coincides with the usual differential operator of integer order, and for $n < 0$, the Caputo differential of negative order can be interpreted as the Riemann-Liouville differential operator of the same order. This Caputo-type differential operator has certain advantages when we try to model real-world phenomena with fractional differential equations (see also Podlubny [16] or Diethelm [3]).

For $t > a$, the Caputo time fractional derivative is defined as

$$ {}_t^C D_{*a}^{\beta} f(t) \;:= \frac{1}{\Gamma(n-\beta)} \int_a^t (t-\tau)^{n-\beta-1} \left(\frac{d}{d\tau}\right)^n f(\tau)\, d\tau \tag{5}$$

where $0 < \beta < n,\ n \in \mathbb{Z}_+$ (the set of positive integers).

The Eigen functions $\psi_n(x)$ corresponding to the Eigen-values λ_n satisfy following Eigen-value problem (see Hinchey [6, p.541])

$$D_x[\psi_n(x)] + i\lambda_n \psi_n(x) = 0,\ \left(D_x \equiv \frac{d}{dx}\right),\ x,\lambda_n \in \mathbb{R}\ (n = 0,1,2,3,\ldots.) \tag{6}$$

2. Series of Bilinear Eigen Functions

We define a series of bilinear Eigen functions which is a three variable function $F(x,y,t)$ possesses a formal power series containing bilinear Eigen functions in the form

$$F(x,y,t) = \sum_{n=0}^{\infty} f_n(t)\, \psi_n(x)\, \psi_n(y),\ x,y \in R,\ t > 0. \tag{7}$$

Particularly, if in (7) we put $f_n(t) = \gamma_n t^n, \forall n = 0,1,2,3,\ldots,\ \{\gamma_n\}$ is independent of x,y,t, then $F(x,y,t)$ becomes a bilinear generating function for the set $\{\psi_n\}_{n=0}^{\infty}$ (see Srivastava and Manocha [21]).

Here, in our work we present an initial-value problem of generalized anomalous diffusion equation involving second order in two dimensional space and fractional order in time derivatives which is taken in terms of Caputo derivative and then solve it on making an application of a three variable function $F(x,y,t)$ given in (7). We find a numerical approximation formula and also discuss some of its special cases.

3. Main Problem

In this section, we formulate following generalized anomalous diffusion problem

$$\frac{1}{k}\,{}_{t}^{C}D_{*0}^{\alpha}u(x,y,t)=D_{x}^{2}u(x,y,t)+D_{y}^{2}u(x,y,t),$$
$$k\neq 0,\ 0<\alpha\leq 1,\ a,\ a',b,\ b'\in\mathbb{R},\ t>0,$$
$$x\in(a,b)\ \text{and}\ \ y\in\left(a',b'\right) \tag{8}$$
$$u(x,y,t)=u_{0}(x,y),\ \text{at}\ t=0 \tag{9}$$

and

$$\lim_{x,y\to\pm\infty}u(x,y,t)=0 \tag{10}$$

Particularly, when set $\alpha=1$ in the problem (8)-(10), we get the well-known two dimensional diffusion problem of Sneddon [18, p. 305, Misc. Prob. 14].

When put $\alpha=1$ in the problem (8)-(10) and set $u(x,y,t)$ as independent of y, we get the well-known one dimensional diffusion problem of Kumar [10].

When α is arbitrary and we consider $u(x,y,t)$ as independent of y in the problem (8)-(10) then it becomes time-fractional diffusion problem of Mathai, Saxena and Haubold [11, p. 175, Theorem 6.3]

Before solving above problem (8)-(10), we present the following theorem:

Theorem 3.1 (A). *If the series of bilinear Eigen functions*

$$u(x,y,t)=\sum_{n=0}^{\infty}u_{n}(t)\,\psi_{n}(x)\,\psi_{n}(y)$$

satisfies the generalized anomalous diffusion problem ((8)-(10))*,* $x\in(a,b)$ *and* $y\in(a',b'),a,a',b,b'\in R,t>0$ *and ,* $\psi_{n}(x)$ *satisfy the Eigen value problem* (6) *for non-zero Eigen values* $\lambda_{n},(\forall n=0,1,2,3,\ldots)$*, then there exists a sequence of integrals and given by*

$$u_{n}(t)=u_{0n}E_{\alpha,1}\left(-2k\lambda_{n}^{2}t^{\alpha}\right),(n=0,1,2,3,\ldots) \tag{11}$$

