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# SOME PROPERTIES OF TWO-FOLD SYMMETRIC ANALYTIC FUNCTIONS

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In this paper, we introduce a new class of two-fold symmetric functions analytic in the unit disc. We prove such results as subordination and superordination properties, convolution properties, distortion theorems, and inequality properties of this new class.

## 1. Introduction

Let $\mathcal{A}(m)$ denote the class of functions $f$:

$$f(z) = z + \sum_{k=m+1}^{\infty} a_k z^k, \quad m \in \mathbb{N} = \{1, 2, \dots\}, \tag{1}$$

which are analytic in the open unit disc $E = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Also, let $\mathcal{H}[a, m+1]$ be the class of analytic functions of the form

$$f(z) = a + a_{m+1} z^{m+1} + a_{m+2} z^{m+2} + \dots, \quad z \in E.$$

If $f$ and $g$ are analytic in $E$, we say that $f$ is subordinate to $g$, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function $w$ with $w(0) = 0$ and $|w(z)| < 1$

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in $E$ such that $f(z) = g(w(z))$. Furthermore, if the function $g(z)$ is univalent in $E$, then we have the following equivalence holds, see [4, 5]

$$f(z) \prec g(z) \;\; (z \in E) \Longleftrightarrow f(0) = g(0) \text{ and } f(E) \subset g(E).$$

For function $f$, $g \in \mathcal{A}(m)$, where $f$ is given by (1) and $g$ is defined by

$$g(z) = z + \sum_{k=m+1}^{\infty} b_k z^k, \quad m \in \mathbb{N} = \{1, 2, \ldots\},$$

then the Hadamard product (or convolution) $f * g$ of the function $f$ and $g$ is defined by

$$(f * g)(z) = z + \sum_{k=m+1}^{\infty} a_k b_k z^k = (g * f)(z).$$

In [8], Sakaguchi defined the class of starlike functions with respect to symmetrical points as follows:

Let $f \in \mathcal{A}$. Then $f$ is said to be starlike with respect to symmetrical points in $E$ if, and only if,

$$Re \, \frac{zf'(z)}{f(z) - f(-z)} > 0, \, z \in E.$$

Obviously, it forms a subclass of close-to-convex functions and hence univalent. Moreover, this class includes the class of convex functions and odd starlike functions with respect to the origin, see [8].

**Definition 1.1.** A function $f \in \mathcal{A}(m)$ is said to be in the class $\mathcal{B}^{\lambda,\mu}(m, A, B)$, if it satisfies the following subordination condition:

$$(1-\lambda)\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu} + \lambda \frac{z(f'(z)+f'(-z))}{f(z)-f(-z)}\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu} \prec \frac{1+Az}{1+Bz}, \tag{2}$$

where and throughout this paper unless otherwise mention the parameters $\lambda, \mu$, $A$ and $B$ are constrained as follows:

$$\lambda \in \mathbb{C} : \; Re\,(\mu) > 0 : \; -1 \leq B \leq 1, \; A \neq B, \; A \in \mathbb{R} \text{ and } m \in \mathbb{N},$$

and all powers are understood as principal values.

In this paper, we prove such results as subordination and superordination properties, convolution properties, distortion theorems, and inequality properties of the class $\mathcal{B}^{\lambda,\mu}(m, A, B)$.

For interested readers see the work done by the authors [1, 2, 10–13].

## 2. Preliminary Results

**Definition 2.1.** Let $\mathcal{Q}$ be the set of all functions $f$ that are analytic and injective on $\overline{E}\backslash U(f)$, where

$$U(f) = \left\{ \zeta \in \partial E : \lim_{z \to \zeta} f(z) = \infty \right\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial E \backslash U(f)$.

To establish our main results we need the following Lemmas.

