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PRESERVING PROPERTIES OF SUBORDINATION AND SUPERORDINATION OF ANALYTIC FUNCTIONS INVOLVING THE WRIGHT GENERALIZED HYPERGEOMETRIC FUNCTION

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In this paper, we obtain some subordination and superordination - preserving results of analytic functions associated with the Wright generalized hypergeometric function. Sandwich-type result involving this operator is also derived.

1. Introduction

Let $H(U)$ be the class of functions analytic in $U = \{z : z \in C \text{and } |z| < 1\}$ and $H[a,k]$ be the subclass of $H(U)$ consisting of functions of the form $f(z) = a + a_k z^k + a_{k+1} z^{k+1} + ...$, with $H_0 \equiv H[0,1]$ and $H \equiv H[1,1]$.
Let A_p denote the class of functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \, (p, \in \mathbb{N} = \{1,2,3,...\}; z \in U), \tag{1}$$

which are analytic in the open unit disk U.
Let f and F be members of $H(U)$, the function $f(z)$ is said to be subordinate to $F(z)$, or $F(z)$ is said to be superordinate to $f(z)$, if there exists a function $w(z)$

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analytic in U with $w(0)=0$ and $|w(z)|<1 (z\in U)$, such that $f(z)=F(w(z))$. In such a case we write $f(z)\prec F(z)$. In particular, if F is univalent, then $f(z)\prec F(z)$ if and only if $f(0)=F(0)$ and $f(U)\subset F(U)$(see [1,2]).
Let $\Psi: C^2\times U\to C$ and let h be univalent in U. If p is analytic in U and satisfies the first order differential subordination

$$\Psi\left(p(z), zp'(z); z\right)\prec h(z) \, (z\in U), \tag{2}$$

then p is called a solution of the differential subordination (2).
The univalent function q is called a dominant solutions of the differential subordination (2) if $p\prec q$ for all p satisfying (2). A dominant $\tilde{q}$ that satisfies $\tilde{q}\prec q$ for all dominants q of (2) is said to be the best dominant of (2).
Similarly, let $\Phi: C^2\times U\to C$ and let h be univalent in U. If p is analytic in $U$and satisfies the first order differential superordination

$$h(z)\prec \Phi\left(p(z), zp'(z); z\right) (z\in U), \tag{3}$$

then p is called a solution of the differential superordination (3).
The univalent function q is called a subordinant solutions of the differential superordination (3) if $q\prec p$ for all p satisfying (3). A subordinant $\tilde{q}$ that satisfies $q\prec\tilde{q}$ for all subordinant q of (3) is said to be the best subordinant. (see the monograph by Miller and Mocanu [11], and [12]).
Let $\alpha_1, A_1, ..., \alpha_l, A_l$ and $\beta_1, B_1, ..., \beta_m, B_m$ $(l, m\in\mathbb{N}=\{1,2...\})$ be positive real parameters such that

$$1+\sum_{k=1}^{m} B_k-\sum_{k=1}^{l} A_k>0. \tag{4}$$

The Wright generalized hypergeometric function (see [14], [15] and [16])
$_l\Psi_m[(\alpha_1, A_1, ..., \alpha_l, A_l); (\beta_1, B_1, ..., \beta_m, B_m); z]$
$=_L\Psi_m\left[(\alpha_n, A_n)_{1,l}; (\beta_n, B_n)_{1,m}; z\right]$ is defined by

$$\begin{aligned} &{}_l\Psi_m\left[(\alpha_n, A_n)_{1,l}; (\beta_n, B_n)_{1,m}; z\right] \\ &= \sum_{k=0}^{\infty}\left\{\prod_{n=1}^{l}\Gamma(\alpha_n+kA_n)\right\}\left\{\prod_{n=1}^{m}\Gamma(\beta_n+kB_n)\right\}^{-1}\frac{z^k}{k!}\ (z\in U). \end{aligned} \tag{5}$$

If $A_n=1$, $(n=1,\dots,l)$, $B_n=1$, $(n=1,\dots,m)$, we have

$$\Omega\, {}_l\Psi_m\left[(\alpha_n,1)_{1,l}; (\beta_n,1)_{1,m}; z\right]={}_lF_m(\alpha_1, ...\alpha_l, \beta_1, ...\beta_m, z), \tag{6}$$

which is the generalized hypergemetric function where

$$\Omega=\left(\prod_{n=1}^{l}\Gamma(\alpha_n)\right)^{-1}\left(\prod_{n=1}^{m}\Gamma(\beta_n)\right). \tag{7}$$

