PRESERVING PROPERTIES OF SUBORDINATION AND SUPERORDINATION OF ANALYTIC FUNCTIONS INVOLVING THE WRIGHT GENERALIZED HYPERGEOMETRIC FUNCTION

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In this paper, we obtain some subordination and superordination preserving results of analytic functions associated with the Wright generalized hypergeometric function. Sandwich-type result involving this operator is also derived.

1. Introduction

Let $H(U)$ be the class of functions analytic in $U = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \}$ and $H[a,k]$ be the subclass of $H(U)$ consisting of functions of the form $f(z) = a + a_k z^k + a_{k+1} z^{k+1} + \ldots$, with $H_0 \equiv H[0,1]$ and $H \equiv H[1,1]$.

Let $A_p$ denote the class of functions of the form

$$ f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \quad (p, \in \mathbb{N} = \{1,2,3,\ldots\}; z \in U), \quad (1) $$

which are analytic in the open unit disk $U$.

Let $f$ and $F$ be members of $H(U)$, the function $f(z)$ is said to be subordinate to $F(z)$, or $F(z)$ is said to be superordinate to $f(z)$, if there exists a function $w(z)$.

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analytic in $U$ with $w(0) = 0$ and $|w(z)| < 1 (z \in U)$, such that $f(z) = F(w(z))$. In such a case we write $f(z) \prec F(z)$. In particular, if $F$ is univalent, then $f(z) \prec F(z)$ if and only if $f(0) = F(0)$ and $f(U) \subset F(U)$ (see [1,2]). Let $\Psi : C^2 \times U \to C$ and let $h$ be univalent in $U$. If $p$ is analytic in $U$ and satisfies the first order differential subordination

$$\Psi\left(p(z), zp'(z); z\right) \prec h(z) (z \in U),$$

(2)

then $p$ is called a solution of the differential subordination (2). The univalent function $q$ is called a dominant solutions of the differential subordination (2) if $p \prec q$ for all $p$ satisfying (2). A dominant $\bar{q}$ that satisfies $\bar{q} \prec q$ for all dominants $q$ of (2) is said to be the best dominant of (2). Similarly, let $\Phi : C^2 \times U \to C$ and let $h$ be univalent in $U$. If $p$ is analytic in $U$ and satisfies the first order differential superordination

$$h(z) \prec \Phi\left(p(z), zp'(z); z\right) (z \in U),$$

(3)

then $p$ is called a solution of the differential superordination (3). The univalent function $q$ is called a subordinant solutions of the differential superordination (3) if $q \prec p$ for all $p$ satisfying (3). A subordinant $\bar{q}$ that satisfies $q \prec \bar{q}$ for all subordinant $q$ of (3) is said to be the best subordinant. (see the monograph by Miller and Mocanu [11], and [12]). Let $\alpha_1, A_1, ..., \alpha_l, A_l$ and $\beta_1, B_1, ..., \beta_m, B_m$ ($l, m \in \mathbb{N} = \{1, 2, ...\}$) be positive real parameters such that

$$1 + \sum_{k=1}^{m} B_k - \sum_{k=1}^{l} A_k > 0.$$  

(4)

The Wright generalized hypergeometric function (see [14], [15] and [16])

$$I_{\Psi} m\left[\left(\alpha_1, A_1, ..., \alpha_l, A_l\right); \left(\beta_1, B_1, ..., \beta_m, B_m\right); z\right] = L_{\Psi} m\left[\left(\alpha_n, A_n\right)_{1,l}; \left(\beta_n, B_n\right)_{1,m}; z\right]$$

is defined by

$$I_{\Psi} m\left[\left(\alpha_n, A_n\right)_{1,l}; \left(\beta_n, B_n\right)_{1,m}; z\right] = \sum_{k=0}^{\infty} \left\{ \prod_{n=1}^{l} \Gamma(\alpha_n + kA_n) \right\} \left\{ \prod_{n=1}^{m} \Gamma(\beta_n + kB_n) \right\}^{-1} \frac{z^k}{k!} (z \in U).$$

(5)

If $A_n = 1$, $(n = 1, \ldots, l)$, $B_n = 1$, $(n = 1, \ldots, m)$, we have

$$\Omega I_{\Psi} m\left[\left(\alpha_n, 1\right)_{1,l}; \left(\beta_n, 1\right)_{1,m}; z\right] = I_{F} m\left(\alpha_1, ... \alpha_l, \beta_1, ... \beta_m, z\right),$$