**Lemma 2.2** (Miller and Mocanu [4, 5])**.** *Let the function $h(z)$ be analytic and convex (univalent) in $E$ with $h(0) = 1$. Suppose also that the function $\Phi(z)$ given by*

$$\Phi(z) = 1 + c_{m+1}z^{m+1} + c_{m+2}z^{m+2} + \dots$$

*is analytic in $E$,*

$$\Phi(z) + \frac{z\,\Phi'(z)}{\gamma} \prec h(z) \;\; (z \in E;\ Re\ \gamma \geq 0;\ \gamma \neq 0), \tag{3}$$

*then*

$$\Phi(z) \prec \Psi(z) = \frac{\gamma}{(m+1)z^{\frac{\gamma}{m+1}}} \int_0^z t^{\frac{\gamma}{m+1}-1} h(t)dt \prec h(z) \;\; (z \in E),$$

*and $\Psi(z)$ is the best dominant of (3).*

**Lemma 2.3** (Shanmugam et al. [9])**.** *Let $\sigma \in \mathbb{C}$, $\eta \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$ and let $q$ be a convex univalent function in $E$ with*

$$Re\ \left(1 + \frac{zq''(z)}{q'(z)}\right) > \max\left\{0; -Re\ \frac{\sigma}{\eta}\right\},\ z \in E.$$

*If $p$ is analytic in $E$ and*

$$\sigma p(z) + \eta z p'(z) \prec \sigma q(z) + \eta z q'(z), \tag{4}$$

*then $p(z) \prec q(z)$, and $q$ is the best dominant of (4).*

**Lemma 2.4** ([5])**.** *Let $q(z)$ be convex univalent in $E$ and $k \in \mathbb{C}$. Further assume that $Re\ \ k > 0$. If*

$$g(z) \in \mathcal{H}[q(0), 1] \cap \mathcal{Q},$$

*and*

$$g(z) + kzq'(z) \prec g(z) + kzg'(z),$$

*implies $q(z) \prec g(z)$ and $q(z)$ is the best subordinant.*

**Lemma 2.5** ([3])**.** *Let $F$ be analytic and convex in $E$. If $f,g \in \mathcal{A}(1)$ and $f,g \prec F$, then*

$$\lambda f+(1-\lambda)g \prec F \quad (0 \leq \lambda \leq 1).$$

**Lemma 2.6** ([7])**.** *Let*

$$f(z)=1+\sum_{k=1}^{\infty} a_k z^k$$

*be analytic in $E$ and*

$$g(z)=1+\sum_{k=1}^{\infty} b_k z^k$$

*be analytic and convex in $E$. If $f(z) \prec g(z)$, then*

$$|a_k|<|b_1|, \ \ k \in \mathbb{N}.$$

## 3. Main Results

**Theorem 3.1.** *Let $f(z) \in \mathcal{B}^{\lambda,\mu}(m,A,B)$ with Re $\lambda>0$. Then*

$$\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu} \prec \psi(z)=\frac{\mu}{\lambda(m+1)} \int_0^1 \frac{1+Azu}{1+Bzu} u^{\frac{\mu}{\lambda(m+1)}-1} du \prec \frac{1+Az}{1+Bz}, \quad (5)$$

*and $\psi(z)$ is the best dominant.*

*Proof.* Set

$$\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu}=h(z), \ \ z \in E. \quad (6)$$

Then $h(z)$ is analytic in $E$ with $h(0)=1$.

Logarithmic differentiation of (5) and simple computations yield

$$(1-\lambda)\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu}+\lambda \frac{z(f'(z)+f'(-z))}{f(z)-f(-z)}\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu}$$
$$=h(z)+\frac{\lambda}{\mu} z h'(z) \prec \frac{1+Az}{1+Bz}. \quad (7)$$

Applying Lemma 2.2 to (7) with $\gamma=\frac{\mu}{\lambda}$, we have

$$\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu} \prec \psi(z)=\frac{\mu}{\lambda(m+1)} z^{-\frac{\mu}{\lambda(m+1)}} \int_0^z \frac{1+At}{1+Bt} t^{\frac{\mu}{\lambda(m+1)}-1} dt$$
$$=\frac{\mu}{\lambda(m+1)} \int_0^1 \frac{1+Azu}{1+Bzu} u^{\frac{\mu}{\lambda(m+1)}-1} du \prec \frac{1+Az}{1+Bz}, \quad (8)$$

and $\psi(z)$ is the best dominant. This completes the proof. □

**Theorem 3.2.** *Let $q(z)$ be univalent in E, $\lambda \in \mathbb{C}^*$. Suppose also that $q(z)$ satisfies the following inequality:*

$$Re\left(1+\frac{zq''(z)}{q'(z)}\right) > \max\left\{0; -Re\left(\frac{\mu}{\lambda}\right)\right\}. \tag{9}$$