Using the Wright hypergeometric function, Dziok and Raina ([7] and [8]) introduced the linear operator

$$\theta_p^{l,m}\left[(\alpha_n,A_n)_{1,l};(\beta_n,B_n)_{1,m}\right]:A_p\to A_p,$$

which is defined by the following convolution

$$\theta_p^{l,m}\left[(\alpha_n,A_n)_{1,l};(\beta_n,B_n)_{1,m}\right]f(z)=\phi_p^{l,m}\left[(\alpha_n,A_n)_{1,l};(\beta_n,B_n)_{1,m}\right]*f(z), \quad (8)$$

where

$$\phi_p^{L,m}\left[(\alpha_n,A_n)_{1,L};(\beta_n,B_n)_{1,m}\right]=\Omega z_L^p\Psi_m\left[(\alpha_n,A_n)_{1,L};(\beta_n,B_n)_{1,m};z\right]*f(z), \quad (9)$$

If $f(z)\in A_p$ is given by equation (1), then we have

$$\theta_p^{l,m}\left[(\alpha_n,A_n)_{1,l};(\beta_n,B_n)_{1,m}\right]f(z)=z^p+\sum_{k=1}^{\infty}\sigma(k)a_{k+p}z^{k+p}, \quad (10)$$

where

$$\sigma(k)=\Omega\frac{\prod_{n=1}^{l}\Gamma(\alpha_n+kA_n)}{\prod_{n=1}^{m}\Gamma(\beta_n+kB_n)\,k!}. \quad (11)$$

In order to make the notation simple, we write

$$\theta_p^{l,m}\left[(\alpha_n,A_n)_{1,l};(\beta_n,B_n)_{1,m}\right]=\theta_p^{l,m}[\alpha_1,A_1,B_1]. \quad (12)$$

It is easily verified from (9) that

$$\begin{aligned} z\left(\theta_p^{L,m}[\alpha_1,A_1,B_1]f(z)\right)' &= \frac{\alpha_1}{A_1}\theta_p^{L,m}[\alpha_1+1,A_1,B_1]f(z)\\ &\quad-\left(\frac{\alpha_1}{A_1}-p\right)\theta_p^{L,m}[\alpha_1,A_1,B_1]f(z) \quad (A_1>0). \end{aligned} \quad (13)$$

Not that for $A_n=1$, $(n=1,\dots,l)$, $B_n=1$, $(n=1,\dots,m)$, we have

$$\theta_p^{l,m}[\alpha_1,1,1]=H_p^{l,m}[\alpha_1], \quad (14)$$

where $H_p^{l,m}[\alpha_1]$ is the Dziok–Srivastava operator [5].
It is well known [6] that

$$z\left[H_m^l[\alpha_1]f(z)\right]'=\alpha_1H_m^l[\alpha_1+1]f(z)-(\alpha_1-p)H_m^l[\alpha_1]f(z), \quad (15)$$

where $H_m^l[\alpha_1]f(z)=H_m^l(\alpha_1,\dots\dots,\alpha_l;\beta_1,\dots\dots,\beta_m)f(z)$.
To prove our results, we need the following definitions and lemmas.

Definition 1.1. ([11]) Denote by Q the set of all functions $q(z)$ that are analytic and injective on $\bar{U}/E(q)$ where

$$E(q)=\{\zeta\in\partial U:\lim_{z\to\zeta}q(z)=\infty\},$$

and are such that $q'(\zeta)\neq 0$ for $\zeta\in\partial U/E(q)$. Further, let the subclass of Q for which $q(0)=a$ be denoted by $Q(a)$, $Q(0)\equiv Q_0$ and $Q(1)\equiv Q_1$.

Definition 1.2. ([12]) A function $L(z,t)\,(z\in U, t\geq 0)$ is said to be a subordination chain if $L(0,t)$ is analytic and univalent in $z\in U$ for all $t\geq 0$, $L(z,0)$ is continuously differentiable on $[0;1]$ for all $z\in U$ and $L(z,t_1)\prec L(z,t_2)$ for all $0\leq t_1\leq t_2$.