(6)

which is the generalized hypergeometric function where

$$\Omega = \left( \prod_{n=1}^{l} \Gamma(\alpha_n) \right)^{-1} \left( \prod_{n=1}^{m} \Gamma(\beta_n) \right).$$

(7)
Using the Wright hypergeometric function, Dziok and Raina ([7] and [8]) introduced the linear operator

\[ \theta^l_m \left[ (\alpha_n, A_n)_{1,l}; (\beta_n, B_n)_{1,m} \right] : A_p \to A_p, \]

which is defined by the following convolution

\[ \theta^l_m \left[ (\alpha_n, A_n)_{1,l}; (\beta_n, B_n)_{1,m} \right] f(z) = \phi^l_m \left[ (\alpha_n, A_n)_{1,l}; (\beta_n, B_n)_{1,m} \right] * f(z), \quad (8) \]

where

\[ \phi^l_m \left[ (\alpha_n, A_n)_{1,l}; (\beta_n, B_n)_{1,m} \right] = \Omega(z^p)_m [ \Omega(z^p)_m \left[ (\alpha_n, A_n)_{1,l}; (\beta_n, B_n)_{1,m} ; z \right] * f(z) ] \]  

\[ \text{If } f(z) \in A_p \text{ is given by equation (1), then we have} \]

\[ \theta^l_m \left[ (\alpha_n, A_n)_{1,l}; (\beta_n, B_n)_{1,m} \right] f(z) = z^p + \sum_{k=1}^{\infty} \sigma(k) a_k z^{k+p}, \quad (10) \]

where

\[ \sigma(k) = \Omega \prod_{n=1}^{l} \Gamma(\alpha_n + k A_n) \prod_{n=1}^{m} \Gamma(\beta_n + k B_n) k!. \quad (11) \]

In order to make the notation simple, we write

\[ \theta^l_m \left[ (\alpha_n, A_n)_{1,l}; (\beta_n, B_n)_{1,m} \right] = \theta^l_m \left[ \alpha_1, A_1, B_1 \right]. \quad (12) \]

It is easily verified from (9) that

\[ z \left( \theta^l_m \left[ \alpha_1, A_1, B_1 \right] f(z) \right)' = \frac{\alpha_1}{A_1} \theta^l_m \left[ \alpha_1 + 1, A_1, B_1 \right] f(z) - \left( \frac{\alpha_1}{A_1} - p \right) \theta^l_m \left[ \alpha_1, A_1, B_1 \right] f(z) \quad (A_1 > 0). \quad (13) \]

Not that for \( A_n = 1, (n = 1, \ldots, l), B_n = 1, (n = 1, \ldots, m), \) we have

\[ \theta^l_m \left[ \alpha_1, 1, 1 \right] = H^l_m \left[ \alpha_1 \right], \quad (14) \]

where \( H^l_m \left[ \alpha_1 \right] \) is the Dziok–Srivastava operator [5].

It is well known [6] that

\[ z \left[ H^l_m \left[ \alpha_1 \right] f(z) \right]' = \alpha_1 H^l_m \left[ \alpha_1 + 1 \right] f(z) - (\alpha_1 - p) \alpha_1 H^l_m \left[ \alpha_1 \right] f(z), \quad (15) \]

where \( H^l_m \left[ \alpha_1 \right] f(z) = H^l_m \left( \alpha_1, \ldots, \alpha_i; \beta_1, \ldots, \beta_m \right) f(z). \)

To prove our results, we need the following definitions and lemmas.
Definition 1.1. ([11]) Denote by $Q$ the set of all functions $q(z)$ that are analytic and injective on $\bar{U}/E(q)$ where

$$E(q) = \{ \zeta \in \partial U : \lim_{z \to \zeta} q(z) = \infty \},$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial U/E(q)$. Further, let the subclass of $Q$ for which $q(0) = a$ be denoted by $Q(a), Q(0) \equiv Q_0$ and $Q(1) \equiv Q_1$.

Definition 1.2. ([12]) A function $L(z,t) (z \in U, t \geq 0)$ is said to be a subordination chain if $L(0,t)$ is analytic and univalent in $z \in U$ for all $t \geq 0$, $L(z,0)$ is continuously differentiable on $[0;1]$ for all $z \in U$ and $L(z,t_1) \prec L(z,t_2)$ for all $0 \leq t_1 \leq t_2$.