Here, $0<\alpha\leq 1$ *and* $E_{\alpha,1}(.)$ *is the well-known Mittag-Leffler function (see Srivastava and Manocha [21], Diethelm [3]) and*

$$u_{0n}=\left(\frac{1}{\left[\int_{a}^{b}|\psi_{n}(x)|^{2}dx\right]\left[\int_{a'}^{b'}|\psi_{n}(y)|^{2}dy\right]}\right)\int_{a}^{b}\int_{a'}^{b'}u_{0}(x,y)\,\overline{\psi_{n}(x)\psi_{n}(y)}dxdy$$
$$(n=0,1,2,3,\ldots)$$

Proof. Set $u(x,y,t) = \sum_{n=0}^{\infty} u_n(t)\,\psi_n(x)\,\psi_n(y)$ into equation (8) to get

$$\frac{1}{k}\sum_{n=0}^{\infty} {}_t^C D_{*0}^{\alpha} u_n(t)\,\psi_n(x)\,\psi_n(y) - \sum_{n=0}^{\infty} u_n(t)\left(\psi_n(y)\frac{\partial^2}{\partial x^2}\psi_n(x) + \psi_n(x)\frac{\partial^2}{\partial y^2}\psi_n(y)\right) = 0 \quad (12)$$

Now making an appeal to (6) and (12), we find that

$$\sum_{n=0}^{\infty}\left(\frac{1}{k}{}_t^C D_{*0}^{\alpha} u_n(t) + 2\lambda_n^2 u_n(t)\right)\psi_n(x)\,\psi_n(y) = 0 \quad (13)$$

Then the equation (13) gives us

$$\frac{1}{k}{}_t^C D_{*0}^{\alpha} u_n(t) + 2\lambda_n^2 u_n(t) = 0\,(n = 0,1,2,3,\ldots.) \quad (14)$$

On taking Laplace transform of (14), and making an appeal to the techniques due to Huang and Liu [7], we get

$$\mathcal{L}\{u_n(t)\} = \frac{s^{\alpha-1}u_n(0)}{s^{\alpha} + 2k\lambda_n^2}\,(n = 0,1,2,3,\ldots.) \quad (15)$$

Again on making an appeal to the theorem 4.5 of Diethelm [3, p.72] and equating to the equation (15), we get

$$u_n(t) = u_n(0)\,E_{\alpha,1}\left(-2k\lambda_n^2 t^{\alpha}\right)(n = 0,1,2,3,\ldots.) \quad (16)$$

where $E_{\alpha,1}(.)$ is the well-known Mittag-Leffler function (see Srivastava and Manocha [21], Diethelm [3]).

Further at $t = 0$, the series of bilinear Eigen functions

$$u(x,y,t) = \sum_{n=0}^{\infty} u_n(t)\,\psi_n(x)\,\psi_n(y)$$

gives us

$$u_0(x,y) = \sum_{n=0}^{\infty} u_{0n}\psi_n(x)\,\psi_n(y) \quad (17)$$

Then multiply $\frac{\overline{\psi_m(x)}}{\left[\int_a^b |\psi_m(x)|^2 dx\right]^{1/2}} \cdot \frac{\overline{\psi_m(y)}}{\left[\int_{a'}^{b'} |\psi_m(y)|^2 dy\right]^{1/2}}$ in both sides of (17) and then integrate to that sides with respect to x from a to b and again with respect to y from a' to b' to get

$$\frac{1}{\left[\int_a^b |\psi_m(x)|^2 dx\right]^{1/2}\left[\int_{a'}^{b'} |\psi_m(y)|^2 dy\right]^{1/2}}\int_a^b\int_{a'}^{b'} u_0(x,y)\,\overline{\psi_m(x)\psi_m(y)}dxdy$$

$$= \sum_{n=0}^{\infty} u_{0n} \frac{\int_a^b \psi_n(x)\overline{\psi_m(x)}dx}{\left[\int_a^b |\psi_m(x)|^2 dx\right]^{1/2}} \cdot \frac{\int_{a'}^{b'} \psi_n(y)\overline{\psi_m(y)}dy}{\left[\int_{a'}^{b'} |\psi_m(y)|^2 dy\right]^{1/2}} \tag{18}$$