Lemma 1.3. ([13]) *The function $L(z,t):U\times[0;1]\to\mathbb{C}$ of the form*

$$L(z,t)=a_1(t)z+a_2(t)z^2+\ldots \qquad (a_1(t)\neq 0; t\geq 0),$$

and $\lim_{t\to\infty}|a_1(t)|=\infty$ is a subordination chain if and only if

$$\Re\left\{\frac{z\partial L(z,t)/\partial t}{\partial L(z,t)/\partial t}\right\}>0 \quad (z\in U,\ t\geq 0).$$

Lemma 1.4. ([9]) *Suppose that the function $H:\mathbb{C}^2\to\mathbb{C}$ satisfies the condition*

$$\Re\{H(is;t)\}\leq 0$$

for all real s and for all $t\leq -n(1+s^2)/2, n\in\mathbb{N}$. If the function $p(z)=1+a_nz^n+a_{n+1}z^{n+1}+\ldots$, is analytic in U and $\Re\{H(p(z);zp'(z))\}>0$ $(z\in U)$. then $\Re\{p(z)\}>0$ for $z\in U$.

Lemma 1.5. ([10]) *Let $k,\gamma\in\mathbb{C}$ with $k\neq 0$ and let $h\in H(U)$ with $H(0)=c$. If $\Re\{kh(z)+\gamma\}>0$ $(z\in U)$, then the solution of the following differential equation:*

$$q(z)+\frac{zq'(z)}{kq(z)+\gamma}=h(z)(z\in U;\,q(0)=c),$$

is analytic in U and satisfies $\Re\{kh(z)+\gamma\}>0$ for $z\in U$.

Lemma 1.6. ([11]) *Let $p\in Q(a)$ and let $q(z)=a+a_nz^n+a_{n+1}z^{n+1}+\ldots$, be analytic in U with $q(z)\neq 0$ and $n\geq 1$. If q is not subordinate to p, the there exists two points $z_0=r_0e^{i\theta}\in U$ and $\xi_0\in\partial U/E(q)$ such that $q(U_{r_o})\subset p(U)$; $q(z_0)=p(\xi_0)$ and $z_0p'(z_0)=m\xi_0p(\xi_0)$ $m\geq n$.*

Lemma 1.7. ([12]) *Let $q \in H[a,1]$ and $\phi : \mathbb{C}^2 \to \mathbb{C}$ also $\phi(q(z), zq'(z)) = h(z)$. If $L(z,t) = \phi(q(z), tzq'(z))$ is a subordination chain and $q \in H[a,1] \bigcap Q(a)$, then*

$$h(z) \prec \phi\left(p(z), zp'(z)\right),$$

implies that $q(z) \prec p(z)$. Further, if $\phi(q(z), zq'(z)) = h(z)$ has a univalent solution $q \in Q(a)$, then q is the best subordination.

In the present paper, we aim to prove some subordination-preserving and superordination-preserving properties associated with the fractional differintegral operator $\theta_p^{L,m}[\alpha_1, A_1, B_1]$. Sandwich-type result involving this operator is also derived. A simililar problem for analytic functions was studied by Aouf and Seoudy [3] and [4].

2. Subordination, superordination and sandwich results involving the operator $\theta_p^{L,m}[\alpha_1, A_1, B_1]$

Theorem 2.1. *Let $f, g \in A_p$ and let*

$$\Re\left\{1 + \frac{z\phi''(z)}{\phi'(z)}\right\} > -\delta, \tag{16}$$

where

$$\phi(z) = \left(\frac{\theta_p^{l,m}[\alpha_1+1, A_1, B_1]\, g(z)}{\theta_p^{l,m}[\alpha_1, A_1, B_1]\, g(z)}\right)\left(\frac{\theta_p^{l,m}[\alpha_1, A_1, B_1]\, g(z)}{z^p}\right)^{\mu} (\mu > 0; z \in U), \tag{17}$$

where $\alpha_1, A_1, ..., \alpha_l, A_l$ and $\beta_1, B_1, ..., \beta_m, B_m$ $(l, m \in \mathbb{N} = \{1, 2...\})$ are positive real parameters such that $1 + \sum_{k=1}^{m} B_k - \sum_{k=1}^{l} A_k > 0$, and δ is given by

$$\delta = \frac{A_1^2 + \mu^2\alpha_1^2 - \left|A_1^2 - \mu^2\alpha_1^2\right|}{4\mu A_1 \alpha_1}. \tag{18}$$