Lemma 1.3. ([13]) The function $L(z,t) : U \times [0;1] \to \mathbb{C}$ of the form

$$L(z,t) = a_1(t)z + a_2(t)z^2 + \ldots \quad (a_1(t) \neq 0; t \geq 0),$$

and $\lim_{t \to \infty} |a_1(t)| = \infty$ is a subordination chain if and only if

$$\Re \left\{ \frac{z \partial L(z,t)}{\partial t} \right\} > 0 \quad (z \in U, t \geq 0).$$

Lemma 1.4. ([9]) Suppose that the function $H : \mathbb{C}^2 \to \mathbb{C}$ satisfies the condition

$$\Re \{ H(is,t) \} \leq 0$$

for all real $s$ and for all $t \leq -n(1+s^2)/2, n \in \mathbb{N}$. If the function $p(z) = 1 + a_nz^n + a_{n+1}z^{n+1} + \ldots$, is analytic in $U$ and

$$\Re \{ H(p(z); zp'(z)) \} > 0 \quad (z \in U).$$

then $\Re \{ p(z) \} > 0$ for $z \in U$.

Lemma 1.5. ([10]) Let $k, \gamma \in \mathbb{C}$ with $k \neq 0$ and let $h \in H(U)$ with $H(0) = c$. If $\Re \{ kh(z) + \gamma \} > 0$ $(z \in U)$, then the solution of the following differential equation:

$$q(z) + \frac{zq'(z)}{kq(z)} = h(z)(z \in U; q(0) = c),$$

is analytic in $U$ and satisfies $\Re \{ kh(z) + \gamma \} > 0$ for $z \in U$.

Lemma 1.6. ([11]) Let $p \in Q(a)$ and let $q(z) = a + a_nz^n + a_{n+1}z^{n+1} + \ldots$, be analytic in $U$ with $q(z) \neq 0$ and $n \geq 1$. If $q$ is not subordinate to $p$, the there exists two points $z_0 = r_0e^{i\theta} \in U$ and $\xi_0 \in \partial U/E(q)$ such that

$q(U_{r_0}) \subset p(U); q(z_0) = p(\xi_0)$ and $z_0p'(z_0) = m\xi_0p'(\xi_0) m \geq n$. 
Lemma 1.7. ([12]) Let \( q \in H[a, 1] \) and \( \phi : \mathbb{C}^2 \rightarrow \mathbb{C} \) also \( \phi (q(z), tzq'(z)) = h(z) \). If \( L(z,t) = \phi (q(z), tzq'(z)) \) is a subordination chain and \( q \in H[a, 1] \cap \mathcal{Q}(a) \), then
\[
\h(z) \prec \phi (p(z), zp'(z)),
\]
implies that \( q(z) \prec p(z) \). Further, if \( \phi (q(z), tzq'(z)) = h(z) \) has a univalent solution \( q \in \mathcal{Q}(a) \), then \( q \) is the best subordination.

In the present paper, we aim to prove some subordination-preserving and superordination-preserving properties associated with the fractional differintegral operator \( \theta_p^{L,m} [\alpha_1, A_1, B_1] \). Sandwich-type result involving this operator is also derived. A similar problem for analytic functions was studied by Aouf and Seoudy [3] and [4].

2. Subordination, superordination and sandwich results involving the operator \( \theta_p^{L,m} [\alpha_1, A_1, B_1] \)

Theorem 2.1. Let \( f, g \in A_p \) and let
\[
\Re \left\{ 1 + \frac{z\phi''(z)}{\phi'(z)} \right\} > -\delta,
\]
where
\[
\phi(z) = \left( \frac{\theta_p^{l,m} [\alpha_1 + 1, A_1, B_1] g(z)}{\theta_p^{l,m} [\alpha_1, A_1, B_1] g(z)} \right) \left( \frac{\theta_p^{l,m} [\alpha_1, A_1, B_1] g(z)}{z^p} \right)^\mu \quad (\mu > 0; z \in U),
\]
where \( \alpha_1, A_1, \ldots, \alpha_l, A_l \) and \( \beta_1, B_1, \ldots, \beta_m, B_m \) \((l, m \in \mathbb{N} = \{1, 2, \ldots \})\) are positive real parameters such that \( 1 + \sum_{k=1}^{m} B_k - \sum_{k=1}^{l} A_k > 0 \), and \( \delta \) is given by
\[
\delta = \frac{A_1^2 + \mu^2 \alpha_1^2 - A_1^2 - \mu^2 \alpha_1^2}{4\mu A_1 \alpha_1}.
\]