Now in the right hand side of (18) apply the orthogonal property to find that

$$u_{0n} = \frac{1}{\left[\int_a^b |\psi_n(x)|^2 dx\right]\left[\int_{a'}^{b'} |\psi_n(y)|^2 dy\right]} \int_a^b \int_{a'}^{b'} u_0(x,y)\overline{\psi_n(x)\psi_n(y)}dxdy$$

$$(n = 0,1,2,3,\ldots.) \tag{19}$$

After making some manipulations, we get $u_n(0) = u_{0n}$, therefore with the help of (16) and (19), we evaluate (11). □

Solution of the problem stated in the equations (8)**-**(10)

On making an appeal to the theorem A, the solution of the problem stated in the equations (8)-(10) is then evaluated as

$$u(x,y,t) = \sum_{n=0}^{\infty} u_n(t)\,\psi_n(x)\,\psi_n(y) \tag{20}$$

Here

$$u_n(t) = u_{0n} E_{\alpha,1}\left(-2k\lambda_n^2 t^\alpha\right), (n = 0,1,2,3,\ldots.) \tag{21}$$

and

$$u_{0n} = \left(\frac{1}{\left[\int_a^b |\psi_n(x)|^2 dx\right]\left[\int_{a'}^{b'} |\psi_n(y)|^2 dy\right]}\right) \int_a^b \int_{a'}^{b'} u_0(x,y)\overline{\psi_n(x)\psi_n(y)}dxdy$$

$$(n = 0,1,2,3,\ldots.) \tag{22}$$

4. Numerical Approximation Formula

In this section, we present the numerical solution of the problem ((8)-(10)) by applying Gr*ü*nwald approximation formula [5] for fractional order derivative (also see Oldham and Spainier [15]) given by

$$D_0^\alpha f(x) = \lim_{N\to\infty} \left\{ \left[\frac{x}{N}\right]^{-\alpha} \sum_{j=0}^{N-1} \frac{\Gamma(j-\alpha)}{\Gamma(-\alpha)\Gamma(j+1)} f\left(x - \frac{j}{N}x\right) \right\},$$

$$N \in \mathbb{N}, N-1 < \alpha < N \tag{23}$$

Now make an appeal to the equation (23) and the formulae given in the equations (3) and (4), we get the approximation formula for Caputo derivative in the form

$$
{}_t^C D_{*0}^{\alpha} f(t) = \lim_{N\to\infty} \left\{ \left[\frac{x}{N}\right]^{-\alpha} \sum_{j=0}^{N-1} \frac{\Gamma(j-\alpha)}{\Gamma(-\alpha)\Gamma(j+1)} f\left(t-\frac{j}{N}t\right) \right\} - \sum_{k=0}^{N-1} \frac{f^{(k)}(0)}{k!} \frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)} \quad (24)
$$

Then make an appeal to (14) and (24) to find the numerical approximation of the problem ((2.1)-(10))

$$
\begin{aligned}
u_n(t) = \left(-\frac{1}{2k\lambda_n^2}\right) \left\{ \left[\frac{N}{t}\right]^{\alpha} \sum_{j=0}^{N-1} \frac{\Gamma(j-\alpha)}{\Gamma(-\alpha)\Gamma(j+1)} u_n\left(t-\frac{j}{N}t\right) \right. \\
\left. - \sum_{k=0}^{N-1} \frac{u_n{}^{(k)}(0)}{k!} \frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)} \right\}, \\
N \text{ is large}, t > 0, \lambda_n \neq 0, \text{ and } k \neq 0\, (n = 0,1,2,3,\ldots.) \quad (25)
\end{aligned}
$$