Then the subordination condition

$$\left(\frac{\theta_p^{l,m}[\alpha_1+1, A_1, B_1]\, f(z)}{\theta_p^{l,m}[\alpha_1, A_1, B_1]\, f(z)}\right)\left(\frac{\theta_p^{l,m}[\alpha_1, A_1, B_1]\, f(z)}{z^p}\right)^{\mu}$$

$$\prec \left(\frac{\theta_p^{l,m}[\alpha_1+1, A_1, B_1]\, g(z)}{\theta_p^{l,m}[\alpha_1, A_1, B_1]\, g(z)}\right)\left(\frac{\theta_p^{l,m}[\alpha_1, A_1, B_1]\, g(z)}{z^p}\right)^{\mu},$$

implies that

$$\left(\frac{\theta_p^{l,m}[\alpha_1,A_1,B_1]f(z)}{z^p}\right)^{\mu} \prec \left(\frac{\theta_p^{l,m}[\alpha_1,A_1,B_1]g(z)}{z^p}\right)^{\mu},$$

and the function $\left(\frac{\theta_p^{l,m}[\alpha_1,A_1,B_1]g(z)}{z^p}\right)^{\mu}$ *is the best dominant.*

Proof. Let us define the functions $F(z)$ and $G(z)$ in U by

$$F(z)=\left(\frac{\theta_p^{l,m}[\alpha_1,A_1,B_1]f(z)}{z^p}\right)^{\mu} \text{ and } G(z)=\left(\frac{\theta_p^{l,m}[\alpha_1,A_1,B_1]g(z)}{z^p}\right)^{\mu} \quad (19)$$

we assume here, without loss of generality, that $G(z)$ is analytic and univalent on $\overline{U}$ and

$$G'(\zeta)\neq 0 \quad (|\zeta|=1).$$

If not, then we replace $F(z)$ and $G(z)$ by $F(\rho z)$ and $G(\rho z)$, respectively, with $0<\rho<1$. These new functions have the desired properties on $\overline{U}$, and we can use them in the proof of our result. Therefore, the results would follow by letting $\rho\to 1$. We first show that, if

$$q(z)=1+\frac{zG''(z)}{G'(z)} \quad (z\in U), \quad (20)$$

then

$$\Re\{q(z)\}>0 \quad (z\in U).$$

From (11) and the definition of the functions G, ϕ, we obtain that

$$\phi(z)=G(z)+\frac{A_1zG'(z)}{\mu\alpha_1}. \quad (21)$$

Differentiating both side of (21) with respect to z yields

$$\phi'(z)=\left(1+\frac{A_1}{\mu\alpha_1}\right)G'(z)+\frac{A_1zG'(z)}{\mu\alpha_1}. \quad (22)$$

Combining (20) and (22), we easily get

$$1+\frac{z\phi''(z)}{\phi'(z)}=q(z)+\frac{A_1zq'(z)}{q(z)+\mu\alpha_1}=h(z) \quad (z\in U). \quad (23)$$

It follows from (16) and (23) that

$$\Re\left\{h(z)+\frac{\mu\alpha_1}{A}\right\}>0 \ \ (z\in U). \tag{24}$$

Moreover, by using Lemma 1.5, we conclude that the differential equation (23) has a solution $q(z)\in H(U)$ with $h(0)=q(0)=1$. Let

$$H(u,v)=u+\frac{A_1v}{A_1u+\mu\alpha_1}+\delta,$$

where δ is given by (18). From (23) and (24), we obtain

$$\Re\left\{H(q(z);zq'(z))\right\}>0 \ \ (z\in U).$$

To verify the condition that

$$\Re\{H(is;t)\}\leq 0 \qquad \left(t\leq -(1+s^2)/2;\, s\in\mathbb{R}\right). \tag{25}$$

we proceed it as follows:

$$\Re\{H(is;t)\}=\Re\left\{is+\frac{A_1t}{A_1is+\mu\alpha_1}+\delta\right\}=\frac{A_1\alpha_1t\mu}{A_1^2s^2+\mu^2\alpha_1^2}+\delta$$

$$\leq -\frac{A_1^2\psi_p(\alpha_1,\,\mu,A_1,\,s)}{2\left[A_1^2s^2+\mu^2\alpha_1^2\right]},$$

where

$$\psi_p(\alpha_1,\,\mu,A_1,\,s)=\left[\frac{\mu\alpha_1}{A_1}-2\delta\right]s^2-2\delta\frac{\mu^2\alpha_1^2}{A_1^2}+\frac{\mu\alpha_1}{A_1}. \tag{26}$$