Then the subordination condition
\[
\left( \frac{\theta_p^{l,m} [\alpha_1 + 1, A_1, B_1] f(z)}{\theta_p^{l,m} [\alpha_1, A_1, B_1] f(z)} \right) \left( \frac{\theta_p^{l,m} [\alpha_1, A_1, B_1] f(z)}{z^p} \right)^\mu
\]
\[
\prec \left( \frac{\theta_p^{l,m} [\alpha_1 + 1, A_1, B_1] g(z)}{\theta_p^{l,m} [\alpha_1, A_1, B_1] g(z)} \right) \left( \frac{\theta_p^{l,m} [\alpha_1, A_1, B_1] g(z)}{z^p} \right)^\mu
\]
implies that
\[
\left( \frac{\theta^{l,m}_{p}[\alpha_1,A_1,B_1]f(z)}{z^p} \right)^{\mu} < \left( \frac{\theta^{l,m}_{p}[\alpha_1,A_1,B_1]g(z)}{z^p} \right)^{\mu},
\]
and the function \( \left( \frac{\theta^{l,m}_{p}[\alpha_1,A_1,B_1]g(z)}{z^p} \right)^{\mu} \) is the best dominant.

**Proof.** Let us define the functions \( F(z) \) and \( G(z) \) in \( U \) by
\[
F(z) = \left( \frac{\theta^{l,m}_{p}[\alpha_1,A_1,B_1]f(z)}{z^p} \right)^{\mu} \quad \text{and} \quad G(z) = \left( \frac{\theta^{l,m}_{p}[\alpha_1,A_1,B_1]g(z)}{z^p} \right)^{\mu}\]
(19)
we assume here, without loss of generality, that \( G(z) \) is analytic and univalent on \( U \) and
\[
G'(\zeta) \neq 0 \quad (|\zeta| = 1).
\]
If not, then we replace \( F(z) \) and \( G(z) \) by \( F(\rho z) \) and \( G(\rho z) \), respectively, with \( 0 < \rho < 1 \). These new functions have the desired properties on \( \bar{U} \), and we can use them in the proof of our result. Therefore, the results would follow by letting \( \rho \to 1 \). We first show that, if
\[
q(z) = 1 + \frac{zG''(z)}{G'(z)} \quad (z \in U),
\]
(20)
then
\[
\Re \{ q(z) \} > 0 \quad (z \in U).
\]
From (11) and the definition of the functions \( G, \phi \), we obtain that
\[
\phi(z) = G(z) + \frac{A_1zG'(z)}{\mu \alpha_1}.
\]
(21)
Differentiating both side of (21) with respect to \( z \) yields
\[
\phi''(z) = \left( 1 + \frac{A_1}{\mu \alpha_1} \right) G'(z) + \frac{A_1zG'(z)}{\mu \alpha_1}.
\]
(22)
Combining (20) and (22), we easily get
\[
1 + \frac{z\phi''(z)}{\phi'(z)} = q(z) + \frac{A_1zq'(z)}{q(z) + \mu \alpha_1} = h(z) \quad (z \in U).
\]
(23)
It follows from (16) and (23) that
\[ \Re \left\{ h(z) + \frac{A_1 \alpha_1}{A} \right\} > 0 \quad (z \in U). \]  

(24)

Moreover, by using Lemma 1.5, we conclude that the differential equation (23) has a solution \( q(z) \in H(U) \) with \( h(0) = q(0) = 1 \). Let

\[ H(u, v) = u + \frac{A_1 v}{A_1 u + \mu \alpha_1} + \delta, \]

where \( \delta \) is given by (18). From (23) and (24), we obtain
\[ \Re \{ H(q(z); zq'(z)) \} > 0 \quad (z \in U). \]

To verify the condition that \( \Re \{ H(is; t) \} \leq 0 \quad (t \leq -(1+s^2)/2; s \in \mathbb{R}) \),