*If $f \in \mathcal{A}(m)$ satisfies the following subordination:*

$$(1-\lambda)\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu}+\lambda\frac{z(f'(z)+f'(-z))}{f(z)-f(-z)}\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu} \prec q(z)+\frac{\lambda}{\mu}zq'(z), \tag{10}$$

*then*

$$\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu} \prec q(z),$$

*and $q(z)$ is the best dominant.*

*Proof.* Let the function $h(z)$ be defined by (6). We know that the first part of (7) holds true. Combining (7) and (10), we have

$$h(z)+\frac{\lambda}{\mu}zh'(z) \prec q(z)+\frac{\lambda}{\mu}zq'(z). \tag{11}$$

By using Lemma 2.3 and (11), we easily get the assertion of Theorem 3.2. □

**Corollary 3.3.** *Let $\lambda \in \mathbb{C}^*$ and $-1 \le B < A \le 1$. Suppose also that*

$$Re\left(\frac{1-Bz}{1+Bz}\right) > \max\left\{0, -Re\left(\frac{\mu}{\lambda}\right)\right\}.$$

*If $f \in \mathcal{A}(m)$ satisfies the following subordination:*

$$\begin{aligned}&(1-\lambda)\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu}+\lambda\frac{z(f'(z)+f'(-z))}{f(z)-f(-z)}\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu}\\ &\prec \frac{1+Az}{1+Bz}+\lambda\frac{(A-B)z}{(1+Bz)^2},\end{aligned}$$

*then*

$$\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu} \prec \frac{1+Az}{1+Bz},$$

*and $\frac{1+Az}{1+Bz}$ is the best dominant.*

If $f$ is subordinate to $F$, then $F$ is superordinate to $f$. We now derive the following superordination result for the class $\mathcal{B}^{\lambda,\mu}(m,A,B)$.

**Theorem 3.4.** *Let $q$ be convex univalent in $E, \lambda \in \mathbb{C}$ with $Re\ \lambda > 0$. Also let*

$$\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu} \in \mathcal{H}[q(0),1] \cap \mathcal{Q}$$

*and*

$$(1-\lambda)\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu} + \lambda \frac{z(f'(z)+f'(-z))}{f(z)-f(-z)} \left(\frac{f(z)-f(-z)}{2z}\right)^{\mu}$$

*be univalent in $E$. If*

$$q(z) + \frac{\lambda}{\mu} z q'(z)$$
$$\prec (1-\lambda)\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu} + \lambda \frac{z(f'(z)+f'(-z))}{f(z)-f(-z)} \left(\frac{f(z)-f(-z)}{2z}\right)^{\mu},$$

*then*

$$q(z) \prec \left(\frac{f(z)-f(-z)}{2z}\right)^{\mu},$$

*and $q$ is the best subordinant.*

*Proof.* Let the function $h(z)$ be defined by (6). Then

$$q(z) + \frac{\lambda}{\mu} z q'(z)$$
$$\prec (1-\lambda)\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu} + \lambda \frac{z(f'(z)+f'(-z))}{f(z)-f(-z)} \left(\frac{f(z)-f(-z)}{2z}\right)^{\mu}$$
$$= h(z) + \frac{\lambda}{\mu} z h'(z).$$

An application of Lemma 2.4 yields the assertion of Theorem 3.4. □

Taking $q(z) = \frac{1+Az}{1+Bz}$ in Theorem 3.4, we obtain the following corollary.