For δ given by (18), we note that the expression $\psi_p(\alpha_1,\,\mu,A_1,\,s)$ in (26) is a positive, which implies that (25) holds. Thus, by using Lemma 1.4, we conclude that

$$\Re\{q(z)\}>0 \ \ (z\in U).$$

By the definition of $q(z)$, we know that G is convex. To prove $F\prec G$, let the function $L(z,t)$ be defined by

$$L(z,t)=G(z)+\frac{(1+t)A_1zG'(z)}{\mu\alpha_1} \quad (0\leq t<\infty;\, z\in U). \tag{27}$$

Since G is convex, then

$$\left.\frac{\partial L(z,t)}{\partial z}\right|_{z=0}=G'(0)\left(1+\frac{A_1(1+t)}{\mu\alpha_1}\right)\neq 0 \quad (0\leq t<\infty;\, z\in U)$$

and

$$\Re\left\{\frac{z\partial L(z,t)/\partial t}{\partial L(z,t)/\partial t}\right\}=\Re\left\{\frac{\mu\alpha_1}{A_1}+(1+t)q(z)\right\}>0\ (0\le t<\infty;\,z\in U).$$

Therefore, by using Lemma 1.3, we deduce that $L(z,t)$ is a subordination chain. It follows from the definition of subordination chain that

$$\phi(z)=G(z)+\frac{A_1zG'(z)}{\mu\alpha_1}=L(z,0),$$

and

$$L(z,0)\prec L(z,t)\quad(0\le t<\infty),$$

which implies

$$L(\zeta,t)\notin L(U,0)\quad(0\le t<\infty;\zeta\in\partial U), \tag{28}$$

If F is not subordinate to G, by using Lemma 1.6, we know that there exist two points $z_0\in U$and $\zeta_0\in\partial U$such that

$$F(z_0)=G(\zeta_0)\ \textit{and}\ z_0F'(z_0)=(1+t)\zeta_0p(\zeta_0)\quad(0\le t<\infty). \tag{29}$$

Hence, by virtue of (11) and (29), we have

$$L(\zeta_0,t)=G(\zeta_0)+\frac{(1+t)A_1zG'(\zeta_0)}{\mu\alpha_1}=F(z_0)+\frac{A_1z_0F'(z_0)}{\mu\alpha_1}$$

$$=\left(\frac{\theta_p^{l,m}[\alpha_1+1,A_1,B_1]f(z_0)}{\theta_p^{l,m}[\alpha_1,A_1,B_1]f(z_0)}\right)\left(\frac{\theta_p^{l,m}[\alpha_1,A_1,B_1]f(z_0)}{z^p}\right)^{\mu}\in\phi(U).$$

This contradicts to (28). Thus, we deduce that $F\prec G$. Considering $F=G$, we see that the function G is the best dominant.

□

By taking $A_n=1$, $(n=1,\ldots,l)$ and $B_n=1$, $(n=1,\ldots,m)$, in Theorem 2.1 and using the relation (14) we get the following corollary

Corollary 2.2. *Let f, $g\in A_p$ and let*

$$\Re\left\{1+\frac{z\phi''(z)}{\phi'(z)}\right\}>-\delta, \tag{30}$$

where

$$\phi(z)=\left(\frac{H_p^{l,m}[\alpha_1+1]g(z)}{H_p^{l,m}[\alpha_1]g(z)}\right)\left(\frac{H_p^{l,m}[\alpha_1]g(z)}{z^p}\right)^{\mu}\ (\mu>0;z\in U), \tag{31}$$

where $\alpha_1,...,\alpha_l$ *and* $\beta_1,...,\beta_m (L,m \in \mathbb{N} = \{1,2...\})$ *are positive real parameters and* δ *is given by*

$$\delta = \frac{1+\mu^2\alpha_1^2 - \left|1-\mu^2\alpha_1^2\right|}{4\mu\alpha_1}. \tag{32}$$