(25)

we proceed it as follows:
\[ \Re \{ H(is; t) \} = \Re \left\{ is + \frac{A_1 t}{A_1 is + \mu \alpha_1} + \delta \right\} = \frac{A_1 \alpha_1 t \mu}{A_1^2 s^2 + \mu^2 \alpha_1^2} + \delta \]
\[ \leq -\frac{A_1^2 \psi_p(\alpha_1, \mu, A_1, s)}{2 [A_1^2 s^2 + \mu^2 \alpha_1^2]}, \]

where
\[ \psi_p(\alpha_1, \mu, A_1, s) = \left[ \frac{\mu \alpha_1}{A_1} - 2\delta \right] s^2 - 2\delta \frac{\mu^2 \alpha_1^2}{A_1^2} + \frac{\mu \alpha_1}{A_1}. \]  

(26)

For \( \delta \) given by (18), we note that the expression \( \psi_p(\alpha_1, \mu, A_1, s) \) in (26) is a positive, which implies that (25) holds. Thus, by using Lemma 1.4, we conclude that
\[ \Re \{ q(z) \} > 0 \quad (z \in U). \]

By the definition of \( q(z) \), we know that \( G \) is convex. To prove \( F \prec G \), let the function \( L(z, t) \) be defined by
\[ L(z, t) = G(z) + \frac{(1+t)A_1 z G'(z)}{\mu \alpha_1} \quad (0 \leq t < \infty; z \in U). \]  

(27)

Since \( G \) is convex, then
\[ \frac{\partial L(z, t)}{\partial z} \bigg|_{z=0} = G'(0) \left( 1 + \frac{A_1(1+t)}{\mu \alpha_1} \right) \neq 0 \quad (0 \leq t < \infty; z \in U). \]
Therefore, by using Lemma 1.3, we deduce that \( L(z, t) \) is a subordination chain. It follows from the definition of subordination chain that

\[
\phi(z) = G(z) + \frac{A_1 z G'(z)}{\mu \alpha_1} = L(z, 0),
\]

and

\[
L(z, 0) \prec L(z, t) \quad (0 \leq t < \infty),
\]

which implies

\[
L(\zeta, t) \notin L(U, 0) \quad (0 \leq t < \infty; \zeta \in \partial U),
\]

(28)

If \( F \) is not subordinate to \( G \), by using Lemma 1.6, we know that there exist two points \( z_0 \in U \) and \( \zeta_0 \in \partial U \) such that

\[
F(z_0) = G(\zeta_0) \quad \text{and} \quad z_0 F'(z_0) = (1 + t) \zeta_0 p(\zeta_0) \quad (0 \leq t < \infty).
\]

(29)

Hence, by virtue of (11) and (29), we have

\[
L(\zeta_0, t) = G(\zeta_0) + \frac{(1 + t) A_1 z G'(\zeta_0)}{\mu \alpha_1} \quad = F(z_0) + \frac{A_1 z_0 F'(z_0)}{\mu \alpha_1}
\]

\[
= \left( \frac{\theta_p^{l,m} [\alpha_1 + 1, A_1, B_1] f(z_0)}{\theta_p^{l,m} [\alpha_1, A_1, B_1] f(z_0)} \right) \mu \phi(U).
\]

This contradicts to (28). Thus, we deduce that \( F \prec G \). Considering \( F = G \), we see that the function \( G \) is the best dominant.

By taking \( A_n = 1 \), \( (n = 1, \ldots, l) \) and \( B_n = 1 \), \( (n = 1, \ldots, m) \), in Theorem 2.1 and using the relation (14) we get the following corollary

**Corollary 2.2.** Let \( f, g \in A_p \) and let

\[
\Re \left\{ 1 + \frac{z \phi''(z)}{\phi'(z)} \right\} > -\delta,
\]

(30)

where

\[
\phi(z) = \left( \frac{H_p^{l,m} [\alpha_1 + 1] g(z)}{H_p^{l,m} [\alpha_1] g(z)} \right) \left( \frac{H_p^{l,m} [\alpha_1] g(z)}{z^p} \right) (\mu > 0; z \in U),
\]

(31)
where \( \alpha_1, \ldots, \alpha_l \) and \( \beta_1, \ldots, \beta_m \) are positive real parameters and \( \delta \) is given by
\[
\delta = \frac{1 + \mu^2 \alpha_1^2 - |1 - \mu^2 \alpha_1^2|}{4 \mu \alpha_1}.
\] (32)