**Corollary 3.5.** *Let $q(z)$ be convex univalent in $E$ and $-1 \leq B < A \leq 1$, $\lambda \in \mathbb{C}$ with $Re\ \lambda > 0$. Also let*

$$0 \neq \left(\frac{f(z)-f(-z)}{2z}\right)^{\mu} \in \mathcal{H}[q(0),1] \cap \mathcal{Q},$$

*and*

$$(1-\lambda)\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu} + \lambda \frac{z(f'(z)+f'(-z))}{f(z)-f(-z)} \left(\frac{f(z)-f(-z)}{2z}\right)^{\mu}$$

*be univalent in E. If*

$$\frac{1+Az}{1+Bz}+\lambda\frac{(A-B)z}{(1+Bz)^2}$$
$$\prec(1-\lambda)\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu}+\lambda\frac{z(f'(z)+f'(-z))}{f(z)-f(-z)}\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu},$$

*then*

$$\frac{1+Az}{1+Bz}\prec\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu},$$

*and* $\frac{1+Az}{1+Bz}$ *is the best subordinant.*

Combining the above results of subordination and superordination, we easily get the following "sandwich-type result".

**Corollary 3.6.** *Let* $q_1$ *be convex univalent and let* $q_2$ *be univalent in E,* $\lambda\in\mathbb{C}$ *with Re* $\lambda>0$*. Let* $q_2$ *satisfy (9). If*

$$0\neq\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu}\in\mathcal{H}[q(0),1]\cap\mathcal{Q},$$

*and*

$$(1-\lambda)\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu}+\lambda\frac{z(f'(z)+f'(-z))}{f(z)-f(-z)}\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu}$$

*is univalent in E, also*

$$q_1(z)+\frac{\lambda zq_1'(z)}{\mu}$$
$$\prec(1-\lambda)\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu}+\lambda\frac{z(f'(z)+f'(-z))}{f(z)-f(-z)}\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu}$$
$$\prec q_2(z)+\frac{\lambda zq_2'(z)}{\mu},$$

*then*

$$q_1(z)\prec\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu}\prec q_2(z),$$

*and* $q_1$ *and* $q_2$ *are, respectively, the best subordinant and dominant.*

**Theorem 3.7.** *If* $\lambda\in\mathbb{C}$*,* $\mu>0$ *and* $f(z)\in\mathcal{B}^{0,\mu}(m,1-2\rho,-1)$ $(0\leq\rho<1)$*, then* $f(z)\in\mathcal{B}^{\lambda,\mu}(m,1-2\rho,-1)$ *for* $|z|<R$*,*

*where*

$$R = \left( \left( \sqrt{\left( \frac{|\lambda|(m+1)}{\mu} \right)^2 + 1} \right) - \frac{|\lambda|(m+1)}{\mu} \right)^{\frac{1}{m+1}}. \tag{12}$$

*The bound R is best possible.*

*Proof.* Set

$$\left( \frac{f(z)-f(-z)}{2z} \right)^{\mu} = (1-\rho)h(z)+\rho, \quad z \in E, \quad 0 \le \rho < 1. \tag{13}$$

Then, clearly the function $h(z)$ is analytic in $E$ with $h(0)=1$. Proceeding as an Theorem 3.1, we have

$$\frac{1}{1-\rho} \left\{ (1-\lambda) \left( \frac{f(z)-f(-z)}{2z} \right)^{\mu} + \lambda \frac{z(f'(z)+f'(-z))}{f(z)-f(-z)} \left( \frac{f(z)-f(-z)}{2z} \right)^{\mu} - \rho \right\} = h(z) + \frac{\lambda z h'(z)}{\mu}. \tag{14}$$

Using the following well-known estimate, see [6]

$$\left| zh'(z) \right| \le \frac{2(m+1)r^{m+1} Re\,(h(z))}{(1-r^{2(m+1)})} \quad (|z| = r < 1)$$

in (14), we obtain that

$$Re\, \frac{1}{1-\rho} \left\{ (1-\lambda) \left( \frac{f(z)-f(-z)}{2z} \right)^{\mu} + \lambda \frac{z(f'(z)+f'(-z))}{f(z)-f(-z)} \left( \frac{f(z)-f(-z)}{2z} \right)^{\mu} - \rho \right\} \ge Re\, h(z) \left\{ 1 - \frac{2|\lambda|(m+1)r^{m+1}}{\mu(1-r^{2(m+1)})} \right\}. \tag{15}$$

Right hand side of (15) is positive, provided that $r < R$, where $R$ is given by (12).