Then the subordination condition

$$\left(\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}\right)\left(\frac{H_p^{l,m}[\alpha_1]f(z)}{z^p}\right)^{\mu}$$

$$\prec \left(\frac{H_p^{l,m}[\alpha_1+1]g(z)}{H_p^{l,m}[\alpha_1]g(z)}\right)\left(\frac{H_p^{l,m}[\alpha_1]g(z)}{z^p}\right)^{\mu}$$

implies that

$$\left(\frac{H_p^{l,m}[\alpha_1]f(z)}{z^p}\right)^{\mu} \prec \left(\frac{H_p^{l,m}[\alpha_1]g(z)}{z^p}\right)^{\mu},$$

and the function $\left(\frac{H_p^{l,m}[\alpha_1]g(z)}{z^p}\right)^{\mu}$ *is the best dominant.*

We now derive the following superordination result.

Theorem 2.3. *Let* $f, g \in A_p$ *and let*

$$\Re\left\{1+\frac{z\phi''(z)}{\phi'(z)}\right\} > -\delta, \tag{33}$$

where

$$\phi(z) = \left(\frac{\theta_p^{l,m}[\alpha_1+1,A_1,B_1]g(z)}{\theta_p^{l,m}[\alpha_1,A_1,B_1]g(z)}\right)\left(\frac{\theta_p^{l,m}[\alpha_1,A_1,B_1]g(z)}{z^p}\right)^{\mu} (\mu > 0; z \in U), \tag{34}$$

where $\alpha_1,A_1,...,\alpha_l,A_l$ *and* $\beta_1,B_1,...,\beta_m,B_m$ $(l,m \in \mathbb{N} = \{1,2...\})$ *are positive real parameters such that* $1+\sum_{k=1}^{m}B_k - \sum_{k=1}^{l}A_k > 0$, *and* δ *is given by* (18). *If the function* $\left(\frac{\theta_p^{l,m}[\alpha_1+1,A_1,B_1]f(z)}{\theta_p^{l,m}[\alpha_1,A_1,B_1]f(z)}\right)\left(\frac{\theta_p^{l,m}[\alpha_1,A_1,B_1]f(z)}{z^p}\right)^{\mu}$ *is univalent in* U *and* $\left(\frac{\theta_p^{l,m}[\alpha_1,A_1,B_1]f(z)}{z^p}\right)^{\mu} \in Q$, *then the superordination condition*

$$\left(\frac{\theta_p^{l,m}[\alpha_1+1,A_1,B_1]g(z)}{\theta_p^{l,m}[\alpha_1,A_1,B_1]f(z)}\right)\left(\frac{\theta_p^{l,m}[\alpha_1,A_1,B_1]g(z)}{z^p}\right)^{\mu}$$

$$\prec \left(\frac{\theta_p^{l,m}[\alpha_1+1,A_1,B_1]f(z)}{\theta_p^{l,m}[\alpha_1,A_1,B_1]f(z)}\right)\left(\frac{\theta_p^{l,m}[\alpha_1,A_1,B_1]f(z)}{z^p}\right)^{\mu},$$

implies that

$$\left(\frac{\theta_p^{l,m}[\alpha_1,A_1,B_1]\,g(z)}{z^p}\right)^{\mu} \prec \left(\frac{\theta_p^{l,m}[\alpha_1,A_1,B_1]\,f(z)}{z^p}\right)^{\mu},$$

and the function $\left(\frac{\theta_p^{l,m}[\alpha_1,A_1,B_1]g(z)}{z^p}\right)^{\mu}$ *is the best subordinant.*

Proof. Suppose that the functions F, G and q are defined by (19) and (20), respectively. By applying the similar method as in the proof of Theorem 2.1, we get

$$\Re\{q(z)\} > 0 \ \ (z \in U).$$

Next, to arrive at our desired result, we show that $G \prec F$. For this, we suppose that the function $L(z,t)$ be defined by (27).
Since G is convex, by applying a similar method as in Theorem 2.1, we deduce that $L(z,t)$ is subordination chain. Therefore, by using Lemma 1.7, we conclude that $G \prec F$. Moreover, since the differential equation

$$\phi(z) = G(z) + \frac{A_1 z G'(z)}{\mu\alpha_1} = \varphi\left(G(z),\, zG'(z)\right)$$

has a univalent solution G, it is the best subordinant. □

By taking $A_n = 1$, $(n = 1,\ldots,l)$ and $B_n = 1$, $(n = 1,\ldots,m)$, in Theorem 2.3 and using the relation (14) we get the following corollary