Particularly, for $0 < \alpha < 1$ from the equation (25) and the formula due to Oldham and Spainier [15, eq. (3.4.5)], we get the numerical approximation formula

$$
\begin{aligned}
u_n(t) = \left(\frac{1}{2k\lambda_n^2}\right) \left\{ u_n(0) \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \right. \\
\left. - \lim_{N\to\infty} \left[\frac{t}{N}\right]^{-1} \sum_{j=0}^{1} \frac{\Gamma(j-\alpha)}{\Gamma(-\alpha)\Gamma(j+1)} u_n\left(t-\frac{j}{N}t\right) \right\}, \\
t > 0,\ \lambda_n \neq 0, \text{ and } k \neq 0\ (n = 0,1,2,3,\ldots.). \quad (26)
\end{aligned}
$$

5. Special Cases

Set $\psi_n(x) = e^{inx}$, $a,\ a' = -\pi$ and $b, b' = \pi$ in (20)-(22) to get

$$
u(x,y,t) = \sum_{n=0}^{\infty} u_n(t) e^{inx} e^{iny} \quad (27)
$$

Here

$$
\begin{aligned}
u_n(t) = E_{\alpha,1}\left(-2k\lambda_n^2 t^{\alpha}\right) \left(\frac{1}{[2\pi]^2}\right) \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} u_0(x,y) e^{-in(x+y)} dxdy, \\
t > 0,\ (n = 0,1,2,3,\ldots.). \quad (28)
\end{aligned}
$$

Again, set $\psi_n(x) = e^{-inx}, a, a' = 0$ and $b, b' = \pi$ in (20)-(22) to get

$$u(x,y,t) = \sum_{n=0}^{\infty} u_n(t)\, e^{-inx} e^{-iny} \tag{29}$$

Here

$$u_n(t) = E_{\alpha,1}\left(-2k\lambda_n^2 t^\alpha\right)\left(\frac{1}{[\pi]^2}\right)\int_0^\pi \int_0^\pi u_0(x,y)\, e^{in(x+y)} dxdy,$$
$$t > 0,\ (n = 0,1,2,3,\ldots). \tag{30}$$

Now set

$$u_0(x,y) = (\sin x)^{w_1-1}(\sin y)^{w_2-1} F\begin{matrix} E:F;F' \\ G:H;H' \end{matrix}\left[\begin{matrix} (e):(f);(f');\beta_1(\sin x)^{2\rho_1} \\ (g):(h);(h');\beta_2(\sin y)^{2\rho_2} \end{matrix}\right]$$
$$\times H \begin{matrix} 0,\lambda : (\mu',\nu');\ldots;(\mu^{(r)},\nu^{(r)}) \\ A,C:(B',D');\ldots;(B^{(r)},D^{(r)}) \end{matrix}$$
$$\left[\begin{matrix} [(a):\xi',\ldots,\xi^{(r)}]:[(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}]; \\ [(c):\eta',\ldots,\eta^{(r)}]:[(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}]; \end{matrix}\ \begin{matrix} z_1(\sin x)^{2\sigma_1}(\sin y)^{2\gamma_1} \\ \vdots \\ z_r(\sin x)^{2\sigma_r}(\sin y)^{2\gamma_r} \end{matrix}\right] \tag{31}$$

where,

$$F\begin{matrix} E:F;F' \\ G:H;H' \end{matrix}[.]$$

is a Kampé de Fériet function [8] (see also Exton [4], Srivastava and Manocha [21] and Srivastava and Karlsson [20]) and

$$H\begin{matrix} 0,\lambda : (\mu',\nu');\ldots;\left(\mu^{(r)},\nu^{(r)}\right) \\ A,C:(B',D');\ldots;\left(B^{(r)},D^{(r)}\right) \end{matrix}[.]$$

is a multivariable H-function of Srivastava and Panda [22, 23, 24] (see also Srivastava, Gupta and Goyal [19] and Mathai, Saxena and Haubold [11]).