In order to show that the bound $R$ is best possible, we consider the function $f(z) \in \mathcal{A}(m)$ defined by

$$\left( \frac{f(z)-f(-z)}{2z} \right)^{\mu} = (1-\rho) \frac{1+z^{m+1}}{1-z^{m+1}} + \rho, \;\; z \in E, 0 \le \rho < 1.$$

We note that

$$\frac{1}{1-\rho}\left\{(1-\lambda)\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu} + \lambda\frac{z(f'(z)+f'(-z))}{f(z)-f(-z)}\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu} - \rho\right\} = \frac{1+z^{m+1}}{1-z^{m+1}} + \frac{2|\lambda|(m+1)z^{m+1}}{\mu(1-z^{m+1})^2} = 0,$$

for $|z| = R$, we conclude that the bound is the best possible and this proves the theorem. □

**Theorem 3.8.** *Let* $0 \le \lambda_1 \le \lambda_2$ *and* $-1 \le B_1 \le B_2 < A_2 \le A_1 \le 1$. *Then*

$$\mathcal{B}^{\lambda_2,\mu}(m,A_2,B_2) \subset \mathcal{B}^{\lambda_1,\mu}(m,A_1,B_1). \tag{16}$$

*Proof.* Suppose that $f \in \mathcal{B}^{\lambda_2,\mu}(m,A_2,B_2)$. We know that

$$\left\{(1-\lambda_2)\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu} + \lambda_2\frac{z(f'(z)+f'(-z))}{f(z)-f(-z)}\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu}\right\} \prec \frac{1+A_2z}{1+B_2z}.$$

Since $-1 \le B_1 \le B_2 < A_2 \le A_1 \le 1$, we easily find that

$$\left\{(1-\lambda_2)\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu} + \lambda_2\frac{z(f'(z)+f'(-z))}{f(z)-f(-z)}\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu}\right\} \prec \frac{1+A_2z}{1+B_2z} \prec \frac{1+A_1z}{1+B_1z}, \tag{17}$$

that is $f \in \mathcal{B}^{\lambda_2,\mu}(m,A_1,B_1)$.
Thus the assertion (16) holds true for $0 \le \lambda_1 = \lambda_2$. If $\lambda_2 > \lambda_1 \ge 0$, by Theorem 3.1 and (17), we know that $f \in \mathcal{B}^{0,\mu}(m,A_2,B_2)$, that is,

$$\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu} \prec \frac{1+A_1z}{1+B_1z}. \tag{18}$$

At the same time, we have

$$\left\{(1-\lambda_1)\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu} + \lambda_1\frac{z(f'(z)+f'(-z))}{f(z)-f(-z)}\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu}\right\} = \frac{\lambda_1}{\lambda_2}\left[(1-\lambda_2)\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu} + \lambda_2\frac{z(f'(z)+f'(-z))}{f(z)-f(-z)}\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu}\right] + \left(1-\frac{\lambda_1}{\lambda_2}\right)\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu}. \tag{19}$$

Moreover,

$$0 \le \frac{\lambda_1}{\lambda_2} < 1,$$

and the function $\frac{1+A_1 z}{1+B_1 z}$, $-1 \le B_1 < A_1 \le 1$, $z \in E$ is analytic and convex in $E$. Combining (17-19) and Lemma 2.5, we find that

$$\left\{(1-\lambda_1)\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu} + \lambda_1 \frac{z(f'(z)+f'(-z))}{f(z)-f(-z)}\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu}\right\} \prec \frac{1+A_1 z}{1+B_1 z},$$

that is $f \in \mathcal{B}^{\lambda_1,\mu}(m,A_1,B_1)$, which implies that the assertion (16) of Theorem 3.8 holds and this completes the proof. □

**Theorem 3.9.** *Let $f \in \mathcal{B}^{\lambda,\mu}(m,A,B)$ with $\lambda > 0$ and $-1 \le B_1 < A_1 \le 1$. Then*

$$\frac{\mu}{\lambda(m+1)} \int_0^1 \frac{1-Au}{1-Bu} u^{\frac{\mu}{\lambda(m+1)}-1} du$$

$$< Re\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu} < \frac{\mu}{\lambda(m+1)} \int_0^1 \frac{1+Au}{1+Bu} u^{\frac{\mu}{\lambda(m+1)}-1} du. \tag{20}$$