Corollary 2.4. *Let* $f, g \in A_p$ *and let*

$$\Re\left\{1 + \frac{z\phi''(z)}{\phi'(z)}\right\} > -\delta, \tag{35}$$

where

$$\phi(z) = \left(\frac{H_p^{l,m}[\alpha_1+1]\,g(z)}{H_p^{l,m}[\alpha_1]\,g(z)}\right)\left(\frac{H_p^{l,m}[\alpha_1]\,g(z)}{z^p}\right)^{\mu} (\mu > 0; z \in U), \tag{36}$$

where $\alpha_1,...,\alpha_l$ *and* $\beta_1,...,\beta_m(l,m \in \mathbb{N} = \{1,2...\})$ *are positive real parameters and* δ *is given by* (32).
If the function $\left(\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}\right)\left(\frac{H_p^{l,m}[\alpha_1]f(z)}{z^p}\right)^{\mu}$ *is univalent in* U
and $\left(\frac{H_p^{l,m}[\alpha_1]f(z)}{z^p}\right)^{\mu} \in Q$*, then the superordination condition*

$$\left(\frac{H_p^{l,m}[\alpha_1+1]g(z)}{H_p^{l,m}[\alpha_1]g(z)}\right)\left(\frac{H_p^{l,m}[\alpha_1]g(z)}{z^p}\right)^{\mu}$$
$$\prec\left(\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}\right)\left(\frac{H_p^{l,m}[\alpha_1]f(z)}{z^p}\right)^{\mu}$$

implies that

$$\left(\frac{H_p^{l,m}[\alpha_1]g(z)}{z^p}\right)^{\mu}\prec\left(\frac{H_p^{l,m}[\alpha_1]f(z)}{z^p}\right)^{\mu},$$

and the function $\left(\frac{H_p^{l,m}[\alpha_1]g(z)}{z^p}\right)^{\mu}$ *is the best subordinant.*

Combining Theorems 2.1 and 2.3, we obtain the following "sandwich-type result".

Theorem 2.5. *Let* $f, g_i \in A_p\ (j=1,2)$ *and let*

$$\Re\left\{1+\frac{z\phi_j''(z)}{\phi_j'(z)}\right\}>-\delta, \tag{37}$$

where

$$\phi_j(z)=\left(\frac{\theta_p^{l,m}[\alpha_1+1,A_1,B_1]g_j(z)}{\theta_p^{l,m}[\alpha_1,A_1,B_1]g_j(z)}\right) \cdot\left(\frac{\theta_p^{l,m}[\alpha_1,A_1,B_1]g_j(z)}{z^p}\right)^{\mu}\ (\mu>0; z\in U) \tag{38}$$

where $\alpha_1,A_1,...,\alpha_l,A_l$ *and* $\beta_1,B_1,...,\beta_m,B_m$ $(L,m\in\mathbb{N}=\{1,2...\})$ *are positive real parameters such that* $1+\sum_{k=1}^{m}B_k-\sum_{k=1}^{l}A_k>0$, *and* δ *is given by* (18). *If the function* $\left(\frac{\theta_p^{l,m}[\alpha_1+1,A_1,B_1]f(z)}{\theta_p^{l,m}[\alpha_1,A_1,B_1]f(z)}\right)\left(\frac{\theta_p^{l,m}[\alpha_1,A_1,B_1]f(z)}{z^p}\right)^{\mu}$ *is univalent in* U*and* $\left(\frac{\theta_p^{l,m}[\alpha_1,A_1,B_1]f(z)}{z^p}\right)^{\mu}\in Q$, *then the condition*

$$\left(\frac{\theta_p^{l,m}[\alpha_1+1,A_1,B_1]g_1(z)}{\theta_p^{l,m}[\alpha_1,A_1,B_1]g_1(z)}\right)\left(\frac{\theta_p^{l,m}[\alpha_1,A_1,B_1]g_1(z)}{z^p}\right)^{\mu}$$
$$\prec\left(\frac{\theta_p^{l,m}[\alpha_1+1,A_1,B_1]f(z)}{\theta_p^{l,m}[\alpha_1,A_1,B_1]f(z)}\right)\left(\frac{\theta_p^{l,m}[\alpha_1,A_1,B_1]f(z)}{z^p}\right)^{\mu}$$