Then the subordination condition
\[
\left( \frac{H_p^{l,m}[\alpha_1 + 1] f(z)}{H_p^{l,m}[\alpha_1] f(z)} \right)^\mu \prec \left( \frac{H_p^{l,m}[\alpha_1 + 1] g(z)}{H_p^{l,m}[\alpha_1] g(z)} \right)^\mu,
\]
implies that
\[
\left( \frac{H_p^{l,m}[\alpha_1] f(z)}{z^\mu} \right) \prec \left( \frac{H_p^{l,m}[\alpha_1] g(z)}{z^\mu} \right),
\]
and the function \( \left( \frac{H_p^{l,m}[\alpha_1] g(z)}{z^\mu} \right)^\mu \) is the best dominant.

We now derive the following superordination result.

**Theorem 2.3.** Let \( f, g \in A_p \) and let
\[
\mathcal{R} \left\{ 1 + \frac{z \phi''(z)}{\phi'(z)} \right\} > -\delta,
\] (33)
where
\[
\phi(z) = \left( \frac{\theta_p^{l,m}[\alpha_1 + 1, A_1, B_1] g(z)}{\theta_p^{l,m}[\alpha_1, A_1, B_1] g(z)} \right)^\mu \left( \frac{\theta_p^{l,m}[\alpha_1, A_1, B_1] g(z)}{z^\mu} \right) (\mu > 0; z \in U),
\] (34)
where \( \alpha_1, A_1, \ldots, \alpha_l, A_l \) and \( \beta_1, B_1, \ldots, \beta_m, B_m \) are positive real parameters such that \( 1 + \sum_{k=1}^m B_k - \sum_{k=1}^l A_k > 0 \), and \( \delta \) is given by (18).

If the function
\[
\left( \frac{\theta_p^{l,m}[\alpha_1 + 1, A_1, B_1] f(z)}{\theta_p^{l,m}[\alpha_1, A_1, B_1] f(z)} \right)^\mu \left( \frac{\theta_p^{l,m}[\alpha_1, A_1, B_1] f(z)}{z^\mu} \right)
\]
is univalent in \( U \) and
\[
\left( \frac{\theta_p^{l,m}[\alpha_1 + 1, A_1, B_1] f(z)}{\theta_p^{l,m}[\alpha_1, A_1, B_1] f(z)} \right)^\mu \in Q,
\]
then the superordination condition
\[
\left( \frac{\theta_p^{l,m}[\alpha_1 + 1, A_1, B_1] f(z)}{\theta_p^{l,m}[\alpha_1, A_1, B_1] f(z)} \right)^\mu \prec \left( \frac{\theta_p^{l,m}[\alpha_1, A_1, B_1] f(z)}{z^\mu} \right)^\mu,
\]
implies that
\[
\left( \frac{\theta_p^{l,m}[\alpha_1 + 1, A_1, B_1] f(z)}{\theta_p^{l,m}[\alpha_1, A_1, B_1] f(z)} \right)^\mu \prec \left( \frac{\theta_p^{l,m}[\alpha_1, A_1, B_1] f(z)}{z^\mu} \right)^\mu,
\]
implies that
\[
\left( \frac{\theta_{p,l,m}^{l,m} [\alpha_1, A_1, B_1] g (z)}{z^p} \right)^\mu \prec \left( \frac{\theta_{p,l,m}^{l,m} [\alpha_1, A_1, B_1] f (z)}{z^p} \right)^\mu,
\]
and the function \( \left( \frac{\theta_{p,l,m}^{l,m} [\alpha_1, A_1, B_1] g (z)}{z^p} \right)^\mu \) is the best subordinant.

**Proof.** Suppose that the functions \( F, G \) and \( q \) are defined by (19) and (20), respectively. By applying the similar method as in the proof of Theorem 2.1, we get
\[
\Re \{ q(z) \} > 0 \ (z \in U).
\]
Next, to arrive at our desired result, we show that \( G \prec F \). For this, we suppose that the function \( L (z, t) \) be defined by (27).
Since \( G \) is convex, by applying a similar method as in Theorem 2.1, we deduce that \( L (z, t) \) is subordination chain. Therefore, by using Lemma 1.7, we conclude that \( G \prec F \). Moreover, since the differential equation
\[
\phi(z) = G(z) + \frac{A_1 z G'(z)}{\mu \alpha_1} = \phi \left( G(z), zG'(z) \right)
\]
has a univalent solution \( G \), it is the best subordinant. \( \square \)