*The extremal function of (20) is defined by*

$$F_{\lambda,\mu,m,A,B}(z) = 2z\left(\frac{\mu}{\lambda(m+1)} \int_0^1 \frac{1+Azu}{1+Bzu} u^{\frac{\mu}{\lambda(m+1)}-1} du\right)^{\frac{1}{\mu}}. \tag{21}$$

*Proof.* Let $f \in \mathcal{B}^{\lambda,\mu}(m,A,B)$ with $\lambda > 0$. From Theorem 3.1, we know that (5) holds, which implies that

$$Re\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu} < \sup_{z\in E} Re\left\{\frac{\mu}{\lambda(m+1)} \int_0^1 \frac{1+Azu}{1+Bzu} u^{\frac{\mu}{\lambda(m+1)}-1} du\right\}$$

$$\le \left\{\frac{\mu}{\lambda(m+1)} \int_0^1 \sup_{z\in E} Re\left(\frac{1+Azu}{1+Bzu}\right) u^{\frac{\mu}{\lambda(m+1)}-1} du\right\}$$

$$< \frac{\mu}{\lambda(m+1)} \int_0^1 \frac{1+Au}{1+Bu} u^{\frac{\mu}{\lambda(m+1)}-1} du, \tag{22}$$

and

$$Re\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu} > \inf_{z\in E} Re\left\{\frac{\mu}{\lambda(m+1)}\int_0^1 \frac{1+Azu}{1+Bzu}u^{\frac{\mu}{\lambda(m+1)}-1}du\right\}$$
$$\geq \left\{\frac{\mu}{\lambda(m+1)}\int_0^1 \inf_{z\in E} Re\left(\frac{1+Azu}{1+Bzu}\right)u^{\frac{\mu}{\lambda(m+1)}-1}du\right\}$$
$$> \frac{\mu}{\lambda(m+1)}\int_0^1 \frac{1+Au}{1+Bu}u^{\frac{\mu}{\lambda(m+1)}-1}du. \quad (23)$$

Combining (22) and (23), we obtain (20). Noting that the function $F_{\lambda,\mu,m,A,B}(z)$ defined by (21) belongs to the class $\mathcal{B}^{\lambda,\mu}(m,A,B)$, we get that inequality (20) is sharp. This completes the proof. □

In view of Theorem 3.9, we have the following distortion theorems for the class $\mathcal{B}^{\lambda,\mu}(m,A,B)$.

**Corollary 3.10.** *Let $f(z)\in\mathcal{B}^{\lambda,\mu}(m,A,B)$ with $\lambda>0$ and $-1\leq B<A\leq 1$. Then for $|z|=r<1$, we have*

$$2r\left(\frac{\mu}{\lambda(m+1)}\int_0^1 \frac{1-Aur}{1-Bur}u^{\frac{\mu}{\lambda(m+1)}-1}du\right)^{\frac{1}{\mu}}$$
$$< |f(z)-f(-z)| < 2r\left(\frac{\mu}{\lambda(m+1)}\int_0^1 \frac{1+Aur}{1+Bur}u^{\frac{\mu}{\lambda(m+1)}-1}du\right)^{\frac{1}{\mu}}. \quad (24)$$

*The extremal function of (24) is defined by (21).*

*By noting that*

$$(Re\,(v))^{\frac{1}{2}} \leq Re\,(v^{\frac{1}{2}}) \leq |v|^{\frac{1}{2}},\ \ v\in\mathbb{C}; Re\,v\geq 0.$$

From Theorem 3.9, we can easily derive the following result.

**Corollary 3.11.** *Let $f(z)\in\mathcal{B}^{\lambda,\mu}(m,A,B)$ with $\lambda>0$ and $-1\leq B<A\leq 1$. Then*

$$\left(\frac{\mu}{\lambda(m+1)}\int_0^1 \frac{1-Au}{1-Bu}u^{\frac{\mu}{\lambda(m+1)}-1}du\right)^{\frac{1}{2}}$$
$$< Re\left(\frac{f(z)-f(-z)}{2z}\right)^{\frac{\mu}{2}} < \left(\frac{\mu}{\lambda(m+1)}\int_0^1 \frac{1+Au}{1+Bu}u^{\frac{\mu}{\lambda(m+1)}-1}du\right)^{\frac{1}{2}}.$$