Then use (30) in (29) and apply the techniques of Chandel, Agarwal and

Kumar [2] to get

$$u_n(t) = \left(\frac{e^{in\pi/2}}{2^{w_1+w_2-2}}\right)\left\{E_{\alpha,1}\left(-2k\lambda_n^2 t^\alpha\right)\right\}$$
$$\sum_{p,\ p'=0}^{\infty} \frac{\prod_{q=1}^{E}(e_q)_{p+p'}\prod_{q=1}^{F}(f_q)_p\prod_{q=1}^{E}(f'_q)_{p'}}{\prod_{q=1}^{G}(g_q)_{p+p'}\prod_{q=1}^{H}(h_q)_p\prod_{q=1}^{H'}(h'_q)_{p'}}$$
$$\times \frac{(\beta_1/4^{\rho_1})^p}{p!}\frac{(\beta_2/4^{\rho_2})^{p'}}{p'!} H \begin{matrix} 0,\lambda+2:(\mu',\nu');\ldots;\left(\mu^{(r)},\nu^{(r)}\right) \\ A+2,C+4:(B',D');\ldots;\left(B^{(r)},D^{(r)}\right) \end{matrix} \Bigg[$$
$$\begin{matrix} \left[(a):\xi',\ldots,\xi^{(r)}\right], \left[1-w_1-2p\rho_1:2\sigma_1,\ldots,2\sigma_r\right], \\ \left[(c):\eta',\ldots,\eta^{(r)}\right], \left[\frac{1}{2}-\frac{w_1+2p\rho_1\pm n}{2}:\sigma_1,\ldots,\sigma_r\right], \end{matrix}$$
$$\begin{matrix} [1-w_2-2p'\rho_2:2\gamma_1,\ldots,2\gamma_r]:[(b'):\phi'];\ldots;\left[\left(b^{(r)}\right):\phi^{(r)}\right]; \\ \left[\frac{1}{2}-\frac{w_2+2p'\rho_2\pm n}{2}:\gamma_1,\ldots,\gamma_r\right]:[(d'):\delta'];\ldots;\left[\left(d^{(r)}\right):\delta^{(r)}\right]; \end{matrix} \begin{matrix} \frac{z_1}{4^{\sigma_1+\gamma_1}} \\ \vdots \\ \frac{z_r}{4^{\sigma_r+\gamma_r}} \end{matrix} \Bigg],$$
$$t>0,\ (n=0,1,2,3,\ldots). \quad (32)$$

provided that

$$\left|\frac{z_\varepsilon}{4^{\sigma_\varepsilon+\gamma_\varepsilon}}\right| < \frac{\pi}{2}\triangle_\varepsilon, \forall\varepsilon=1,2,\ldots,r \text{ and } \triangle_\varepsilon = \sum_{J=1}^{\lambda}\xi_J^{(\varepsilon)} - \sum_{J=\lambda+1}^{A}\xi_J^{(\varepsilon)} + \sum_{J=1}^{\nu^{(\varepsilon)}}\phi_J^{(\varepsilon)}$$
$$-\sum_{J=\nu^{(\varepsilon)}+1}^{B^{(\varepsilon)}}\phi_J^{(\varepsilon)} - \sum_{J=1}^{C}\eta_J^{(\varepsilon)} + \sum_{J=1}^{\mu^{(\varepsilon)}}\delta_J^{(\varepsilon)} - \sum_{J=\mu^{(\varepsilon)}+1}^{D^{(\varepsilon)}}\delta_J^{(\varepsilon)} > 0 \quad (33)$$

On making some manipulations in parameters of multivariable H-function of (30) and (31), we may obtain $u_n(t)$ in terms of a double series involving many hypergeometric functions cited in the literature of generalized special functions.

6. Conclusions

If we set $\psi_n(x) = sin\left(\frac{n\pi x}{b}\right),\ \psi_n(y) = sin\left(\frac{n\pi y}{b'}\right)$ *and* $\alpha = 1$, a = 0, and a' = 0 in (20)-(22) we get the solution

$$u(x,y,t) = \sum_{n=1}^{\infty} u_{0n} exp\left(-2k\lambda_n^2 t\right)\sin\left(\frac{n\pi x}{b}\right)\sin\left(\frac{n\pi y}{b'}\right) \quad (34)$$

Here

$$u_{0n} = \left(\frac{4}{bb'}\right)\int_0^b\int_0^{b'} u_0(x,y)\, sin\left(\frac{n\pi x}{b}\right) sin\left(\frac{n\pi y}{b'}\right) dxdy\ \ (n=0,1,2,3,\ldots)$$