By taking \( A_n = 1 \), \((n = 1, \ldots, l)\) and \( B_n = 1 \), \((n = 1, \ldots, m)\), in Theorem 2.3 and using the relation (14) we get the following corollary

**Corollary 2.4.** Let \( f, g \in \mathcal{A}_p \) and let
\[
\Re \left\{ 1 + \frac{z \phi''(z)}{\phi'(z)} \right\} > -\delta,
\]
where
\[
\phi(z) = \left( \frac{H_{l,m}^{l,m} [\alpha_1 + 1] g (z)}{H_{l,m}^{l,m} [\alpha_1] g (z)} \right) \left( \frac{H_{l,m}^{l,m} [\alpha_1] g (z)}{z^p} \right)^\mu (\mu > 0; z \in U),
\]
where \( \alpha_1, \ldots, \alpha_l \) and \( \beta_1, \ldots, \beta_m (l, m \in \mathbb{N} = \{1, 2, \ldots\}) \) are positive real parameters and \( \delta \) is given by (32).
If the function
\[
\left( \frac{H_{l,m}^{l,m} [\alpha_1 + 1] f (z)}{H_{l,m}^{l,m} [\alpha_1] f (z)} \right) \left( \frac{H_{l,m}^{l,m} [\alpha_1] f (z)}{z^p} \right)^\mu
\]
is univalent in \( U \)
and \( \left( \frac{H_{l,m}^{l,m} [\alpha_1] f (z)}{z^p} \right)^\mu \in \mathcal{Q}, \) then the superordination condition
\[
\left( \frac{H_p^{1,m} [\alpha_1 + 1] g(z)}{H_p^{1,m} [\alpha_1] g(z)} \right) \left( \frac{H_p^{1,m} [\alpha_1] g(z)}{z^p} \right) \mu
\]

\[
\prec \left( \frac{H_p^{1,m} [\alpha_1 + 1] f(z)}{H_p^{1,m} [\alpha_1] f(z)} \right) \left( \frac{H_p^{1,m} [\alpha_1] f(z)}{z^p} \right) \mu
\]

implies that

\[
\left( \frac{H_p^{1,m} [\alpha_1] g(z)}{z^p} \right) \mu \prec \left( \frac{H_p^{1,m} [\alpha_1] f(z)}{z^p} \right) \mu,
\]

and the function \( \left( \frac{H_p^{1,m} [\alpha_1] g(z)}{z^p} \right) \mu \) is the best subordinant.

Combining Theorems 2.1 and 2.3, we obtain the following “sandwich-type result”.

**Theorem 2.5.** Let \( f, g_i \in A_p \ (j = 1, 2) \) and let

\[
\Re \left\{ 1 + \frac{z \phi_j'' (z)}{\phi_j (z)} \right\} > -\delta,
\]

where

\[
\phi_j(z) = \left( \frac{\theta_p^{1,m} [\alpha_1 + 1, A_1, B_1] g_j(z)}{\theta_p^{1,m} [\alpha_1, A_1, B_1] g_j(z)} \right) \cdot \left( \frac{\theta_p^{1,m} [\alpha_1, A_1, B_1] g_j(z)}{z^p} \right) \mu \ (\mu > 0; z \in U)
\]

where \( \alpha_1, \ldots, \alpha_i, \alpha_i \) and \( \beta_j, B_1, \ldots, B_m, B_m \ (L, m \in \mathbb{N} = \{1, 2, \ldots\}) \) are positive real parameters such that \( 1 + \sum_{k=1}^m B_k - \sum_{k=1}^m A_k > 0 \), and \( \delta \) is given by (18).