**Theorem 3.12.** *Let*

$$f(z)=z+\sum_{k=m+1}^{\infty}a_kz^k\in\mathcal{B}^{\lambda,\mu}(m,A,B),\ m\in\mathbb{N}. \tag{25}$$

*Then*

$$|a_{m+1}|\le\left|\frac{2(A-B)}{\lambda(m+1)+2\mu}\right|. \tag{26}$$

*The inequality (26) is sharp, with the extremal function defined by (21).*

*Proof.* Combining (2) and (25), we have

$$\begin{aligned}&(1-\lambda)\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu}+\lambda\frac{z(f'(z)+f'(-z))}{f(z)-f(-z)}\left(\frac{f(z)-f(-z)}{2z}\right)^{\mu}\\&=1+[1+\frac{\lambda(m+1)}{2\mu}]\mu a_{m+1}\,z^{m+1}+\cdots\prec\frac{1+Az}{1+Bz}\\&=1+(A-B)z+\ldots\end{aligned} \tag{27}$$

An application of Lemma 2.5 to (27) yields

$$\left|[1+\frac{\lambda(m+1)}{2\mu}]\mu a_{m+1}\right|\le|A-B|. \tag{28}$$

Thus, from (28), we easily arrive at (26) asserted by Theorem 3.12. □

**Theorem 3.13.** *Let $f(z)\in\mathcal{B}^{\lambda,\mu}(m,A,0)$ with $Re\,\lambda>0, A>0$ and $|\lambda|\left(1+Re\,\frac{\mu}{\lambda(m+1)}\right)>A\mu$. Then*

$$\left|\frac{z(f'(z)+f'(-z))}{f(z)-f(-z)}-1\right|<\frac{A\left[|\lambda|\left(m+1+Re\,\frac{\mu}{\lambda}\right)+\mu\right]}{|\lambda|\left[|\lambda|(m+1+Re\,\frac{\mu}{\lambda}-A\mu\right]}.$$

*Proof.* Let $h(z)$ be defined by (5). It follows from (6) that

$$h(z)+\frac{\lambda zh'(z)}{\mu}=1+Aw(z), \tag{29}$$

where

$$w(z)=\sum_{k=m+1}^{\infty}w_kz^k,\ \ m\in\mathbb{N},$$

is analytic in $E$ with $|w(z)|<1$, $z\in E$. From (29), we can get

$$\begin{aligned}h(z)&=1+A\frac{\mu}{\lambda}\int_0^1t^{\frac{\mu}{\lambda}-1}w(tz)dt\\&=1+A\frac{\mu}{\lambda}\sum_{k=m+1}^{\infty}\frac{1}{k+\frac{\mu}{\lambda}}w_kz^k.\end{aligned} \tag{30}$$

It follows from (30) that

$$
\begin{aligned}
(zh(z))' &= 1 + A\frac{\mu}{\lambda}\sum_{k=m+1}^{\infty}\frac{k+1}{k+\frac{\mu}{\lambda}}w_k z^k \\
&= 1 + A\frac{\mu}{\lambda}\sum_{k=m+1}^{\infty}\frac{1}{k+\frac{\mu}{\lambda}}w_k z^k \\
&\quad + A\frac{\mu}{\lambda}\left(w(z) - \frac{\mu}{\lambda}\int_0^1 t^{\frac{\mu}{\lambda}-1}w(tz)dt\right).
\end{aligned} \tag{31}
$$

We now find from (30) and (31) that

$$
zh'(z) = A\frac{\mu}{\lambda}\left(w(z) - \frac{\mu}{\lambda}\int_0^1 t^{\frac{\mu}{\lambda}-1}w(tz)dt\right). \tag{32}
$$

Combining (30) and (32), we can get

$$
\left|\frac{zh'(z)}{h(z)}\right| < \frac{A\mu\left[|\lambda|\left(m+1+Re\,\frac{\mu}{\lambda}\right)+\mu\right]}{|\lambda|\left[|\lambda|\left(m+1+Re\,\frac{\mu}{\lambda}-A\mu\right]\right.}. \tag{33}
$$

Thus, from (6) and (33), we easily arrive at the assertion of Theorem 3.13. □

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