This solution is equivalent to the problem of Sneddon [18, p. 305, Misc. Prob. 14] for

$$\lambda_n^2 = \frac{1}{2} n^2 \pi^2 \left(\frac{1}{b^2} + \frac{1}{b'^2} \right), (n = 0, 1, 2, 3, \ldots.) \tag{35}$$

Again, set y $\to 0$, $w_2 = 1, \alpha = 1,\ \beta_1 \to 0,\ \beta_2 \to 0,\ \gamma_1 = 0,\ \ldots,\ \gamma_r = 0$ in (29)-(32) to get solution of one dimensional diffusion equation (see Kumar [10]) in x-direction of the form

$$u(x,t) = \sum_{n=0}^{\infty} u_n(t)\, e^{-inx} \tag{36}$$

where

$$u_n(t) = \left(\frac{e^{in\pi/2}}{2^{w_1 - 1}} \right) \left\{ exp\left(-2k\lambda_n^2 t \right) \right\}$$
$$\times H \begin{matrix} 0, \lambda+1 : (\mu', \nu'); \ldots; \left(\mu^{(r)}, \nu^{(r)}\right) \\ A+1, C+2 : (B', D'); \ldots; \left(B^{(r)}, D^{(r)}\right) \end{matrix} \left[\begin{matrix} [(a) : \xi', \ldots, \xi^{(r)}], \\ [(c) : \eta', \ldots, \eta^{(r)}], \end{matrix} \right.$$
$$\begin{matrix} [1 - w_1 : 2\sigma_1, \ldots, 2\sigma_r] : \\ \left[\frac{1}{2} - \frac{w_1 \pm n}{2} : \sigma_1, \ldots, \sigma_r\right] : \end{matrix}$$
$$\left. \begin{matrix} [(b') : \phi']; \ldots; \left[\left(b^{(r)}\right) : \phi^{(r)}\right]; \\ [(d') : \delta']; \ldots; \left[\left(d^{(r)}\right) : \delta^{(r)}\right]; \end{matrix} \quad \begin{matrix} \frac{z_1}{4^{\sigma_1}} \\ \vdots \\ \frac{z_r}{4^{\sigma_r}} \end{matrix} \right], t > 0, (n = 0, 1, 2, 3, \ldots.) \tag{37}$$

provided that the conditions due to (33) are satisfied.[5]

Again, set x $\to 0$, $w_1 = 1, \alpha = 1,\ \beta_1 \to 0,\ \beta_2 \to 0,\ \sigma_1 = 0,\ \ldots,\ \sigma_r = 0$ in (29)-(32), we get solution of one dimensional diffusion equation in y-direction of the form

$$u(y,t) = \sum_{n=0}^{\infty} u_n(t)\, e^{-iny} \tag{38}$$

where

$$u_n(t) = \left(\frac{e^{in\pi/2}}{2^{w_2-1}}\right)\left\{exp\left(-2k\lambda_n^2 t\right)\right\}$$

$$\times H\begin{matrix} 0,\lambda+1:(\mu',\nu');\ldots;\left(\mu^{(r)},\nu^{(r)}\right) \\ A+1,C+2:(B',D');\ldots;\left(B^{(r)},D^{(r)}\right)\end{matrix}\left[\begin{matrix}\left[(a):\xi',\ldots,\xi^{(r)}\right], \\ \left[(c):\eta',\ldots,\eta^{(r)}\right],\end{matrix}\right.$$

$$\begin{matrix}\left[1-w_2:2\gamma_1,\ldots,2\gamma_r\right]: \\ \left[\frac{1}{2}-\frac{w_2\pm n}{2}:\gamma_1,\ldots,\gamma_r\right]:\end{matrix}$$

$$\left.\begin{matrix}\left[(b'):\phi'\right];\ldots;\left[\left(b^{(r)}\right):\phi^{(r)}\right]; \\ \left[(d'):\delta'\right];\ldots;\left[\left(d^{(r)}\right):\delta^{(r)}\right];\end{matrix}\begin{matrix}\frac{z_1}{4\gamma_1} \\ \vdots \\ \frac{z_r}{4\gamma_r}\end{matrix}\right], t>0, (n=0,1,2,3,\ldots) \quad (39)$$

provided that the conditions due to (33) are satisfied.

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