$$\prec \left(\frac{\theta_p^{l,m}[\alpha_1+1,A_1,B_1]\,g_2(z)}{\theta_p^{l,m}[\alpha_1,A_1,B_1]\,g_2(z)}\right)\left(\frac{\theta_p^{l,m}[\alpha_1,A_1,B_1]\,g_2(z)}{z^p}\right)^{\mu},$$

implies that

$$\left(\frac{\theta_p^{l,m}[\alpha_1,A_1,B_1]\,g_1(z)}{z^p}\right)^{\mu} \prec \left(\frac{\theta_p^{l,m}[\alpha_1,A_1,B_1]\,f(z)}{z^p}\right)^{\mu}$$

$$\prec \left(\frac{\theta_p^{l,m}[\alpha_1,A_1,B_1]\,g_1(z)}{z^p}\right)^{\mu},$$

and the function $\left(\frac{\theta_p^{l,m}[\alpha_1,A_1,B_1]g_1(z)}{z^p}\right)^{\mu}$ *and* $\left(\frac{\theta_p^{l,m}[\alpha_1,A_1,B_1]g_2(z)}{z^p}\right)^{\mu}$ *are, respectively, the best subordinant and the best dominant.*

By taking $A_n = 1$, $(n = 1,\ldots,l)$ and $B_n = 1$, $(n = 1,\ldots,m)$, in Theorem 2.5 and using the relation (14) we get the following corollary

Corollary 2.6. *Let* $f,\, g_i \in A_p \;\; (j = 1,2)$ *and let*

$$\Re\left\{1+\frac{z\phi_j''(z)}{\phi_j'(z)}\right\} > -\delta, \tag{39}$$

where

$$\phi_j(z) = \left(\frac{H_p^{l,m}[\alpha_1+1]\,g_j(z)}{H_p^{l,m}[\alpha_1]\,g_j(z)}\right)\left(\frac{H_p^{l,m}[\alpha_1]\,g_j(z)}{z^p}\right)^{\mu} \quad (\mu>0;\, z\in U) \tag{40}$$

where $\alpha_1,\ldots,\alpha_l$ *and* $\beta_1,\ldots,\beta_m(l,m\in\mathbb{N}=\{1,2\ldots\})$ *are positive real parameters and* δ *is given by* (32). *If the function* $\left(\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}\right)\left(\frac{H_p^{l,m}[\alpha_1]f(z)}{z^p}\right)^{\mu}$ *is univalent in* U *and* $\left(\frac{H_p^{l,m}[\alpha_1]f(z)}{z^p}\right)^{\mu} \in Q$, *then the condition*

$$\left(\frac{H_p^{l,m}[\alpha_1+1]\,g_1(z)}{H_p^{l,m}[\alpha_1]\,g_1(z)}\right)\left(\frac{H_p^{l,m}[\alpha_1]\,g_1(z)}{z^p}\right)^{\mu}$$

$$\prec \left(\frac{H_p^{l,m}[\alpha_1+1]\,f(z)}{H_p^{l,m}[\alpha_1]\,f(z)}\right)\left(\frac{H_p^{l,m}[\alpha_1]\,f(z)}{z^p}\right)^{\mu}$$

$$\prec \left(\frac{H_p^{l,m}[\alpha_1+1]\,g_2(z)}{H_p^{l,m}[\alpha_1]\,g_2(z)}\right)\left(\frac{H_p^{l,m}[\alpha_1]\,g_2(z)}{z^p}\right)^{\mu}$$

implies that

$$\left(\frac{H_p^{l,m}[\alpha_1]\,g_1(z)}{z^p}\right)^{\mu} \prec \left(\frac{H_p^{l,m}[\alpha_1]\,f(z)}{z^p}\right)^{\mu} \prec \left(\frac{H_p^{l,m}[\alpha_1]\,g_1(z)}{z^p}\right)^{\mu},$$

and the function $\left(\frac{H_p^{l,m}[\alpha_1]g_1(z)}{z^p}\right)^{\mu}$ *and* $\left(\frac{H_p^{l,m}[\alpha_1]g_2(z)}{z^p}\right)^{\mu}$ *are, respectively, the best subordinant and the best dominant.*

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