If the function

\[
\left( \frac{\theta_p^{1,m} [\alpha_1 + 1, A_1, B_1] f(z)}{\theta_p^{1,m} [\alpha_1, A_1, B_1] f(z)} \right) \mu
\]

is univalent in \( U \) and

\[
\left( \frac{\theta_p^{1,m} [\alpha_1, A_1, B_1] f(z)}{z^p} \right)^\mu \in Q,
\]

then the condition

\[
\left( \frac{\theta_p^{1,m} [\alpha_1 + 1, A_1, B_1] g_1(z)}{\theta_p^{1,m} [\alpha_1, A_1, B_1] g_1(z)} \right) \mu
\]

\[
\prec \left( \frac{\theta_p^{1,m} [\alpha_1 + 1, A_1, B_1] f(z)}{\theta_p^{1,m} [\alpha_1, A_1, B_1] f(z)} \right) \mu
\]

implies that

\[
\left( \frac{\theta_p^{1,m} [\alpha_1, A_1, B_1] g_1(z)}{z^p} \right) \mu \prec \left( \frac{\theta_p^{1,m} [\alpha_1, A_1, B_1] f(z)}{z^p} \right) \mu.
\]
By taking \( A \) the best subordinant and the best dominant.

\[
\left( \frac{\theta_{p}^{l,m}[\alpha_{1} + 1, A_{1}, B_{1}] g_{2}(z)}{\theta_{p}^{l,m}[\alpha_{1}, A_{1}, B_{1}] g_{2}(z)} \right)^{\mu} \left( \frac{\theta_{p}^{l,m}[\alpha_{1}, A_{1}, B_{1}] g_{2}(z)}{z^{p}} \right) = 0, 
\]

implies that

\[
\left( \frac{\theta_{p}^{l,m}[\alpha_{1}, A_{1}, B_{1}] g_{1}(z)}{z^{p}} \right)^{\mu} \left( \frac{\theta_{p}^{l,m}[\alpha_{1}, A_{1}, B_{1}] f(z)}{z^{p}} \right) = \left( \frac{\theta_{p}^{l,m}[\alpha_{1}, A_{1}, B_{1}] g_{1}(z)}{z^{p}} \right)^{\mu}, 
\]

and the function \( \left( \frac{\theta_{p}^{l,m}[\alpha_{1}, A_{1}, B_{1}] g_{1}(z)}{z^{p}} \right)^{\mu} \) and \( \left( \frac{\theta_{p}^{l,m}[\alpha_{1}, A_{1}, B_{1}] g_{2}(z)}{z^{p}} \right)^{\mu} \) are, respectively, the best subordinant and the best dominant.

By taking \( A_{n} = 1, (n = 1, \ldots, l) \) and \( B_{n} = 1, (n = 1, \ldots, m), \) in Theorem 2.5 and using the relation (14) we get the following corollary

**Corollary 2.6.** Let \( f, g_{j} \in A_{p} \ (j = 1, 2) \) and let

\[
\Re \left\{ 1 + \frac{z \phi_{j}''(z)}{\phi_{j}'(z)} \right\} > -\delta, \quad (39)
\]

where

\[
\phi_{j}(z) = \left( \frac{H_{p}^{l,m}[\alpha_{1} + 1] g_{j}(z)}{H_{p}^{l,m}[\alpha_{1}] g_{j}(z)} \right)^{\mu} \left( \frac{H_{p}^{l,m}[\alpha_{1}] g_{j}(z)}{z^{p}} \right) \ (\mu > 0; z \in U) \quad (40)
\]

where \( \alpha_{1}, \ldots, \alpha_{l} \) and \( \beta_{1}, \ldots, \beta_{m} (l, m \in \mathbb{N} = \{1, 2, \ldots\}) \) are positive real parameters and \( \delta \) is given by (32). If the function \( \left( \frac{H_{p}^{l,m} g_{1}(z)}{H_{p}^{l,m} f(z)} \right)^{\mu} \) is univalent in \( U \) and \( \left( \frac{H_{p}^{l,m} g_{1}(z)}{z^{p}} \right)^{\mu} \) is a part of the relation (39).
implies that
\[
\left( \frac{H_{p}^{l,m}[\alpha_{1}]g_{1}(z)}{z^{p}} \right)^{\mu} \prec \left( \frac{H_{p}^{l,m}[\alpha_{1}]f(z)}{z^{p}} \right)^{\mu} \prec \left( \frac{H_{p}^{l,m}[\alpha_{1}]g_{1}(z)}{z^{p}} \right)^{\mu},
\]
and the function \( \left( \frac{H_{p}^{l,m}[\alpha_{1}]g_{1}(z)}{z^{p}} \right)^{\mu} \) and \( \left( \frac{H_{p}^{l,m}[\alpha_{1}]g_{2}(z)}{z^{p}} \right)^{\mu} \) are, respectively, the best subordinant and the best dominant.

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REFERENCES


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